Pure Mathematics 1

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PREFACE

Recent developments in the Mathematics syllabuses of many boards at advanced level have created a need for a new approach to Pure Mathematics.

This book, the first of two volumes, together with *Applied Mathematics Volume I*, covers the work necessary for a traditional single subject Mathematics course at advanced level. The second volume completes coverage of an advanced level course in Pure Mathematics, and together with *Applied Mathematics Volume II*, provides a course for Further Mathematics at advanced level.

We have incorporated many modern approaches to mathematical understanding whilst keeping the best of the traditional methods by making a feature of many and varied worked examples and exercises to illustrate each new concept at each stage of its development. The student will find stimulation in examples to be found in the miscellaneous exercises at the end of most chapters, some of which are suitable for the student of 'special paper' calibre.

The starting point for an advanced level text on Pure Mathematics always presents problems, especially as there are so many varied syllabuses at ordinary level. We have assumed knowledge only of a 'common core' syllabus, begining most topics from first principles.

The teaching order presents another problem. The arrangement of topics in this book forms a logical sequence suitable for a student working on his own. However the course teacher will find considerable flexibility.

There are many people who have helped us in the writing of this book and we would like to thank all of them. In particular we are very grateful to Miss J. Broughton for her helpful criticism and correction of the text and to Mr. C. Eva for undertaking the mammoth task of working all the exercises. We would also like to thank Miss E. Clarke for typing the manuscript.

Finally we are grateful to the following Examination boards for permission to reproduce questions from their past examination papers (part questions are indicated by the suffix p):
University of London (U of L)
Joint Matriculation Board (J M B)
University of Cambridge Local Examinations Syndicate (C)
Oxford Delegacy of Local Examinations (O)
The Associated Examining Board (A E B)

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NOTES ON USE OF THE BOOK

Notation

$=$	is equal to
$\equiv$	is identical to
$\simeq$	is approximately equal to*
$>$	is greater than
$\geqslant$	is greater than or equal to
$<$	is less than
$\leqslant$	is less than or equal to
∞	infinitely large
$\rightarrow$	approaches
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Longleftrightarrow$	implies and is implied by

A stroke through any of the above symbols negates it. i.e. $\neq$ means 'is not equal to', $\not>$ means 'is not greater than'.

Abbreviations

$\parallel$	parallel
+ve	positive
−ve	negative
w.r.t.	with respect to

*Practical problems rarely have exact answers. Where numerical answers are given they are correct to two or three decimal places depending on their context, e.g. π is 3.142 correct to 3 d.p. and although we write $\pi = 3.142$ it is understood that this is not an exact value. We reserve the symbol $\simeq$ for those cases where the approximation being made is part of the method used.

Instructions for answering Multiple Choice Exercises

These exercises are at the end of most chapters. The questions are set in groups, each group representing one of the variations that may arise in examination papers. The answering techniques are different for each type of question and are classified as follows:

TYPE I

These questions consist of a problem followed by several alternative answers, only *one* of which is correct.

Write down the letter corresponding to the correct answer.

TYPE II

In this type of question some information is given and is followed by a num-number of possible responses. *One or more* of the suggested responses follow(s) directly and necessarily from the information given.

Write down the letter(s) corresponding to the correct response(s).

e.g. PQR is a triangle

(a) $\hat{P} + \hat{Q} + \hat{R} = 180°$

(b) PQ + QR is less than PR

(c) if $\hat{P}$ is obtuse, $\hat{Q}$ and $\hat{R}$ must both be acute.

(d) $\hat{P} = 90°$, $\hat{Q} = 45°$, $\hat{R} = 45°$.

The correct responses are (a) and (c).

(b) is definitely incorrect and (d) may or may not be true of triangle PQR, i.e. it does not follow directly and necessarily from the information given. Responses of this kind should not be regarded as correct.

TYPE III

Each problem contains two independent statements (a) and (b).

1) If (a) always implies (b) but (b) does not always imply (a)	write A.
2) If (b) always implies (a) but (a) does not always imply (b)	write B.
3) If (a) always implies (b) *and* (b) always implies (a)	write C.
4) If (a) denies (b) and (b) denies (a)	write D.
5) If none of the first four relationships apply	write E.

TYPE IV

A problem is introduced and followed by a number of pieces of information. You are not required to solve the problem but to decide whether:

1) the given information is *all* needed to solve the problem. In this case write A;

2) the total amount of information is insufficient to solve the problem. If so write I;
3) the problem can be solved without using one or more of the given pieces of information. In this case write down the letter(s) corresponding to the items not needed.

TYPE V

A single statement is made. Write T if it is true and F if it is false.

CONTENTS

CHAPTER 1

ALGEBRAIC RELATIONSHIPS

THE NATURE OF ALGEBRAIC EXPRESSIONS

Consider the expressions $(x+2)^2 = 2x+7$ [1]

$(x+2)^2 = x^2+4x+4$ [2]

$(x-2) > 1$ [3]

$(x+2)^2$ [4]

By inspection we see that these four expressions are all different in nature and by investigating each of them we shall try to identify these differences.

(1) $(x+2)^2 = 2x+7$

Substituting 1 for x in both the left hand side (LHS) and right hand side (RHS) separately we find

$$\text{LHS} = (1+2)^2 = 3^2 = 9$$

$$\text{RHS} = 2+7 = 9$$

i.e. LHS = RHS when $x = 1$.

However substituting 2 for x in a similar way we find

$$\text{LHS} = (2+2)^2 = 16$$

$$\text{RHS} = 4+7 = 11$$

i.e. LHS $\neq$ RHS when $x = 2$.

Now rearranging the original expression gives

$$x^2 + 4x + 4 = 2x + 7$$

$$\Rightarrow \quad x^2 + 2x - 3 = 0$$

i.e. $$(x + 3)(x - 1) = 0$$

from which we see that LHS = RHS if and only if

either $x + 3 = 0$ i.e. $x = -3$

or $x - 1 = 0$ i.e. $x = 1$

Thus $(x + 2)^2 = 2x + 7$ only if $x = -3$ or if $x = 1$ and the equality is not true for any other value of x.

Expressions of this type are called equations and the equality is true only for a number of distinct values of the unknown quantity (or quantities).

The process of finding these values is referred to as solving the equation.

(2) $(x + 2)^2 = x^2 + 4x + 4$

If we substitute 1 for x in both sides of this expression we find

$$\text{LHS} = (1 + 2)^2 = 9$$

$$\text{RHS} = 1^2 + 4.1 + 4 = 9$$

i.e. LHS = RHS when $x = 1$

If we substitute -1 for x as before we find

$$\text{LHS} = (-1 + 2)^2 = 1$$

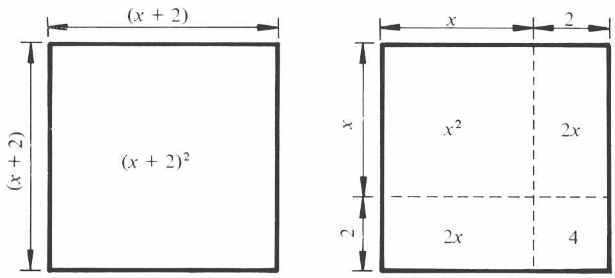
$$\text{RHS} = (-1)^2 + (4)(-1) + 4 = 1$$

i.e. LHS = RHS when $x = -1$.

Whatever other numerical value we substitute for x we find that LHS = RHS, so it appears (but is not proved) that LHS = RHS for all values of x.

The following geometric illustration confirms that this suspicion is correct.

Consider a square of side $x + 2$ units



Since these two squares are identical, their areas are identical.
Hence $(x+2)^2 = x^2 + 4x + 4$ for all values of x and we say that $(x+2)^2$ *is identical to* $x^2 + 4x + 4$.
Using the symbol ≡ for 'is identical to' the relationship is written

$$(x+2)^2 \equiv x^2 + 4x + 4.$$

Relationships of this type are called identities and both sides are equal for any value of the unknown quantity. They are, in fact, two forms for the same expression, and we shall use the identity symbol whenever we are dealing with an identity relationship and we strongly recommend that the reader does the same.

(3) $(x-2) > 1$

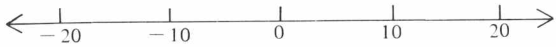
Reading from left to right the symbol $>$ means greater than (and $<$ means less than). This relationship is obviously different from the first two and it is called an inequality.
By inspection we see that for $x - 2$ to have a value greater than one then x must have a value greater than three.

i.e. if $$x - 2 > 1$$

then $$x > 3$$

Consider a line as being made up of adjacent points. We can represent all the real values that x, the variable, can have by the positions of points on a line, (known as a *number line*).

negative values of x



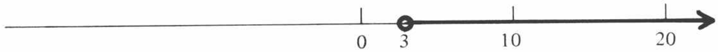
positive values of x

The position of a point to the *left* of a second point corresponds to a value of x *less than* the value of x at the second point.

i.e. $-99 < 1, \quad -10 < -5 \quad \ldots$ etc.

The values of x given by the statement $x > 3$ can then be represented by a section of this line.

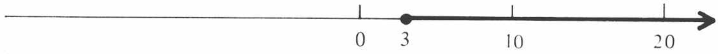
i.e.



From this we see that *not* all values of x satisfy the inequality but that there is an infinite set of values that do, i.e. the solution of an inequality is a range (or ranges) of values of the variable involved.

Note that $x = 3$ is *not* included in the range and this is indicated by an open circle at $x = 3$. For $x \geqslant 3$, which means x is greater than or equal to 3, the value $x = 3$ *is* included in the range. This is indicated on a number line by a solid circle.

i.e.



(4) $(x+2)^2$

This expression is not related to any other expression and can take different values depending on the value given to x.

Such an expression is called a function (of x in this case). Using the symbol $f(x)$ to represent a function of x, we write

$$f(x) \equiv (x+2)^2$$

To represent the value of this function when $x = 1$, say, we write $f(1)$, where

$$f(1) = (1+2)^2 = 9$$

similarly $$f(3) = (3+2)^2 = 25$$

and $$f(-2) = (-2+2)^2 = 0$$

Note that the values both of x and $f(x)$ are variable but whereas x can be given *any* value, the value of $f(x)$ depends on that of x, so x *is referred to as the independent variable* and $f(x)$ *as the dependent variable*.

EXERCISE 1a

1) State which of the following are equations and which are identities

(a) $x^2 - 3 = 2$ (b) $x^2 - 9 = (x-3)(x+3)$ (c) $\dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x^2+x}$

(d) $\dfrac{1}{x-1} + \dfrac{1}{x+1} = \dfrac{2}{x^2-1}$ (e) $p^2 + 2p - 3 = 3 - 2p - p^2$

(f) $(x+y)(x-y) = x^2 - y^2$ (g) $y - 1 = \dfrac{1}{y}$ (h) $\dfrac{2q}{q^2-1} = \dfrac{1}{q-1} + \dfrac{1}{q+1}$

2) Find the range of values of x for which the following inequalities are true and illustrate the range on a number line.

(a) $x - 5 > 0$ (b) $x + 1 \leqslant -1$ (c) $0 \geqslant x - 4$ (d) $3 < 4 - x$

3) Find the values of $f(0), f(1), f(-2), f(5)$ where $f(x)$ is:

(a) $x^2 - 3x$ (b) $\dfrac{1}{x-2}$ (c) $(x-7)(x+2)$ (d) $\dfrac{1}{x+1} - \dfrac{2}{2x-3}$

POLYNOMIALS AND FRACTIONAL FUNCTIONS

If each individual term of a function is of the form ax^n, where a is a constant (i.e. has a fixed numerical value) and n is a positive integer, the function is called a *polynomial*. Thus $x^2 - 2x$, $3x^6 - 7x^4 + 6$, $(x-4)^2$ are polynomials but $\sqrt{x}$, $\dfrac{1}{x}$, $\sqrt{x^2 - 2}$ are not.

The highest power of x that occurs in a polynomial defines *the degree or order of the polynomial.*
Thus $5x^6 - 7x^3 + 6x$ is a polynomial of degree 6.
A fractional function of the form $\dfrac{3x^2-7}{x^3+1}$ where both the numerator and denominator are polynomials, is referred to as a 'proper' fraction if the degree of the numerator is less than the degree of the denominator. However if the degree of the numerator is greater than, or equal to, the degree of the denominator the fraction is referred to as 'improper'.
An improper numerical fraction such as $\dfrac{9}{7}$ may be written as $\dfrac{7+2}{7} = 1 + \dfrac{2}{7}$
Similarly an algebraic fraction such as $\dfrac{x^2-1}{x^2+1}$ may be written as

$$\frac{x^2+1-2}{x^2+1} \equiv \frac{x^2+1}{x^2+1} - \frac{2}{x^2+1} \equiv 1 - \frac{2}{x^2+1}$$

PARTIAL FRACTIONS

Consider first a function such as $f(x) \equiv \dfrac{2}{x+1} + \dfrac{x}{x^2+1}$.
$f(x)$ may be expressed as a single fraction with a common denominator thus

$$f(x) \equiv \frac{2}{x+1} + \frac{x}{x^2+1} \equiv \frac{2(x^2+1)+x(x+1)}{(x+1)(x^2+1)} \equiv \frac{3x^2+x+2}{(x+1)(x^2+1)}$$

It is often useful to be able to reverse this operation, that is to take a function such as $f(x) \equiv \dfrac{x-2}{(x+3)(x-4)}$ and express $f(x)$ as the sum of two (or in some cases more) separate fractions.
This process is called expressing $f(x)$ in partial fractions.
If the original fraction is 'proper' then the separate (or partial) fractions will also be 'proper'.

Thus $\dfrac{x+3}{(x-2)(x+4)}$ can be expressed as $\dfrac{A}{x-2} + \dfrac{B}{x+4}$

and $\dfrac{x+3}{(x-2)(x^2+4)}$ can be expressed as $\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$ where A, B and C are constants to be determined. The method for evaluating these constants depends to some extent on the factors in the denominator.

EXAMPLES 1b

1) Express $\dfrac{x+3}{(x-2)(x+4)}$ in partial fractions.

This example is a proper fraction with *linear* (of degree one) *factors* only and so its partial fractions are also proper. As these partial fractions have linear denominators their numerators contain only one constant.

$$\frac{x+3}{(x-2)(x+4)} \equiv \frac{A}{x-2} + \frac{B}{x+4}$$

so
$$\frac{x+3}{(x-2)(x+4)} \equiv \frac{A(x+4)+B(x-2)}{(x-2)(x+4)}$$

As the denominators are obviously identical, the numerators must also be identical,

i.e.
$$x+3 \equiv A(x+4)+B(x-2)$$

Now LHS = RHS for any value of x.
Choosing to substitute 2 for x (to eliminate B) gives

$$2+3 = A(2+4)+B(0)$$

$\Rightarrow$
$$A = \frac{5}{6}$$

substituting -4 for x (to eliminate A) gives

$$-4+3 = B(-4-2)$$

$\Rightarrow$
$$B = \frac{1}{6}$$

Therefore
$$\frac{x+3}{(x-2)(x+4)} \equiv \frac{5}{6(x-2)} + \frac{1}{6(x+4)}$$

2) Express $\dfrac{x^2-3}{(x-1)(x^2+1)}$ in partial fractions.

This example contains a *quadratic factor* in the denominator,

therefore
$$\frac{x^2-3}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

each numerator on RHS being chosen so that each partial fraction is proper.

$\Rightarrow$
$$\frac{x^2-3}{(x-1)(x^2+1)} \equiv \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

therefore
$$x^2-3 \equiv A(x^2+1)+(Bx+C)(x-1) \qquad [1]$$

substituting 1 for x (so eliminating B and C) gives

$$1^2-3 = A(1^2+1)$$

$\Rightarrow$
$$A = -1$$

There is no value which we can substitute for x to eliminate A (as there is no value of x for which $x^2+1=0$).
But substituting 0 for x will eliminate B giving

$$-3 = A(1) + C(-1)$$

As $A = -1$, $\quad -3 = -1 - C$

$\Rightarrow \quad C = 2$

Any other value can now be substituted for x to find B, a small value being sensible.
Substituting 2 for x gives

$$(2)^2 - 3 = A(2^2 + 1) + (2B + C)(2 - 1)$$

$$1 = 5A + 2B + C$$

But $A = -1$ and $C = 2$ so $B = 2$

Therefore $$\frac{x^2 - 3}{(x-1)(x^2+1)} \equiv \frac{-1}{x-1} + \frac{2x+2}{x^2+1}$$

Alternatively the value of B may be found as follows.
Expanding the RHS of [1] gives

$$x^2 - 3 \equiv Ax^2 + A + Bx^2 - Bx + Cx - C$$

or $$x^2 - 3 \equiv (A+B)x^2 + (C-B)x + A - C \qquad [2]$$

As this is an identity, the coefficients (quantity) of x^2 on the LHS and RHS of [2] must be equal.

Therefore $$1 = A + B$$

(this is referred to as comparing the coefficients of x^2).
As $A = -1$, $B = 2$.
In practice a mixture of these two methods will give a simple solution.

3) Express $\dfrac{x-1}{(x+1)(x-2)^2}$ in partial fractions.

As $(x-2)^2 \equiv (x-2)(x-2)$, it is called a *repeated factor*, but it is also quadratic so we may initially think of $\dfrac{x-1}{(x+1)(x-2)^2}$ as $\dfrac{A}{x+1} + \dfrac{Bx+C}{(x-2)^2}$.
But this is not the simplest partial fraction form, as we shall see.
Considering just the fraction $\dfrac{Bx+C}{(x-2)^2}$ and letting $C = -2B + D$

then $$\frac{Bx+C}{(x-2)^2} \equiv \frac{Bx - 2B + D}{(x-2)^2}$$

$$\equiv \frac{B(x-2)}{(x-2)^2} + \frac{D}{(x-2)^2}$$

$$\equiv \frac{B}{x-2} + \frac{D}{(x-2)^2}$$