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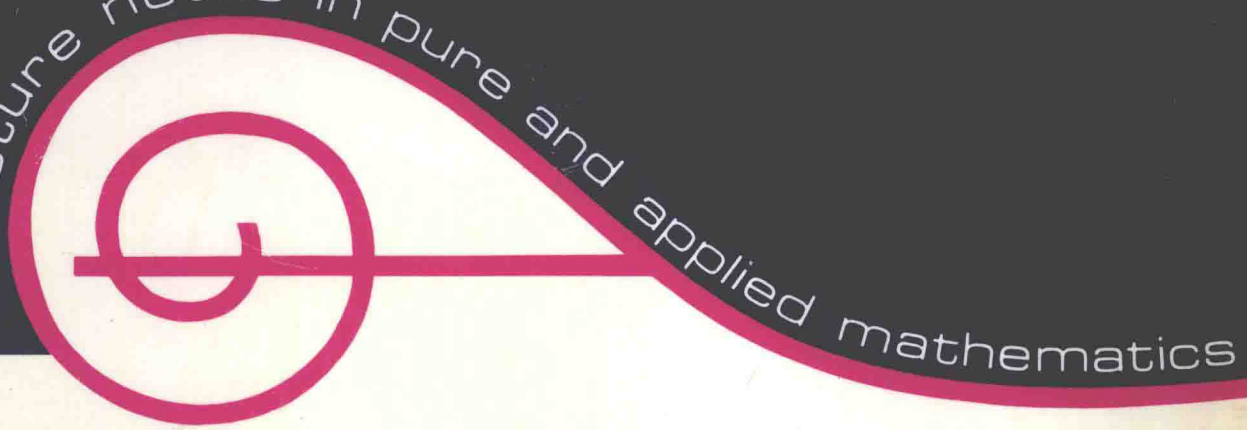

finite element galerkin methods for differential equations

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Finite Element Galerkin Methods for Differential Equations

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Preface

This volume provides a comprehensive account of finite element Galerkin methods for the approximate solution of two-point boundary value problems and partial differential equations. The emphasis throughout is on the theoretical aspects of such methods; few computational details are provided.

The purpose of the first chapter is to introduce, by means of a simple example, the ideas underlying finite element Galerkin methods, and to summarize the basic concepts and theorems of analysis which are required in subsequent chapters. Readers unfamiliar with Sobolev spaces and the general properties of the finite dimensional spaces used in finite element Galerkin methods are advised to omit Sections 1.2-1.4 on a first reading and use these sections for reference in later chapters. Finite element Galerkin methods for two-point boundary value problems are treated in Chapter 2, which also includes a discussion of the spaces of piecewise polynomial functions most commonly used in problems involving one space variable. Multivariate piecewise polynomial functions are introduced in Chapter 3, which is concerned with elliptic boundary value problems. One of the notable features of the book is its extensive treatment of time-dependent problems. Chapters 4 and 5 are devoted to parabolic and hyperbolic problems respectively, and Chapter 6 to alternating direction Galerkin methods for these problems. An extensive list of references is provided at the end of each chapter.

This book is an extended and updated version of a set of notes prepared for a series of lectures given at the South African Council for Scientific and Industrial Research (CSIR) in 1971, and published as a CSIR Special Report in 1973. Professor C. Jacobsz, former Director of the National Research Institute for Mathematical Sciences (NRIMS) of the CSIR, provided me with the opportunity to visit CSIR originally. It is largely due to his interest and encouragement that this book has been written. I should like to express my thanks to him, and to his successor, Professor David H. Jacobson, who kindly arranged a return visit to CSIR during which the book was completed.

I owe a great deal to Julio Cesar Diaz and Annie Saylor of the University of Kentucky who read the entire handwritten manuscript and made many valuable comments and suggestions. Also I have benefited greatly from discussions with Mary F. Wheeler of Rice University, and Daniel B. Kotlow of the University of Kentucky. I am grateful to Wanda Jones who typed the original manuscript in her usual flawless manner, and to Petra Ras and Marie de Villiers of CSIR, and Pat Nichols of the University of Kentucky for their expert assistance and inexhaustible patience in typing the revisions. The typing of the manuscript was par lally supported by a grant from the University of Kentucky Research Foundation.

Finally, I am especially grateful to my wife and children for their support, encouragement, and patient understanding throughout this endeavor. To them, and to my family in Scotland, this book is dedicated.

Graeme Fairweather

CONTENTS

Chapter 1

PRELIMINARIES

1.1 Introduction In recent years the Galerkin method has proved in practice to be a powerful technique for the approximate solution of problems involving differential and integral equations. The basic idea of the method can be explained quite briefly. Suppose one seeks a solution to a problem involving the operator equation $L(u) = f$. First one chooses a suitable finite dimensional space M, the trial space, and a basis $\{w_1,\ldots,w_N\}$ of M. The Galerkin method then consists in finding an approximation to u of the form

$$U = \sum_{j=1}^{N} \alpha_j w_j,$$

where the unknown coefficients α_j, $j=1,\ldots,N$, are determined so that

$$(L(U) - f,V) = 0,$$

for all $V \in M$, where $(\cdot,\cdot)$ denotes an inner product defined on M, (usually the L^2 inner product). The function U is called the Galerkin approximation to u in the space M, whose elements are called trial functions. In time dependent problems, the coefficients α_j, $j = 1,\ldots,N$, are unknown functions of time while the basis functions are functions of the space variables only.

This idea is not new - it was first proposed by Galerkin [7] in 1915. What is new and has made the method so successful is the choice of the space M and the construction of appropriate basis functions. If M is a space of piecewise polynomial functions a basis for M can

be chosen such that each basis function is a piecewise polynomial on a very small region of the spatial domain outside of which it is zero. Such a basis is called a local, or patch, basis. The Galerkin method with a space of piecewise polynomial functions and a local basis yields an approximation of high order accuracy to a smooth solution u. This approximation is determined by solving a sparse system of equations which can be done efficiently by specially designed techniques.

As a simple example of the method consider the two-point boundary value problem

$$L(u) = -u'' = 1, \qquad x \in (0,1), \tag{1.1a}$$

$$u(0) = u(1) = 0. \tag{1.1b}$$

Let M be the space spanned by the functions

$$w_i(x) = \begin{cases} (x-x_{i-1})/(x_i-x_{i-1}), & x \in [x_{i-1},x_i], \\ (x_{i+1}-x)/(x_{i+1}-x_i), & x \in [x_i,x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

for $i = 1,2,\ldots,N$, where $x_0,x_1,\ldots,x_{N+1}$ are N+1 points in [0,1] such that

$$0 = x_0 < x_1 < \cdots < x_{N+1} = 1.$$

Notice that $w_i(0) = w_i(1) = 0$, $i = 1,\ldots,N$, and $w_i(x_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Let $(\cdot,\cdot)$ denote the inner product defined by

$$(f,g) = \int_0^1 f(x)g(x)dx .$$

If U denotes the Galerkin approximation to u in M then, on using the formal integration by parts formula

$$\int_0^1 U''w_i \, dx = - \int_0^1 U'w_i' \, dx ,$$

we find that U satisfies

$$\int_0^1 U'w_i' \, dx = \int_0^1 w_i \, dx , \qquad i = 1,\ldots,N. \tag{1.3}$$

On substituting

$$U(x) = \sum_{j=1}^{N} \alpha_j w_j(x) \tag{1.4}$$

into (1.3) we obtain a system of linear algebraic equations, the Galerkin equations ,

$$\sum_{j=1}^{N} \left(\int_0^1 w_i'w_j' dx \right) \alpha_j = \int_0^1 w_i dx, \qquad i = 1,\ldots,N, \tag{1.5}$$

for the coefficients α_j, j=1,...,N. If we denote the coefficient matrix by $A = (a_{ij})$ it is a simple matter to show that A is tridiagonal, and

$$a_{ii} = (x_{i+1}-x_{i-1})/[(x_i-x_{i-1})(x_{i+1}-x_i)] ,$$

$$a_{i,i-1} = -1/(x_i-x_{i-1}) ,$$

$$a_{i,i+1} = -1/(x_{i+1}-x_i) .$$

Also

$$\int_0^1 w_i dx = \frac{1}{2}(x_{i+1}-x_{i-1}) .$$

Moreover, A is diagonally dominant (see [13]), and hence the solution of the Galerkin equations can be obtained using the well-known algorithm for solving tridiagonal systems ([13], page 195).

Note that since $w_i(x_j) = \delta_{ij}$, we have $U(x_i) = \alpha_i$, and hence in the case in which the points x_i are equally spaced, that is, $x_i = ih$, $i = 0,\ldots,N+1$, we obtain a well-known finite difference approximation to (1.1),[10]. As we shall see in Chapter 2, the Galerkin method is of the same accuracy as this finite difference method even when the points x_j are not equally spaced.

This example exhibits the main ideas of the Galerkin method for solving equilibrium problems with homogeneous boundary conditions. The elements of the space M satisfy these boundary conditions and consequently so does the Galerkin approximation $U \in M$. A local basis for M is chosen which yields a sparse system of linear algebraic equations for the coefficients in the expansion of U in terms of the basis functions. In time dependent problems the Galerkin method is applied in the space variables only, and gives rise to an initial value problem for a system of ordinary differential equations, whose solution defines the so-called continuous-time Galerkin approximation to u in M. In practical computations this initial value problem is solved approximately using a discrete variable method known as a discrete-time Galerkin method, and the resulting approximation to u is called the discrete-time Galerkin approximation to u in M.

The Galerkin method is closely related to the Rayleigh-Ritz method, and in many cases they are equivalent in the sense that they lead to the same approximate solution. The Rayleigh-Ritz method is based on the fact that many boundary value problems can be formulated in terms of variational problems. That is, some functional is to be minimised over an appropriate space of admissible functions. For example the solution of (1.1) can be viewed as the function in H, the space of all functions which are twice continuously differentiable

on the interval [0,1] and satisfy the boundary conditions (1.1b), which minimises the functional

$$I(u) = \int_0^1 [(u')^2 - 2u]dx \ .$$

The Rayleigh-Ritz method involves seeking a minimum to I(u) in M rather than in H. For this example it is easy to show that the element $U \in M$ such that

$$I(U) = \min_{\chi \in M} I(\chi)$$

satisfies (1.4), and hence the Rayleigh-Ritz and Galerkin methods yield the same solution in this case. The advantage of the Galerkin formulation is that there is no need to find the variational problem equivalent to the original problem. In this book we will not consider the Rayleigh-Ritz approach.

In the engineering literature since the mid-1950's, and recently in the numerical analysis literature, the use of piecewise polynomial spaces in the Rayleigh-Ritz or Galerkin procedure,particularly in multi-dimensional problems in irregular regions,has been called the finite element method. Herein, by the finite element Galerkin method, or simply the Galerkin method, we shall mean the combination of the classical technique of Galerkin [7] and spaces of piecewise polynomial functions.

1.2 Notation and some inequalities All functions considered in the following are real-valued. Let Ω be a bounded domain in $\mathbf{R}^n$, n-dimensional Euclidean space, and denote by $\partial\Omega$ the boundary of Ω. Let $x = \{x_1,\ldots,x_n\}$ denote a variable point in Ω and let $dx = dx_1.....dx_n$. If $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n)$ is an n-tuple with nonnegative integer components we write

$$D^{\gamma} = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \ldots \partial x_n^{\gamma_n}} ,$$

where $|\gamma| = \sum_{i=1}^{n} \gamma_i$, and D^0 denotes the identity operator.

For $1 \leq p < \infty$, $L^p(\Omega)$ denotes the space of functions u on Ω such that $\int_\Omega |u(x)|^p dx$ exists and is finite. The norm on $L^p(\Omega)$ is given by

$$||u||_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{1/p}, \quad 1 \leq p < \infty .$$

It is well-known that $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v) = \int_\Omega u(x)v(x)dx .$$

When it is clear from the context, $||\cdot||_{L^2(\Omega)}$ will be written $||\cdot||$. By $L^\infty(\Omega)$ we denote the space of functions u on Ω such that

$$||u||_{L^\infty(\Omega)} \equiv \operatorname{ess\,sup}_{x\in\Omega} |u(x)| < \infty .$$

For any nonnegative integer r, $C^r(\Omega)$ (respectively, $C^r(\bar{\Omega})$) is the space of functions with continuous (respectively, uniformly continuous) derivatives up to and including order r in Ω. Also, $C_o^\infty(\Omega)$ is the space of all infinitely differentiable functions which vanish identically outside some compact set contained in Ω. The Sobolev space of order r on Ω, $H^r(\Omega)$, (respectively $H_0^r(\Omega)$) is defined as the closure of $C^r(\bar{\Omega})$ (respectively $C_o^\infty(\Omega)$) with respect to the norm

$$||u||_{r,\Omega} = (\sum_{|\gamma|\leq r} ||D^\gamma u||^2)^{1/2} .$$

Equivalently, $H^r(\Omega)$ is the space of all functions whose distributional derivatives of order less than or equal to r are in $L^2(\Omega)$. It can be shown that if $\partial\Omega$ is sufficiently smooth and $u \in H_0^r(\Omega) \cap C^{r-1}(\bar{\Omega})$ then

$$\frac{\partial^j u}{\partial n^j} = 0 \qquad \text{on } \partial\Omega$$

for $0 \leq j \leq r-1$, where $\partial/\partial n$ denotes differentiation in the direction of the outward normal to the boundary. The space $H^r(R^n)$ is the completion of $C_o^\infty(R^n)$ in the above norm.

Sometimes we shall use the notation $L^2(\Omega) = H^o(\Omega)$. Also we shall make use of the semi-norm

$$||u||_{r,o,\Omega} = \Big(\sum_{|\gamma|=r} ||D^\gamma u||^2_{L^2(\Omega)} \Big)^{1/2} .$$

When no risk of confusion exists we shall substitute H^r, H_0^r, $||u||_r$, and $||u||_{r,0}$ for $H^r(\Omega)$, $H_0^r(\Omega)$, $||u||_{r,\Omega}$, and $||u||_{r,0,\Omega}$, respectively.

For $r > 0$ and not an integer, $H^r(\Omega)$ is defined by real interpolation between successive integers (see [11]). For $r < 0$ any real number, $H^r(\Omega)$ will denote the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$||u||_r = \sup\{(u,v)/||v||_{-r,\Omega};\ v \in H^{-r}(\Omega),\ ||v||_{-r,\Omega} \neq 0\}.$$

The function space $H^r(\partial\Omega)$ for all real r is the Sobolev space of order r on $\partial\Omega$. For a precise definition we refer the reader to [11]. We shall denote by $|\cdot|_{r,\partial\Omega}$, or $|\cdot|_r$, the norm on $H^r(\partial\Omega)$. Note that $H^o(\partial\Omega) = L^2(\partial\Omega)$, and in this case we denote the inner product by

$$\langle u,v \rangle = \int_{\partial\Omega} uv\,d\sigma ,$$

where $d\sigma$ denotes the surface measure on $\partial\Omega$, and the norm by $|\cdot|$.

In the analysis of time dependent problems the following notation is convenient. If X is a normed space with norm $||\cdot||_X$ and $\phi\colon [0,T] \to X$, then

$$||\phi||_{L^p(X)} = \left(\int_0^T ||\phi(t)||_X^p \, dt\right)^{1/p} , \quad 1 \leq p < \infty ,$$

$$||\phi||_{L^\infty(X)} = \max_{0\leq t\leq T} ||\phi(t)||_X .$$

The space $L^p(X)$ is the set of all ϕ such that the appropriate norm is finite.

Throughout this book C will denote a generic constant with possibly different values in different contexts.

In the following theorems we present, without proof, inequalities which will be of frequent use in error analyses in subsequent chapters.

Theorem 1.1 Suppose that Ω is a bounded domain in R^n and that r is a positive integer. If $0 \leq k < r$, then there exists a constant $C_{k,r}$ such that

$$||u||_{k,0} \leq C_{k,r}||u||_{r,0} , \tag{1.6}$$

for all $u \in H_0^r(\Omega)$.

This result, the Poincaré Inequality for functions in $H_0^r(\Omega)$, is proved in [1]. Using Theorem 1.1 it is easy to show that $||\cdot||_{r,0,\Omega}$ defines a norm on $H_0^r(\Omega)$ equivalent to the norm $||\cdot||_{r,\Omega}$.

A one-dimensional case of this inequality is the Rayleigh-Ritz Inequality given in the following theorem, (see [9]).

Theorem 1.2 If $u \in H_0^1(I)$, where $I \equiv (a,b)$ and $-\infty < a < b < \infty$, then

$$\pi||u|| \leq (b-a)||Du|| . \tag{1.7}$$

A domain Ω has the restricted cone property if $\partial\Omega$ has a locally finite open covering $\{O_i\}$ and corresponding cones $\{C_i\}$ with vertices at the origin and the property that $x+C_i \subset \Omega$ for $x \in \Omega \cap O_i$. A proof of the following theorem can be found in [1].

Theorem 1.3 (Sobolev's Inequality) Let Ω be a bounded domain in R^n having the restricted cone property, let $u \in H^r(\Omega)$, where r is a positive integer such that $r > n/2$, and let $\ell = r - [n/2] - 1$. Then u can be modified on a set of measure zero so that $u \in C^{\ell}(\bar{\Omega})$. Moreover, for any $\delta \geq 1$, $|\gamma| \leq \ell$, $x \in \Omega$,

$$|D^{\gamma}u(x)| \leq C\delta^{-(r-\frac{1}{2}n-|\gamma|)} (||u||_r + \delta^r ||u||) , \tag{1.8}$$

where C is a constant that depends only on Ω and r.

In one space variable the following special case of this result is elementary to establish, (cf. [14], page 26).

Theorem 1.4 If $u \in H_0^1(I)$, where $I \equiv (a,b)$ and $-\infty < a < b < \infty$, then

$$||u||_{L^{\infty}(I)} \leq \frac{1}{2}(b-a)^{1/2}||Du|| . \tag{1.9}$$

1.3 Approximation-theoretic results In subsequent chapters it will be shown that the accuracy of the Galerkin procedure depends on the approximation properties of the chosen spaces M. Usually these spaces are selected in some systematic fashion depending on some parameter, such as the spacing of nodes associated with the functions of a particular basis for the space. Bramble and Schatz [3] formulated a useful definition that isolates exactly the properties of these spaces needed to derive optimal error estimates. Let $\{M_h\}_{0<h\leq 1}$ be a one-parameter family of finite dimensional vector spaces. For given integers k and r with $0 \leq k < r$ we shall say that $\{M_h\}_{0<h\leq 1}$ is of class $S_{k,r}(\Omega)$ if $M_h \subset H^k(\Omega)$ for each h and if for any $u \in H^r(\Omega)$ there exists a $\bar{u} \in M_h$ and a constant C independent of h and u such that