Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Georgia Benkart J. Marshall Osborn (Eds.)

Lie Algebras, Madison 1987

Proceedings

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Editors

Georgia Benkart
J. Marshall Osborn
Department of Mathematics, University of Wisconsin
Madison, WI 53706-1388, USA

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Preface

During the academic year 1987-1988 the University of Wisconsin, Madison hosted a Special Year of Lie Algebras. A Workshop on Lie Algebras, of which these are the proceedings, inaugurated the special year. The principal focus of the year and of the workshop was the long-standing problem of classifying the simple finite dimensional Lie algebras over algebraically closed fields of prime characteristic. However, other lectures at the workshop dealt with the related areas of algebraic groups, representation theory, and Kac-Moody Lie algebras.

The titles of the fourteen papers presented at the workshop can be found at the end, followed by a list of the participants in the workshop and their addresses. Nine of these papers, (eight research articles and one expository article), comprise this volume. The first paper, by Strade, develops the notion of the absolute toral rank of a modular Lie algebra. This new concept combines earlier approaches of Block-Wilson and Benkart-Osborn, and it seems to play a critical role in determining the structure of simple Lie algebras of prime characteristic. The next three papers investigate various topics related to the classification problem: embeddings of generalized Witt algebras; isomorphism classes of Hamiltonian Lie algebras; and Lie algebras with subalgebras of codimension one and their relationship to forms of Zassenhaus Lie algebras. The determination of the restricted simple Lie algebras over algebraically closed fields of prime characteristic has been accomplished recently by Block and Wilson. Serconek and Wilson use this result as the starting point for their discussion of forms of restricted simple algebras. In the next paper Varea employs the classification of the rank one simple Lie algebras to investigate the subalgebra lattice of supersolvable Lie algebras. The final three papers treat problems in Kac-Moody algebras.

We would like to take this opportunity to express our deep appreciation to the National Science Foundation for its support (through grant #DMS 87-02928) of the Special Year of Lie Algebras. Without its support this workshop and the other activities of the special year would not have been possible. Thanks also go to Rolf Farnsteiner, David Finston, Thomas Gregory, Helmut Strade, and Robert Wilson for their participation in the events of the special year, and to the typists, Dee Frana and Diane Reppert, and referees who helped in the preparation of this volume.

Georgia Benkart and J. Marshall Osborn June 1, 1988

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THE ABSOLUTE TORAL RANK OF A LIE ALGEBRA

Helmut Strade

Abstract: The new concept of an absolute toral rank for subalgebras in arbitrary modular Lie algebras is introduced. All nonsimple Lie algebras of absolute toral rank ≤ 2 are determined in terms of smaller constituents. The final result is the first step towards the classification of all simple modular Lie algebras.

Introduction.

In the theory of modular Lie algebras there are several fundamental concepts and objects based on the notion of a "toral rank". In a restricted Lie algebra, for example, the toral rank of a torus gives the GF(p)–dimension of the root lattice determined by this torus. Tori of maximal toral rank occur in the classification theory as very important objects. In the theory of non–restricted semisimple Lie algebras the toral rank of a Cartan subalgebra (CSA) is a frequently used concept, and the particular case that a Lie algebra has a CSA of toral rank one is a very central one.

In this note I shall introduce in §1 the concept of an "absolute toral rank TR(G,L) of a subalgebra G in the Lie algebra L" for arbitrary Lie algebras over any field of characteristic $p \neq 0$. This generalizes all of the above–mentioned concepts to arbitrary Lie algebras. Important tools in this context are the concepts of a "restrictable Lie algebra" and a "p–envelope" of an arbitrary Lie algebra [5] as well as their structure theory [7].

In §2 we will prove several results on the absolute toral rank, which generalize well–known results on the dimension of tori in restricted Lie algebras.

In §3 we consider subalgebras $C_L(T)$ of a modular Lie algebra L for a torus T in some p–envelope of L. The main result in this section is a far–reaching extension of [8, Theorem 2.1] which yields that under suitable conditions subalgebras of this type act triangulably on L.

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Applications are given in section 4. There we describe all Lie algebras of absolute toral rank ≤ 2 in terms of: solvable algebras, algebras having toral rank 1 with respect to some CSA, and simple algebras of absolute toral rank 2 satisfying some additional assumption concerning their CSA's. The final Theorem 4.8 may be considered a generalization of [3, Theorem 4.1.1] to non–restricted algebras. All Lie algebras under consideration are finite dimensional over an algebraically closed field F of characteristic $p > 0$.

This note extends results of a talk, which was given on the occasion of the opening workshop of the "Special Year on Lie Algebras" in August, 1987 at Madison, Wisconsin.

§1. The toral structure of a Lie algebra.

In this chapter we will use the concept of a p–envelope [5] in order to transfer some methods, which are very fruitful in the case of restricted Lie algebras, to arbitrary algebras.

For convenience we use in this note the following abbreviation:

Let L be a restricted algebra. Put

$$MT(L) := \max\{\dim T \mid T \text{ is a torus of } L\} .$$

Using the notation $C_X(Y) = \{x \in X \mid [Y,x] = 0\}$ we recall

Lemma 1.1: Let L be a restricted Lie algebra. Suppose that T is a torus of L.

1) Any T–invariant subspace $W \subset L$ decomposes

$$W = C_W(T) + [T,W] .$$

2) If I is a restricted ideal of L and L/I is a torus, then there exists a torus $T' \supset T$ such that $L = T' + I$.

Proof: [7,(II.4.4),(II.4.5)] □

Lemma 1.2: Let L be restricted.

1) If I is a restricted ideal of L, then

$$MT(L) = MT(L/I) + MT(I) .$$

2) Suppose that K_1 is a restricted subalgebra and K_2 is a restricted ideal of L. Then

$$MT(K_1+K_2) = MT(K_1) + MT(K_2) - MT(K_1 \cap K_2) .$$

Proof: 1) Let $\pi: L \longrightarrow L/I$ denote the canonical homomorphism.

For any torus T of L, $\pi(T)$ and $T \cap I$ are tori of L/I and I, respectively. This shows that $MT(L) \le MT(L/I) + MT(I)$. In order to obtain the reverse inequality we apply Lemma 1.1: Let T be a torus of I and R a torus of L/I. Then L has a torus T' containing T with $T' + I = \pi^{-1}(R)$ (put in Lemma 1.1 (2) $\pi^{-1}(R)$ for L). Therefore

$$\dim(T') \ge \dim \pi(T') + \dim T' \cap I \ge \dim R + \dim T .$$

This gives the result.

2) Applying 1) we obtain

$$MT(K_1+K_2) = MT(K_1+K_2/K_2) + MT(K_2)$$

$$= MT(K_1/K_1 \cap K_2) + MT(K_2) = MT(K_1) - MT(K_1 \cap K_2) + MT(K_2). \square$$

A p–envelope of an arbitrary modular Lie algebra L is a triple $(H,[p],i)$ consisting of a restricted Lie algebra $(H,[p])$ and an embedding $i: L \longrightarrow H$ such that $i(L)$ generates H as a restricted subalgebra [5]. We often consider L a subalgebra of H and call H a p–envelope of L. The following notation will be used: L is always a finite dimensional Lie algebra and L_p denotes a finite dimensional p–envelope containing L as a subalgebra. If L and L_p are given and G is a subalgebra of L, then G_p denotes the minimal restricted subalgebra of L_p which contains G.

Remarks:

1) Suppose that L is nilpotent. Then L_p is nilpotent. L_p has a unique maximal torus. This torus is contained in $C(L_p)$. It is the set of all semisimple elements [7,(II.1.3), (II.4.2)].

2) Every representation $\rho: L \longrightarrow gl(V)$ can be extended to a representation $\hat{\rho}: L_p \longrightarrow gl(V)$ [7,(V.1.1)].

3) Suppose that G is a subalgebra of L and $V \subset L$ is G–invariant. Then V is G_p–invariant. (This is a direct consequence of 2).)

4) Suppose that $\hat{\rho}$ is a restricted representation, such that $\hat{\rho}(t)$ is a nilpotent transformation for every semisimple element t. Then $\hat{\rho}(L_p)$ consists of nilpotent transformations.

Proof of 4): For any $x \in L_p$ there exists a power $t := x^{[p]^r}$ such that t is a semisimple element of $(L_p,[p])$. (We call this the semisimple part of the element x [7,(II.3.5)]). Our present assumption implies that $\hat{\rho}(t)$ is nilpotent and $\hat{\rho}(x)^{p^r} = \hat{\rho}(x^{[p]^r}) = \hat{\rho}(t)$. Then $\hat{\rho}(x)$ is nilpotent. □

It is well–known that for a semisimple element t, $\hat{\rho}(t)$ is nilpotent if and only if $\hat{\rho}(t) = 0$.

Some main results on p–envelopes are summarized as follows.

Theorem 1.3 ([7]): Let L be a finite dimensional Lie algebra.

1) L possesses a finite dimensional p–envelope. If L is semisimple, it has a finite dimensional semisimple p–envelope.

2) Any two p–envelopes of L of minimal dimension are isomorphic as ordinary Lie algebras.

3) Let $(H_k,[p]_k,i_k)$ $(k=1,2)$ be p–envelopes of L. Then there exists a (non–restricted) homomorphism $f: H_1 \to H_2$ and a subspace $J \subset C(H_2)$ such that

$$H_2 = f(H_1) \oplus J, \quad f \circ i_1 = i_2, \quad \ker(f) \subset C(H_1).$$

Proof: 1) [7,(II.5.6)], 2) [7,(II.5.8)] 3) [7,(II.5.6), (II.5.7), (II.5.5)]. □

Corollary 1.4([4]): Let $(H_k,[p]_k,i_k)$ $(k=1,2)$ be two p–envelopes of L. Then there exists a restricted isomorphism

$$\varphi: H_1/C(H_1) \xrightarrow{\sim} H_2/C(H_2)$$

with $\varphi(i_1(x) + C(H_1)) = i_2(x) + C(H_2)\ \forall x \in L$.

Proof: We employ the notation and the result of Theorem 1.3 (3). Let x be an element of H_1 with $f(x) \in C(H_2)$. Then $f([x,i_1(L)]) = 0$ and hence $[x,i_1(L)] \subset i_1(L) \cap \ker(f) \subset \ker(i_2) = (0)$. This implies, as H_1 is generated by $i_1(L)$, $x \in C(H_1)$. Therefore f induces an isomorphism

$\varphi: H_1/C(H_1) \to H_2/C(H_2)$ with $\varphi(i_1(x) + C(H_1)) = i_2(x) + C(H_2)\ \forall x \in L$.

Moreover for any $h_2 \in H_2$ there exist $h_1 \in H_1$ and $z \in C(H_2)$ such that $h_2 = f(h_1) + z$. Hence we obtain for any $x \in H_1$

$$[f(x^{[p]_1})-f(x)^{[p]_2},h_2] = [f(x^{[p]_1}),f(h_1)] - (\mathrm{ad}\, f(x))^p(f(h_1)) = 0.$$

This shows that $f(x^{[p]_1}) - f(x)^{[p]_2} \in C(H_2)$. Then φ is a restricted isomorphism. □

The above result ensures, that the ensuing definition does not depend on the choice of the p–envelope.

<u>Definition</u>: Let L be a Lie algebra and $(H,[p],i)$ a p–envelope of L. Suppose that G is a subalgebra of L and G_p is the restricted subalgebra of H generated by $i(G)$.

1) $TR(G,L) := \max\{\dim T \mid T$ is a torus of $G_p + C(H)/C(H)\}$ is called the <u>(absolute) toral rank of G in L</u>.

2) For $G = L$ we call $TR(L) := TR(L,L)$ the <u>absolute toral rank of L</u>.

<u>Remarks 1.5</u>:

1) The above definition of the absolute toral rank of a Lie algebra L coincides with that given in [4].

2) If L is restricted and G is a torus of L then $TR(G,L) = \dim G/G \cap C(L)$ is the well–known toral rank of G in L. If L is restricted then $TR(L) = MT(L) - MT(C(L))$, so if L is centerless then $TR(L) = MT(L)$.

3) Suppose that G is a nilpotent subalgebra (or even a CSA of L). Let H denote a p–envelope of L and G_p the restricted subalgebra of H generated by G. Since G_p is nilpotent it has a unique maximal torus T. Then $\dim T/C_T(L)$ is called the toral rank of L with respect to G ([8]). Note that, since L generates H as a restricted algebra,

$$C_T(L) = C_T(H) = T \cap C(H)\ .$$

Therefore,

$$\dim T/C_T(L) = TR(T,H) = TR(G,L)$$

in this case.

In [8] Wilson gave a further description of the toral rank of L with respect to G in case that

G is a CSA of L. Then

$$L = \sum_{\alpha \in \Delta} L_\alpha(G)$$

decomposes into root spaces with respect to G. The dimension of the GF(p)–vector space spanned by Δ is the toral rank of L with respect to G. The same holds if G is nilpotent but not necessarily a CSA of L. This result yields a useful interpretation of TR(G,L) if G is nilpotent.

The concept of an "absolute toral rank of G in L" is the adequate concept which generalizes and unifies several definitions involving the term of a "toral rank". G. B. Seligman recently asked for a clear distinction between the various notions of "toral rank" and "rank". Our definition is a reply to this request.

§2. Properties of the absolute toral rank.

In the following we will prove some properties of the absolute toral rank. Note that for restricted semisimple Lie algebras TR and MT coincide. The following results are generalizations from this more specific situation.

Proposition 2.1.: Let L be a Lie algebra, G a subalgebra, L_p a p–envelope and G_p the p–subalgebra of L_p generated by G.

1) $TR(G,L) = TR(G,L_p) = TR(G_p,L_p)$, so that $TR(L) = TR(L_p)$.
2) $TR(G,L) = MT(G_p/C(L_p) \cap G_p) = MT(G_p) - MT(C(L_p) \cap G_p)$.

Proof: 1) follows directly from the definition.
2) The first equation follows directly from the definition of TR and MT. The second equation is a consequence of Lemma 1.2(1). □

Proposition 2.2: Let $G \subset K \subset L$ be Lie algebras.

1) $TR(G,K) \leq TR(G,L)$
2) $TR(G,L) \leq TR(K,L)$
3) $TR(G) \leq TR(K)$.

Proof: Let L_p denote a p–envelope of L and $G_p \subset K_p \subset L_p$ the restricted subalgebras generated by G and K, respectively. Apply Proposition 2.1(2).

1) The inclusion $C(K_p) \cap G_p \supset C(L_p) \cap G_p$ yields 1).

2) Let $T \subset G_p$ be a torus such that $\dim T/T \cap C(L_p)$ is maximal. T is also a torus of K_p. By definition we obtain $TR(K,L) \geq \dim T/T \cap C(L_p) = TR(G,L)$.

3) Putting K = G and L = K we obtain from 1), 2) $TR(G) \leq TR(G,K)$ and $TR(G,K) \leq TR(K)$, respectively. □

The following is a refinement of [4,(1.2)].

Proposition 2.3: Let I be an ideal of K and K a subalgebra of L. Suppose that L_p is a p–envelope of L and let K_p denote the p–envelope of K in L_p. Put $J := \{x \in K_p \mid [x,K] \subset I\}$. Then

$$TR(K,L) = TR(K/I) + TR(J,L_p) \geq TR(K/I) + TR(I,L) \geq TR(K/I) + TR(I) .$$

Proof: a) We consider K contained in K_p. Then I is an ideal of K_p and J is a p–ideal of K_p. Observe that there is a canonical isomorphism

$$\varphi: K_p/I \,/\, C(K_p/I) \xrightarrow{\sim} K_p/J .$$

Since K_p is restricted, the homomorphic image K_p/I is restrictable [7,(II.2.4)]. Choose any p–mapping [p]' on K_p/I and let G be the restricted subalgebra generated by K/I and [p]'.

For any $x \in K_p$ the element

$$(x+I)^{[p]'} - (x^{[p]}+I)$$

centralizes K_p/I. Thus we proved the implication

$$x + I \in G + C(K_p/I) \Rightarrow x^{[p]} + I \in G + C(K_p/I) .$$

As $K/I \subset G$, this shows that

$$K_p/I = G + C(K_p/I)$$

and hence

$$G \cap C(K_p/I) = C(G) .$$

Therefore

$$G/C(G) \cong K_p/I \,/\, C(K_p/I) .$$

The above remark also proves that an element $(x+I)^{[p]'} + C(G)$ is mapped under this isomorphism onto $(x^{[p]}+I) + C(K_p/I)$. Applying φ we obtain a restricted isomorphism

$$G/C(G) \xrightarrow{\sim} K_p/J .$$

b) Since $C(L_p) \cap K_p = C(L_p) \cap J$ and G is a p–envelope of K/I, application of Proposition 2.1 and Lemma 1.2 yields

$$TR(K,L) = MT(K_p) - MT(C(L_p) \cap K_p) = MT(K_p/J) + MT(J) - MT(C(L_p) \cap J)$$

$$= MT(G/C(G)) + TR(J,L_p) = TR(K/I) + TR(J,L_p) .$$

The remaining inequalities follow from the fact that $I \subset J$ and Proposition 2.2. □

<u>Proposition 2.4</u>: Suppose that I is an ideal of L. Then $TR(I,L) = TR(I)$.

<u>Proof</u>: Let L_p be a p–envelope of L and I_p the p–subalgebra generated by I. Take any torus T of I_p. Then L_p decomposes

$$L_p = C_{L_p}(T) + [T,L_p] .$$

Since I_p is an ideal of L_p, we have $[T,L_p] \subset I_p$. Thus

$$T \cap C(I_p) \subset T \cap C(L_p) \subset T \cap C(I_p) .$$

We conclude that $TR(T,L_p) = TR(T,I_p)$. The definition now yields $TR(I,L) = TR(I)$. □

<u>Proposition 2.5</u>:

1) Suppose that S_1, S_2 are Lie algebras. Then

$$TR(S_1 \oplus S_2) = TR(S_1) + TR(S_2)$$

2) Suppose that S_1, S_2 are ideals of a Lie algebra L. Then

$$TR(S_1+S_2) + TR(S_1 \cap S_2) \leq TR(S_1) + TR(S_2)$$

3) Suppose that S_1 is a subalgebra and S_2 is an ideal of L. Then

$$TR(S_1+S_2) + TR(S_1 \cap S_2) \leq TR(S_1, S_1+S_2) + TR(S_2) .$$

Proof: We prove 3) first. Let L_p be a p–envelope of L and K_i the p–algebra generated by S_i $(i=1,2)$. Then $K := K_1 + K_2$ is a restricted subalgebra of L_p, generated by $S_1 + S_2$. Hence K is a p–envelope of $S_1 + S_2$.

According to Lemma 1.2 we have

$$MT(K) = MT(K_1) + MT(K_2) - MT(K_1 \cap K_2) .$$

Application of Proposition 2.1(2) to this equation yields

$$TR(K) + MT(C(K)) = TR(K_1,K) + MT(C(K) \cap K_1)$$

$$+ TR(K_2) + MT(C(K_2)) - TR(K_1 \cap K_2) - MT(C(K_1 \cap K_2)) .$$

Since $TR(K_1 \cap K_2) \geq TR(S_1 \cap S_2)$ we obtain from Proposition 2.1(1) and this equation

$$TR(S_1,S_1+S_2) + TR(S_2) - TR(S_1+S_2) - TR(S_1 \cap S_2)$$

$$(*) \qquad \geq TR(K_1,K) + TR(K_2) - TR(K) - TR(K_1 \cap K_2)$$

$$= MT(C(K)) - MT(C(K) \cap K_1) - MT(C(K_2)) + MT(C(K_1 \cap K_2)) .$$

Let T_1, T_2 denote the unique maximal tori of $C(K) \cap K_1$, $C(K_2)$, respectively. Note that, as K_2 is an ideal of K,

$$[T_2,K] = [T_2^{[p]},K] \subset (\mathrm{ad}\, T_2)^p(K) \subset [T_2,K_2] = 0 .$$

Then $T_2 \subset C(K)$ and $T_1 + T_2$ is a torus of $C(K)$. Since $T_1 \cap T_2$ is a torus of $C(K_1 \cap K_2)$, we have

$$MT(C(K)) + MT(C(K_1 \cap K_2)) \geq \dim (T_1+T_2) + \dim(T_1 \cap T_2) = \dim T_1 + \dim T_2$$

$$= MT(C(K) \cap K_1) + MT(C(K_2)) .$$

Thus the right hand side of (*) is nonnegative. This proves 3).

2) is a direct consequence of 3) and Proposition 2.4, putting $I = S_1$.

1) Take in Proposition 2.3 $S_1 + S_2$ for K and L and S_1 for I. Observe that $S_1 + S_2/S_1 \cong S_2$. Then Proposition 2.3 yields

$$TR(S_1 \oplus S_2) \geq TR(S_2) + TR(S_1) .$$

In combination with 2) we obtain 1). □

Some subalgebras are of major importance in the classification theory.

Definition: a) Let L be a Lie algebra, L_p a p–envelope of L and T a torus of L_p. Decompose L into eigenspaces with respect to T

$$L = \sum_{\alpha \in \Delta} L_\alpha(T) .$$

A subalgebra

$$K = \sum_{\alpha \in \phi} L_\alpha(T), \quad \Phi \subset \Delta \subset T^*$$

is said to be a k–section (with respect to T) if ϕ is the $GF(p)$–vector space which is spanned by $GF(p)$–independent roots $\alpha_1, \cdots, \alpha_k$ relative to T.

b) Let H be a nilpotent subalgebra of L, H_p the p–subalgebra of L_p generated by H and T_0 the unique maximal torus of H_p. A subalgebra K is called a k–section with respect to H, if it is a k–section with respect to T_0.

Theorem 2.6: Let L be a Lie algebra and L_p a p–envelope of L. Suppose that $T \subset L_p$ is a torus with $TR(T,L_p) = TR(L)$. If K is a k–section with respect to T then

$$TR(K) \leq k .$$

Proof: Let R be the maximal torus of $C(L_p)$. Then $T + R$ is a torus with $TR(T+R,L_p) = TR(L)$ and K is a k–section with respect to $T + R$. Hence we may assume that T contains the maximal torus of $C(L_p)$. Let $K_p \subset L_p$ be the p–subalgebra generated by K and T_0 a torus of K_p. We have to prove that $\dim T_0/C(K_p) \cap T_0 \leq k$. Put $T_1 := C_T(K)$. Since $[T_1,K] = 0$ we have $[T_1,K_p] = 0$ and $[T_1,T_0] = 0$. Therefore $T_0 + T_1$ is a torus of L_p, proving

$$\dim (T_0 + T_1)/C(L_p) \cap (T_0+T_1) \le TR(L) .$$

Observe that $C(L_p) \cap (T_0+T_1)$ is a torus in $C(L_p)$ and hence is contained T. Then

$$C(L_p) \cap (T_0+T_1) \subset C(L_p) \cap T .$$

Thus we obtain

$$\dim T - \dim C(L_p) \cap T = TR(L) \ge \dim(T_0+T_1) - \dim C(L_p) \cap (T_0+T_1)$$

(*)

$$\ge \dim T_0 + \dim T_1 - \dim(T_0 \cap T_1) - \dim C(L_p) \cap T .$$

Let $\alpha_1,\dots,\alpha_k \in \phi$ constitute a basis of the GF(p)–vector space spanned by ϕ. Then $T_1 = \bigcap_{i=1}^{k} \ker(\alpha_i)$, which has codimension k in T:

(**) $$TR(L) = \dim T/C(L_p) \cap T = k + \dim T_1 - \dim C(L_p) \cap T .$$

$T_0 \cap T_1$ is contained in K_p and centralizes K. Then $T_0 \cap T_1 \subset C(K_p)$ and we obtain

(***) $$\dim C(K_p) \cap T_0 \ge \dim (T_0 \cap T_1) .$$

Combining (*), (**), (***) one gets

$$k + \dim T_1 - \dim C(L_p) \cap T = TR(L)$$

$$\ge \dim T_0 + \dim T_1 - \dim (T_0 \cap T_1) - \dim C(L_p) \cap T$$

so that

$$k \ge \dim T_0 - \dim (T_0 \cap T_1) \ge \dim T_0/C(K_p) \cap T_0 .$$

This is the result. □

§3. Nilpotent subalgebras

In the classification theory [3] tori of maximal dimension play an important role. Let L be simple and restricted and suppose that T is a torus of maximal dimension. Then $C_L(T)$ is a CSA and $C_L(T)$ is triangulable [8,9] (see below for definition). Moreover, any torus of a k–section with respect to T has toral rank of most k.

Very surprisingly, things hardly change if we consider non-restricted Lie algebras using the concepts of "p-envelope" and "absolute toral rank". In the following let L denote a Lie algebra with finite dimensional p-envelope L_p. If K is a subalgebra of L, then K_p denotes the p-subalgebra of L_p generated by K.

Lemma 3.1: Let $T \subset T_0 \subset L_p$ be tori of L_p. The following are equivalent:

a) $TR(T_0,L_p) = TR(T,L_p)$

b) $T_0 + C(L_p) = T + C(L_p)$.

Proof: Since $T \subset T_0$ we have $\dim T_0 + C(L_p)/C(L_p) = \dim T + C(L_p)/C(L_p)$ if and only if $T_0 + C(L_p) = T + C(L_p)$. □

Proposition 3.2: Let K be a subalgebra of L and $T \subset K_p \subset L_p$ a torus which is either a maximal torus of K_p or of toral rank $TR(T,L_p) = TR(K,L)$. Then $C_K(T)$ is a nilpotent subalgebra.

Proof: If T is a maximal torus of K_p then it is well-known that $C_{K_p}(T)$ is a CSA of K_p. Therefore $C_K(T) \subset C_{K_p}(T)$ is nilpotent. Next assume that T is a torus of maximal toral rank and $T_0 \subset K_p$ a maximal torus of K_p containing T. The maximality of the toral rank implies that $TR(T_0,L_p) = TR(T,L_p)$. Applying (3.1) we obtain $T_0 + C(L_p) = T + C(L_p)$ and hence $C_K(T_0) = C_K(T)$. The first part of the proof shows that $C_K(T_0)$ is nilpotent. □

Note that in general $C_K(T)$ is not a CSA of K. The following theorem yields a characterization of CSAs.

Proposition 3.3: Let $T \subset L_p$ be a maximal torus or a torus of maximal toral rank. The following conditions are equivalent:

a) $TR(C_L(T),L) = TR(T,L_p)$.

b) $T \subset C_L(T)_p + C(L_p)$, where $C_L(T)_p$ denotes the p-algebra in L_p generated by $C_L(T)$.

c) $C_L(T)$ is a CSA.

Proof: $C_L(T)$ is a nilpotent algebra by Proposition 3.2. Then $C_L(T)_p$ is nilpotent, too, and has a