Lecture Notes in Mathematics
1527

M. I. Freidlin J.-F. Le Gall

Ecole d'Eté de Probabilités de Saint-Flour XX – 1990

Editor: P. L. Hennequin

Springer-Verlag

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Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

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Mathématiques Appliquées
F-63177 Aubière Cedex, France

Mathematics Subject Classification (1991): 60-02, 35A25, 35B40, 35C20, 35K55, 60G17, 60J65, 60J80

ISBN 3-540-56250-8 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-56250-8 Springer-Verlag New York Berlin Heidelberg

Typesetting: Camera ready by author/editor
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
46/3140-543210 - Printed on acid-free paper

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for the publication of monographs

In what follows all references to monographs, are applicable also to multiauthorship volumes such as seminar notes.

§ 1. Lecture Notes aim to report new developments - quickly, informally, and at a high level. Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes manuscripts from journal articles which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for Ph. D. theses to be accepted for the Lecture Notes series.

§ 2. Manuscripts or plans for Lecture Notes volumes should be submitted (preferably in duplicate) either to one of the series editors or to Springer- Verlag, Heidelberg . These proposals are then refereed. A final decision concerning publication can only be made on the basis of the complete manuscript, but a preliminary decision can often be based on partial information: a fairly detailed outline describing the planned contents of each chapter, and an indication of the estimated length, a bibliography, and one or two sample chapters - or a first draft of the manuscript. The editors will try to make the preliminary decision as definite as they can on the basis of the available information.

§ 3. Final manuscripts should preferably be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1527

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INTRODUCTION

Ce volume contient deux des cours donnés à l'Ecole d'Eté de Calcul des Probabilités de Saint-Flour du 1er au 18 Juillet 1990.

Nous avons choisi de les publier sans attendre le troisième cours, "Function Estimation and the White Noise Model" de Monsieur DONOHO, dont la rédaction n'est pas encore complètement achevée et figurera dans le volume suivant.

Nous remercions les auteurs qui ont effectué un gros travail de rédaction définitive qui fait de leurs cours un texte de référence.

L'Ecole a rassemblé soixante six participants dont 32 ont présenté, dans un exposé, leur travail de recherche.

On trouvera ci-dessous la liste des participants et de ces exposés dont un résumé pourra être obtenu sur demande.

Afin de faciliter les recherches concernant les écoles antérieures, nous redonnons ici le numéro du volume des "Lecture Notes" qui leur est consacré :

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SEMI-LINEAR PDE'S AND LIMIT THEOREMS FOR LARGE DEVIATIONS

Mark I. FREIDLIN

SEMI-LINEAR PDE'S AND LIMIT THEOREMS FOR LARGE DEVIATIONS

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Introduction.

We consider two classes of asymptotic problems concerning semi-linear parabolic equations. The common element in both these classes is not only the connections with semi-linear PDE's, but the utilization of different kinds of limit theorems for random processes and fields. The limit theorems for large deviations are especially useful in the problems under consideration.

It is well known that a Markov process X_t with continuous trajectories can be connected with any second order elliptic, maybe degenerate, operator $L = \frac{1}{2} \sum_{i,j=1}^{r} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x) \frac{\partial}{\partial x^i}$. The most convenient, but not unique, way to construct this process is given by stochastic differential equations. The solutions of the natural boundary problems for L or of the initial-boundary problems for the operator $\frac{\partial}{\partial t} - L$ can be written as expectations of the proper functionals of the process X_t. These expectations are often called functional integrals. They, together with the stochastic equations, give more or less in an explicit way the dependence of the solutions on the coefficients of operator or on initial-boundary conditions. This makes the probabilistic representations very convenient instruments for studying the PDE's. The probabilistic approach turns out to be especially useful in many asymptotic problems for PDE's. Limit theorems, which is a traditional area of probability theory, help to solve the asymptotic problems for PDE's.

The probabilistic approach turns out to be useful for nonlinear second order parabolic equations, too.

The first class of problems which we consider here concerns some asymptotic problems for semi-linear parabolic equations and systems of such equations. The main attention is paid to wave front propagation in reaction-diffusion equations (RDE's) and systems (see, for example, [20]).

By an RDE we mean one equation or a system of equations of the following form:

$$\frac{\partial u_k}{\partial t} = L_k u_k(t,x) + f_k(x, u_1, \ldots, u_n), \quad x \in D \subseteq R^r, \quad t > 0, \tag{0.1}$$
$$u_k(0,x) = g_k, \quad k = 1, \ldots, n.$$

Here L_k, $k = 1, \ldots, n$, are second order elliptic, maybe degenerate, linear operators. Some boundary conditions should be supplemented to the problem if $D \neq R^r$.

The simplest example of an RDE is the Kolmogorov-Petrovskii-Piskunov (KPP) equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u(1-u), \quad x \in R^1, \quad t > 0, \quad u(0,x) = g(x).$$

It was proved in [17] that for certain initial functions the solution of the KPP equation for large t is close to a running wave solution $v(x-\alpha t)$. The shape $v(z)$ of the wave and its speed α are defined by the equation.

We consider various generalizations of this result ([6]-[8], [10], [11]) in the first part of these lectures. These generalizations lead to some new effects in the behavior of the solutions, such as jumps of the wave fronts and breaking of the Huygens principle in slowly changing non-homogeneous media, or an increase of the speed of the fronts in the weakly coupled RDE's. In simple situations the motion of the wave front can be described by Huygens principle, in the proper Riemannian or Finsler metric. KPP equation and some generalizations of this equation are considered in [1]-[3], [14], [18].

The RDE system defines a semi-flow $U_t = (u_1(t,\cdot),\dots,u_n(t,\cdot))$ in the space of continuous functions of x. This semi-flow in general has a rich ω-limit set, which consists of the stationary points of the semi-flow, the periodic-in-time solutions, and more complicated subsets of the phase space. Suppose now that the semi-flow is subjected to small random perturbations. Then the solution of the perturbed RDE system $u^{\varepsilon}(t,x)$, (ε characterizes the "strength" of the perturbations) will be a random field. We can look on $u^{\varepsilon}(t,x)$ as on a random process in the functional space, which is a perturbation of the initial semi-flow.

The second class of problems which we consider here concerns the deviations of $u^{\varepsilon}(t,\cdot)$ from U_t. In the case of PDE's there are more ways to introduce perturbations than in the case of finite-dimensional dynamical systems. For example, an interesting problem is the consideration of perturbations of the boundary conditions. We study several classes of perturbations of the semi-flow and establish results of law-of-large-numbers type, of central-limit-theorem type, and limit theorems for large deviations (see [5], [9], [13], [16], [19], [21], [24]).

It is my pleasure to thank Richard Sowers for his assistance in the preparation of the manuscript and many useful remarks.

§1. Markov Processes and Differential Equations.

Let

$$L = \frac{1}{2} \sum_{i,j=1}^{r} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x) \frac{\partial}{\partial x^i}$$

be an elliptic, maybe degenerate, operator. This means that $\sum_{i,j=1}^{r} a^{ij}(x)\lambda_i\lambda_j \geq 0$ for any real $\lambda_1,\ldots,\lambda_r$ and any $x \in R^r$.

We assume that the coefficients are bounded and at least Lipschitz continuous. If the matrix $(a^{ij}(x))$ degenerates we assume that the entries $a^{ij}(x)$ have bounded second order derivatives. This last assumption provides the existence of a matrix $\sigma(x) = (\sigma^i_j(x))_1^r$ with Lipschitz-continuous elements such that $\sigma(x)\sigma^*(x) = (a^{ij}(x))$ (see [8] Ch. 1). In the non-degenerate case, of course, the existence of such a matrix $\sigma(x)$ is provided by the Lipschitz continuity of the entries $a^{ij}(x)$.

Let W_t, $t \geq 0$, is the r-dimensional Wiener process. Consider the stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

Here $b(x) = (b^1(x),\ldots,b^r(x))$; $\sigma(x)$ is introduced above. Since the coefficients of the equation are bounded and Lipschitz continuous, there exists a unique solution X_t^x (the index x points out the starting point $x \in R^r$). The set of random processes $\{X_t^x, \ x \in R^r\}$ form the Markov family X_t^x corresponding to the operator L in the phase space R^r. The family of probability measures P_x in the space $C_{0\infty}(R^r)$ of continuous functions on $[0,\infty)$ with values in R^r induced by the processes X_t^x in $C_{0,\infty}(R^r)$ is called the Markov process corresponding to the operator L.

For any smooth enough function $U(t,x)$, one can write the Ito formula

$$(1.1) \quad U(t,X_t^x) - U(0,x) = \int_0^t [\frac{\partial U}{\partial t}(s,X_s^x) + LU(s,X_s^x)]ds + \int_0^t (\nabla_x U(s,X_s^x), \ \sigma(X_s^x)dW_s).$$

Let B be the Banach space of functions $f(x)$, $x \in \mathbb{R}^r$, which are bounded and measurable with respect to the Borel σ-field. We denote $\|f\| = \sup_{x\in R^r} |f(x)|$. Consider the semi-group T_t corresponding to the family X_t^x (to the Markov process $\{P_x\}$)

$$(T_t f)(x) = Ef(X_t^x) = E_x f(X_t).$$

The subscript x in the sign of expectation points out that we consider integral with respect to the measure P_x. We will use both notations: notations connected with the Markov family $(Ef(X_t^x))$ as well as the notation connected with the process $\{P_x\}$ $(E_x f(X_t))$. The family T_t is a positive contracting semi-group; $T_t f(x) \geq 0$ if $f(x) \geq 0$, and $\|T_t f(x)\| \leq \|f(x)\|$. If $f(x)$ is continuous, then $T_t f(x)$ is also continuous (Feller property). From here, taking into account that the trajectories X_t^x are continuous in t with probability 1 for any x, we conclude (see [8]) that the process $\{P_x\}$ (family X_t^x) has the strong Markov property.

Using (1.1) one can check that the generator $\mathcal{A}$: $\mathcal{A}f = \lim\limits_{t\downarrow 0} \frac{T_t f - f}{t}$ of the semi-group is defined at least for the functions $f = f(x)$ having bounded uniformly continuous second derivatives, and $\mathcal{A}f = Lf$ for such f.

Consider the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x) + c(t,x)u(t,x), \quad t > 0, \quad x \in R^r, \tag{1.2}$$
$$u(0,x) = g(x).$$

Here $c(t,x)$ is a continuous bounded function. The famous Feynman-Kac formula gives the representation of the solution $u(t,x)$ of the problem (1.2) in the form of a functional integral

$$u(t,x) = E_x g(X_t) \exp\left\{\int_0^t c(t-s, X_s)ds\right\}. \tag{1.3}$$

To prove (1.3) assume for a moment that $g(x)$ has bounded continuous second order derivatives, and that the operator L is uniformly elliptic. Then the problem (1.2) has a unique solution $u(t,x)$ which is a bounded function having a first derivative in t and a second derivative in x which is continuous and bounded for $0 \leq t \leq T$, $x \in R^r$. Let $Y_t^x = \int_0^t c(t-s, X_s^x)ds$ and consider the function $u(t-s, X_s^x)\exp\{Y_s^x\}$. By Ito's formula we have

$$u(0, X_t^x)e^{\int_0^t c(t-s, X_s^x)ds} - u(t,x) = \int_0^t (\nabla_x u(t-s, X_s^x),\ \sigma(X_s^x)dW_s) \tag{1.4}$$

$$+\int_0^t [Lu(t-s,X_s) - \frac{\partial u}{\partial t}(t-s,X_s)$$

$$+ c(t-s,X_s^x)u(t-s,X_s^x)]\exp\left\{\int_0^t c(t-s,X_s^x)ds\right\}ds.$$

The first integral in the right side of the last equality has zero expectation. The last integral is equal to zero since $u(t,x)$ is the solution of problem (1.2). Taking into account that $u(0,x) = g(x)$, we derive (1.3) from (1.4).

If $g(x)$ is a continuous bounded function it can be uniformly approximated by a sequence $g_n(x)$ of smooth functions. For solutions $u_n(t,x)$ of problem (1.2) with the initial functions $g_n(x)$ we have the Feynman-Kac formula. From the maximum principle we conclude that $u_n(t,x)\to u(t,x)$ when $g_n\to g$. On the other hand, $E_x g_n(X_t)\exp\left\{\int_0^t c(t-s,X_s)ds\right\}\to$ $E_x g(X_t)\exp\left\{\int_0^t c(t-s,X_s)ds\right\}$ when $n\to\infty$. This leads to (1.3) for continuous bounded $g(x)$.

In general, if the operator L degenerates for some $x \in R^r$, problem (1.2) may not have a classical solution. In this case equality (1.3) defines the generalized solution. It is easy to check that this generalized solution coincides with the generalized solution in the small viscosity sense (see [7] Ch. 3).

Let τ be a Markov time with respect to the family of σ-fields $\mathcal{F}_t = \sigma(X_s, s \le t)$, $\tau_t = \tau\wedge t$. Then taking into account strong Markov property we can conclude from (1.3) that

$$u(t,x) = E_x u(t-\tau_t, X_t)\exp\left\{\int_0^{\tau_t} c(t-s,X_s)ds\right\}. \tag{1.5}$$

Let us consider now an initial-boundary problem

$$\frac{\partial u(t,x)}{\partial t} = Lu + c(t,x)u, \quad t > 0, \quad x \in D \subset R^r$$
$$u(0,x) = g(x), \quad u(t,x)\big|_{x\in\partial D} = \psi(x), \tag{1.6}$$

where D is a bounded domain in R^r with smooth boundary ∂D, and $\psi(x)$ is a continuous function on ∂D. Assume for brevity that the operator L is non-degenerate. Then the problem (1.6) has a unique solution. Denote $\tau = \inf\{t : X_t \notin D\}$, the first exit time from D. The following representation holds for the solution of the problem (1.6)

(1.7)

$$u(t,x) = E_x g(X_t)\chi_{\{\tau>t\}}\exp\left\{\int_0^t c(t-s,X_s)ds\right\} + E_x\psi(X_\tau)\chi_{\{\tau\le t\}}\exp\left\{\int_0^\tau c(t-s,X_s)ds\right\}.$$

This representation can be proved in the same way as (1.3) (see [7], Ch. 2).

There are representations in the form of functional integrals for the solutions of stationary problems for the operator L. For example, the solution of the Dirichlet problem $Lu(x) = 0$, $x \in D \subset R^r$, $u(x)|_{\partial D} = \psi(x)$ can be written in the form

$$u(x) = E_x\psi(X_\tau),$$

where $\tau = \inf\{t : X_t \notin D\}$. The domain D is assumed, for brevity, bounded with smooth boundary; the operator L is elliptic.

One can consider boundary and initial-boundary problems with the Neyman condition or with some more general boundary conditions. In this case the representations in the form of functional integrals are also available, but we should consider random processes in the domain with corresponding boundary conditions (see [7], Ch. 2).

Consider now the Cauchy problem for a quasi-linear parabolic equation of the form

$$(1.8)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{r} a^{ij}(x,u)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x,u)\frac{\partial u}{\partial x^i} + c(x,u) \\ u(0,x) = g(x). \end{cases}$$

We assume that $\sum_{i,j=1}^{r} a^{ij}(x)\lambda_i\lambda_j \ge 0$, that the coefficients of the equation are bounded and Lipschitz continuous, and that the initial function $g(x)$ is continuous and bounded.

Suppose that there exists a solution $u(t,x)$ of the problem (1.8). Then we can consider the Markov family

$$(1.9)\qquad X_s^{t,x} - x = \int_0^s \sigma(X_{s_1}^{t,x}, u(t-s_1, X_{s_1}^{t,x}))dW_{s_1} + \int_0^s \sigma(X_{s_1}^{t,x}, u(t-s_1, X_{s_1}^{t,x}))ds_1.$$

Here $\sigma(x,u)\sigma^*(x,u) = (a^{ij}(x,u))$, $u(s,x) \equiv g(x)$ for $s \le 0$. Using the Feynman-Kac formula we can write down for $u(t,x)$ the following equality

$$(1.10)\qquad u(t,x) = Eg(X_t^{t,x})\exp\left\{\int_0^t c(X_s^{t,x}, u(t-s, X_s^{t,x}))ds\right\}.$$

So if $u(t,x)$ is the unique solution of (1.8), then $u(t,x)$ together with $X_s^{t,x}$ satisfies the system (1.9) - (1.10). Therefore we can introduce the generalized solution of (1.8) as a function $u(t,x)$ which, together with $X_s^{t,x}$, satisfies the system (1.9) - (1.10). Of course, if the matrix $(a^{ij}(x,u))$ is non-degenerate, the existence and the uniqueness of the classical (and thus generalized) solution follows from the a priori bounds for the solution of linear parabolic equations. In the degenerate case the solution $u(t,x)$ (in some weak sense) of problem (1.8) will be continuous for $t > 0$ small enough. The solution may have in general discontinuities for t greater than some $t_0 > 0$. We can give some sufficient conditions for existence and uniqueness and continuity of the generalized solution for all $t > 0$. One can also check that under some additional assumptions this solution will be classical.

I shall mention two classes of conditions which provide the existence of the continuous solution for all $t > 0$ (see [7] Ch. 5).

Let the diffusion matrix in (1.8) be independent of u, and assume for brevity that $c(x,u) = 0$. Consider the Markov family $\tilde{X}_t^x$ which is defined by the following equation

$$(1.11)\qquad d\tilde{X}_t^x = \sigma(\tilde{X}_t^x)dW_t, \quad \tilde{X}_0^x = x,$$

where $\sigma(x)\sigma^*(x) = (a^{ij}(x))$. Assume that there exists a bounded Lipschitz continuous vector field $\varphi(x,u) = (\varphi_1(x,u), \dots, \varphi_r(x,u))$ such that

$$(1.12)\qquad \sigma(x)\varphi(x,u) = b(x,u),$$

where $b(x,u) = (b^1(x,u), \dots, b^r(x,u))$. Then according to the Cameron-Martin-Girsanov formula, measures P_x and $\tilde{P}_x$ in C_{0T}, corresponding to the families X_t^x and $\tilde{X}_t^x$, are absolutely continuous each with respect to the other and

$$\frac{dP_x}{d\tilde{P}_x}(\tilde{X}) = \exp\left\{\int_0^T (\varphi(\tilde{X}_s^x, u(t-s, \tilde{X}_s^x)), dW_s) - \frac{1}{2}\int_0^T |\varphi(\tilde{X}_s^x, u(t-s, \tilde{X}_s^x))|^2 ds\right\}.$$

Taking into account the last formula we can write that

$$(1.13) \quad u(t,x) = Eg(X_t^x) = Eg(\tilde{X}_t^x)\exp\left\{\int_0^T (\varphi(\tilde{X}_s^x, u), dW_s) - \frac{1}{2}\int_0^T |\varphi(\tilde{X}_s, u)|^2 ds\right\}.$$

The system (1.11) - (1.13) is equivalent to (1.9) - (1.10).

After the change of variables $(X_t^x, u) \longrightarrow (\tilde{X}_t, u)$, the system of equations became triangular. The first equation (1.11) can be solved independently of the second one. We can consider the family $\tilde{X}_t^x$ as a given one, and then equation (1.13) will be the equation for generalized solution $u(t,x)$ of the problem.

The equation (1.13), under condition that $\varphi(x,u)$ is Lipschitz continuous in u, can be solved by the successive approximations (see [7], Ch. 5): The approximations

$$u_0(t,x) \equiv g(x), \quad u_{n+1}(t,x) = Eg(\tilde{X}_t^x)\exp\left\{\int_0^t (\varphi(\tilde{X}_s^x, u_n(t-s, \tilde{X}_s), dW_s) - \frac{1}{2}\int_0^t |\varphi(\tilde{X}_s^x, u_n(t-s, \tilde{X}_s))|^2 ds\right\}$$

converge uniformly to the unique solution of the equation (1.13). Under some additional assumptions one can prove smoothness of the generalized solution.

Another type of assumption which also provides the existence and uniqueness and continuity of the generalized solution for all $t > 0$ is as follows. Let, for brevity, the last term in equation (1.8) be linear: $c(x,u) = cu$, $c = \text{const}$. It is not difficult to prove that there exists $t_0 > 0$ such that the generalized solution exists for $t \in [0, t_0]$ provided the coefficients and the initial function are smooth enough. The greater the constant $-c$, the bigger t_0 is. It turns out that a constant $c_{critical} < 0$ exists such that if $c < c_{critical}$ the continuous generalized solution exists for all $t \geq 0$. The exact statement and the proof one can find in §5.2 of [7].

A similar approach is also useful in the initial-boundary problems for quasi-linear equations.