Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

769

Jörg Flum
Martin Ziegler

Topological Model Theory

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

769

Jörg Flum
Martin Ziegler

Topological Model Theory

Springer-Verlag
Berlin Heidelberg New York 1980

Authors

Jörg Flum
Mathematisches Institut
Abt. für math. Logik
Universität Freiburg
D-7800 Freiburg

Martin Ziegler
Mathematisches Institut
Beringstr. 4
D-5300 Bonn

AMS Subject Classifications (1980): 03B60, 03C90, 03D35, 12L99, 20A15, 46A99, 54-02

ISBN 3-540-09732-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09732-5 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data
Flum, Jörg.
Topological model theory.
(Lecture notes in mathematics; 769)
Bibliography: p.
Includes index.
1. Topological spaces. 2. Model theory. I. Ziegler, Martin, joint author. II. Title.
III. Series: Lecture notes in mathematics (Berlin); 769.
QA3.L28 no. 769 [QA611.3] 510s [515.7'3] 79-29724
ISBN 0-387-09732-5

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

To

Siegrid and Gisela

INTRODUCTION

The task of model theory is to investigate mathematical structures with the aid of formal languages. Classical model theory deals with algebraic structures. Topological model theory investigates topological structures. A topological structure is a pair $(\mathfrak{A},\sigma)$ consisting of an algebraic structure $\mathfrak{A}$ and a topology σ on A. Topological groups and topological vector spaces are examples. The formal language in the study of topological structures is L_t. This is the fragment of the (monadic) second-order language (the set variables ranging over the topology σ) obtained by allowing quantification over set variables in the form $\exists X(t \in X \wedge \varphi)$, where t is a term and the second-order variable X occurs only negatively in φ (and dually for the universal quantifier). Intuitively, L_t allows only quantifications over sufficiently small neighborhoods of a point.

The reasons for the distinguished role that L_t plays in topological model theory are twofold. On one hand, many topological notions are expressible in L_t, e.g. most of the freshman calculus formulas as "continuity"

$$\forall x\ \forall Y(fx \in Y \rightarrow \exists X(x \in X \wedge \forall z(z \in X \rightarrow fz \in Y))).$$

On the other hand, the expressive power is not too strong, so that a great deal of classical model theory generalizes to L_t. For example, L_t satisfies a compactness theorem and a Löwenheim-Skolem theorem. In fact, L_t is a maximal logic with these properties ("Lindstöm theorem").
While in the second part we study concrete L_t-theories, the first part contains general model-theoretic results. The exposition shows that it is possible to give a parallel treatment of classical and topological theory, since in many cases the results of topological model theory are obtained using refinements of classical methods. On the other hand there are many new

problems which have no classical counterpart.

The content of the sections is the following.

§ 1 contains preliminaries. While second-order language is too rich to allow a fruitful model theory, central theorems of classical model theory remain true if we restrict to invariant second-order formulas. Here φ is called invariant, if for all topological structures $(\mathfrak{A}, \sigma)$,

$$(\mathfrak{A}, \sigma) \models \varphi \quad \text{iff} \quad (\mathfrak{A}, \tau) \models \varphi \text{ holds for all bases } \tau \text{ of } \sigma.$$

Many topological notions are invariant; e.g. "Hausdorff", since when checking the Hausdorff property it suffices to look at the open sets of a basis.

In section 2 we introduce the language L_t; L_t-formulas are invariant, later on (§ 4) we show the converse: each invariant formula is equivalent to an L_t-formula. -

In section 3 we derive for L_t some results (compactness theorem, Löwenheim-Skolem theorem,...) which follow immediately from the fact that L_t may be viewed as a two-sorted first-order language.

We generalize in section 4 the Ehrenfeucht-Fraïssé characterization of elementary equivalence and the Keisler-Shelah ultrapower theorem. For this we introduce for topological structures back and forth methods, which also will be an important tool later on. In § 5 we prove the L_t interpolation theorem, and derive preservation theorems for some relations between topological structures. In particular, we characterize the sentences which are preserved by dense or open substructures. In § 6 we show that operations like the product and sum operation on topological structures preserve L_t-equivalence.

Section 7 contains the L_t-definability theory. Besides the problem of the explicit definability of relations, which in classical model theory are solved by the theorems of Beth, Svenonius,... , there arises in topological model theory also the problem of the explicit definability of a topology.

In § 8 we first prove a Lindström-type characterization of L_t. - There are natural languages for several other classes of second-order structures like structures on uniform spaces, structures on proximity spaces. All these languages as well as L_t can be interpreted in the language L_m for monotone structures.

The omitting types theorem fails for L_t; we show this in section 9, where we also prove on omitting types theorem for a fragment of L_t, which will be useful in the second part. The last section is devoted to the infinitary language $(L_{\omega_1\omega})_t$. We generalize many results to this language showing that each invariant Σ_1-sentence over $(L_{\omega_1\omega})_2$ is equivalent in countable topological structures to a game sentence, whose countable approximations are in $(L_{\omega_1\omega})_t$.- We remark that some results like Scott's isomorphism theorem do not generalize to $(L_{\omega_1\omega})_t$.

The second part can be read without the complete knowledge of the first part. Essentially only §§ 1 - 4 are presupposed. The content of the sections of the second part is the following:

§ 1 Topological spaces.

We investigate decidability of some theories and determine their (L_t-) elementary types. For many classes of spaces, which do not share strong separation properties like T_3, the (L_t-)theory turns out to be undecidable. For T_3-spaces not only a decision procedure is given, but also a complete description of their elementary types by certain invariants. As a byproduct we get simple characterizations of the finitely axiomatized and of the $\aleph_0$-categorical T_3-spaces.

§ 2 Topological abelian groups.

Three theorems are proved:

1) The theory of all Hausdorff topological abelian groups is hereditarily undecidable.

2) The theory of torsionfree topological abelian groups with continuous (partial) division by all natural numbers is decidable.

3) The theory of all topological abelian groups A for which nA is closed and division by n is continuous is decidable.

§ 3 Topological fields.

We describe the L_t-elementary class of locally bounded topological fields (and other related classes) as class of structures which are L_t-equivalent to a topological field, where the filter of neighborhoods of zero is generated by the non-zero ideals of a proper local subring of K having K as quotient field.

V-topologies correspond to valuation rings. This fact has some applications in the theory of V-topological fields.- Finally we give L_t-axiomatizations of the topological fields $\mathbb{R}$ and $\mathbb{C}$.

§ 4 Topological vector spaces.

We give a simple axiomatization of the L_t-theory of the class of locally bounded real topological vector spaces. If we fix the dimension, then this theory is complete.

The L_t-elementary type of a locally bounded real topological vector space V with a distinguished subspace H is determined by the dimensions of H, $\bar{H}/_H$ and $V/_{\bar{H}}$ (where $\bar{H}$ denotes the closure of H). As an application we show that the L_t-theory of surjective and continuous linear mappings (essentially)can be axiomatized by the open mapping theorem.- Finally we determine the L_t-elementary properties of structures (V,V',[,]), where V is a real normed space, V' its dual space and [,] the canonical bilinear form.

The present book arose fom a course in topological model theory given by the second author at the University of Freiburg during the summer of 1977. We have collected all references and historical remarks on the results in the text in separate sections at the end of the first and the second part.

TABLE OF CONTENTS

§ 1 Preliminaries

We denote similarity types by $L, L', \ldots$. They are sets of predicate symbols ($P, Q, R, \ldots$) and function symbols ($f, g, \ldots$). Sometimes 0-placed function symbols are called constants and denoted by $c, d, \ldots$. – $(\mathfrak{A}, \sigma)$ is called a weak L-structure if $\mathfrak{A}$ is an L-structure in the usual sense and σ is a non-empty subset of the power set $P(A)$ of A. If σ is a topology on A, we call $(\mathfrak{A}, \sigma)$ a topological structure.

By $L_{\omega\omega}$ we denote the first-order language associated with L. It is obtained by introducing (individual) variables $w_o, w_1, \ldots$, forming terms and atomic formulas as usual, closing under the logical operations of $\neg, \wedge, \vee, \exists$ and $\forall$. $\rightarrow$ and $\leftrightarrow$ will be regarded as abbreviations, $x, y, \ldots$ will denote variables. – The (monadic) second-order language L_2 is obtained from $L_{\omega\omega}$ by adding the symbol $\in$ and set variables $W_o, W_1, \ldots$ (denoted by $X, Y, \ldots$). New atomic formulas $t \in X$, where t is a term of L, are allowed. A formation rule is added to those of $L_{\omega\omega}$:

If φ is a formula so are $\exists X\varphi$ and $\forall X\varphi$.

The meaning of a formula of L_2 in a weak structure $(\mathfrak{A}, \sigma)$ is defined in the obvious way: quantified set variables range over σ. (Note that we did not introduce formulas of the form $X = Y$, however they are definable in L_2.)

For the sentence of L_2

$$\varphi_{haus} = \forall x\, \forall y(\neg\, x = y \rightarrow \exists X\, \exists Y(x \in X \wedge y \in Y \wedge \forall z \neg\, (z \in X \wedge \in Y))) ,$$

and any topological structure $(\mathfrak{A}, \sigma)$, we have

$$(\mathfrak{A}, \sigma) \models \varphi_{haus} \quad \text{iff} \quad \sigma \text{ is a Hausdorff topology.}$$

Similarly the notions of a regular, a normal or a connected topology are expressible in L_2.

The logic L_2 (using weak structures as models) is reducible to a suitable (two sorted) first-order logic. Hence L_2 satisfies central model-theoretic theorems such as the compactness theorem, the completeness theorem and the Löwenheim-Skolem theorem, e.g.

1.1 Compactness theorem. A set of L_2-sentences has a weak model if every finite subset does.

This is not true if we restrict to topological structures as models: For

$$\varphi_{disc} = \forall x\ \exists X\ \forall y(y \in X \leftrightarrow y = x)\ ,$$

and any topological structure $(\mathfrak{A},\sigma)$, we have

$$(\mathfrak{A},\sigma) \models \varphi_{disc} \quad \text{iff} \quad \sigma \text{ is the discrete topology on } A$$
$$\text{iff} \quad \sigma = P(A).$$

Therefore, full monadic second-order logic is interpretable if we restrict to topological structures. Hence the compactness theorem, the completeness theorem and the Löwenheim-Skolem theorem do not longer hold. - In particular there is no $\varphi \in L_2$ such that

$$(\mathfrak{A},\sigma) \models \varphi \quad \text{iff} \quad \sigma \text{ is a topology}$$

holds for all weak structures $(\mathfrak{A},\sigma)$.

On the other hand to be the basis of a topology is expressible in L_2: Let

$$\varphi_{bas} = \forall x\ \exists X\ x \in X \wedge \forall x\ \forall X\ \forall Y(x \in X \wedge x \in Y \rightarrow$$
$$\exists Z(x \in Z \wedge \forall z(z \in Z \rightarrow (z \in X \wedge z \in Y)))).$$

Then

$$(\mathfrak{A},\sigma) \models \varphi_{bas} \quad \text{iff} \quad \sigma \text{ is basis of a topology on } A.$$

In the next section we will make use of this fact, when we introduce a sublanguage of L_2 which satisfies the basic modeltheoretic theorems even if we restrict to topological structures.

For $\sigma \subset P(A)$, $\sigma \neq \emptyset$, we denote by $\tilde{\sigma}$ the smallest subset of $P(A)$ containing σ and closed under unions,

$$\tilde{\sigma} = \{\bigcup s \mid s \subset \sigma\}.$$

Hence

$$(\mathfrak{A},\sigma) \models \varphi_{bas} \quad \text{iff} \quad \tilde{\sigma} \text{ is a topology.}$$

To prove that a function is continuous or that a topological space is Hausdorff, it suffices to test or to look at the open sets of a basis. These properties are "invariant for topologies" in the sense of the next definition.

1.2 Definition. Let φ be an L_2-sentence.

(i) φ is invariant if for all $(\mathfrak{A},\sigma)$:

$(\mathfrak{A},\sigma) \models \varphi$ iff $(\mathfrak{A},\tilde{\sigma}) \models \varphi$.

(ii) φ is invariant for topologies if for all $(\mathfrak{A},\sigma)$ such that $\tilde{\sigma}$ is a topology,

$(\mathfrak{A},\sigma) \models \varphi$ iff $(\mathfrak{A},\tilde{\sigma}) \models \varphi$.

Each invariant sentence is invariant for topologies. Note that φ is invariant for topologies if and only if for all topological structures $(\mathfrak{A},\tau)$ and any basis σ of τ one has

$(\mathfrak{A},\sigma) \models \varphi$ iff $(\mathfrak{A},\tau) \models \varphi$.

Each sentence of the sublanguage L_t of L_2 that we introduce in the next section is invariant. Later on we will show the converse: Each invariant (invariant for topologies) L_2-sentence is equivalent (in topological structures) to an L_t-sentence.

1.3 Exercise. (a) Show that the notions "Hausdorff", "regular", "discrete" may be expressed by L_2-sentences that are invariant for toplogies.
(b) For unary $f \in L$, $\forall X\, \forall x(x \in X \to \exists Y(fx \in Y \wedge \forall y(y \in Y \to \exists z \in X\; fz = y)))$ is a sentence invariant for topologies expressing that f is an open map.
(c) For unary $P \in L$, $\exists X\, \forall y(y \in X \leftrightarrow Py)$ is a sentence not invariant for topologies. In topological structures it expresses that P is open (but see 2.5 (b)).
(d) Give an example of an L_2-sentence invariant for topologies that is not invariant.

1.4 Exercise. (Hintikka sets and term models). Suppose L is given. Let C be a countable set of new constants and $\mathfrak{U}$ a countable set of "set constants". Denote by $L(C,\mathfrak{U})_2$ the language defined as $(L \cup C)_2$ but using the additional atomic formulas $t \in U$ (for $U \in \mathfrak{U}$). Basic terms are the terms of the form $fc_1,\dots c_n$ (with $c_1,\dots,c_n \in C$) and the constants in C. Let Ω be a set of $L(C,\mathfrak{U})_2$-sentences in negation normal form (for a definition see the beginning of the next section). Ω is said to be a Hintikka set iff (i) - (x) hold:

(i) For each atomic φ of the form $c_1 = c_2$, $Rc_1 \ldots c_n$ or $c \in U$ (where $c_i, c \in C$ and $U \in \mathfrak{U}$) either $\varphi \notin \Omega$ or $\neg\varphi \notin \Omega$.

(ii) If $\varphi_1 \wedge \varphi_2 \in \Omega$ then $\varphi_1 \in \Omega$ and $\varphi_2 \in \Omega$.

(iii) If $\varphi_1 \vee \varphi_2 \in \Omega$ then $\varphi_1 \in \Omega$ or $\varphi_2 \in \Omega$.

(iv) If $\forall x\, \varphi \in \Omega$ then for all $c \in C$, $\varphi\frac{c}{x} \in \Omega$.

(v) If $\exists x\, \varphi \in \Omega$ then for some $c \in C$, $\varphi\frac{c}{x} \in \Omega$.

(vi) If $\forall X\, \varphi \in \Omega$ then for all $U \in \mathfrak{U}$, $\varphi\frac{U}{X} \in \Omega$.

(vii) If $\exists X\, \varphi \in \Omega$ then for some $U \in \mathfrak{U}$, $\varphi\frac{U}{X} \in \Omega$.

(viii) For all $c \in C$, $c = c \in \Omega$.

(ix) If t is a basic term, then for some $c \in C$, $t = c \in \Omega$.

(x) If φ is atomic or negated atomic and t is a basic term such that for some $c \in C$ and some variable x, $t = c \in \Omega$, and $\varphi\frac{t}{x} \in \Omega$, then $\varphi\frac{c}{x} \in \Omega$.

($\varphi\frac{t}{x}$ and similarly $\varphi\frac{U}{X}$, is obtained by replacing each free occurence of x in φ by t).

Suppose Ω is a Hintikka set. For $c_1, c_2 \in C$, let

$$c_1 \sim c_2 \quad \text{iff} \quad c_1 = c_2 \in \Omega.$$

Show that $\sim$ is an equivalence relation. Let $\bar{c}$ be the equivalence class of c. Define an L-structure $(\mathfrak{A}, \sigma)$ by

$$A = \{\bar{c} \mid c \in \Omega\},$$

for n-ary $R \in L$, $R^{\mathfrak{A}}\bar{c}_1 \ldots \bar{c}_n$ iff $Rc_1 \ldots c_n \in \Omega$

for n-ary $f \in L$, $f^{\mathfrak{A}}(\bar{c}_1, \ldots, \bar{c}_n) = \bar{c}$ iff $fc_1 \ldots c_n = c \in \Omega$

$$\sigma = \{\bar{U} \mid U \in \mathfrak{U}\} \text{ where } \bar{U} = \{\bar{c} \mid \text{"}c \in U\text{"} \in \Omega\}.$$

Show: (a) For atomic φ of the form $Rc_1 \ldots c_n$, $fc_1 \ldots c_n = c$, $c_1 = c_2$ or $c \in U$, one has: $(\mathfrak{A}, \sigma) \models \varphi$ iff $\varphi \in \Omega$.
(when interpreting c by $\bar{c}$ and U by $\bar{U}$).

(b) $(\mathfrak{A}, \sigma) \models \Omega$.

$(\mathfrak{A}, \sigma)$ is called the <u>term model</u> of Ω.

§ 2 The Language L_t

An L_2-formula is said to be in negation normal form, if negation signs in it occur only in front of atomic formulas. Using the logical rules for the negation we can assign canonically to any formula φ its negation normal form, a formula in negation normal form equivalent to φ.

An L_2-formula φ is positive (negative) in the set variable X if each free occurence of X in φ is within the scope of an even (odd) number of negation symbols. Equivalently, φ is positive (negative) in X, if each free occurence of X in the negation normal form of φ is of the form $t \in X$ where $t \in X$ is not preceded by a negation symbol (is of the form $\neg t \in X$). Note that for any X, which is not a free variable of φ, φ is both, positive and negative in X.

The formula

$$\exists X \neg t \in X \vee (c \in X \wedge \neg c \in Y \wedge \exists y(y \in X \wedge y \in Y))$$

is positive in X and neither positive nor negative in Y.

We use $\varphi(x_1,\dots,x_n,X_1,\dots,X_r)$ to denote a formula φ whose free variables are among the distinct variables $x_1,\dots,x_n$ and whose free set variables are among the distinct set variables $X_1,\dots,X_r$. - A simple induction shows

2.1 Lemma. Let $\varphi(x_1,\dots,x_n,X_1,\dots,X_r,Y)$ be an L_2-formula, $(\mathfrak{A},\sigma)$ a weak structure, $a_1,\dots,a_n \in A$ and $U_1,\dots,U_r,U \subset A$.
Assume $(\mathfrak{A},\sigma) \models \varphi[a_1,\dots,a_n,U_1,\dots,U_r,U]$.

(a) If φ is positive in Y, then $(\mathfrak{A},\sigma) \models \varphi[a_1,\dots,a_n,U_1,\dots,U_r,V]$ for any V such that $U \subset V \subset A$.

(b) If φ is negative in Y, then $(\mathfrak{A},\sigma) \models \varphi[a_1,\dots,a_n,U_1,\dots,U_r,V]$ for any V such that $V \subset U$.

In the sequel we use for sequences like $a_1,\dots,a_n$ or $U_1,\dots,U_r$ the abbreviations $\bar{a},\bar{U}$.

2.2 Definition. We denote by L_t the set of L_2-formulas obtained from the atomic formulas of L_2 by the formation rules of $L_{\omega\omega}$ and the rules:

(i) If t is a term and φ is positive in X, then $\forall X(t \in X \rightarrow \varphi)$ is a formula.

(ii) If t is a term and φ is negative in X, then $\exists X(t \in X \wedge \varphi)$ is a formula.

We abbreviate $\forall X(t \in X \to \varphi)$ and $\exists X(t \in X \wedge \varphi)$ by $\forall X \ni t\, \varphi$ resp. $\exists X \ni t\, \varphi$. For example,

$$\underline{\text{bas}} = \forall x\, \exists X \ni x\, \forall x\, \forall X \ni x\, \forall Y \ni x\, \exists Z \ni x\, \forall z(z \in Z \to (z \in X \wedge z \in Y))$$

is an L_t-sentence.

Note that if X is free in a subformula φ of an L_t-sentence then either φ is positive or negative in X. For an L_t-formula φ the notation $\varphi(x_1,\ldots,x_n,X_1^+,\ldots,X_r^+,Y_1^-,\ldots,Y_s^-)$ expresses that φ is positive in $X_1,\ldots,X_r$ and negative in $Y_1,\ldots,Y_s$.

2.3 Theorem. L_t-sentences are invariant.

Proof. For given $(\mathfrak{A},\sigma)$ one shows by induction on φ:

if $\varphi(\bar{x},\bar{X}^+,\bar{Y}^-) \in L_t$, $\bar{a} \in A$, $\bar{U},\bar{V} \subset A$, then

$$(\mathfrak{A},\sigma) \models \varphi[\bar{a},\bar{U},\bar{V}] \quad \text{iff} \quad (\mathfrak{A},\tilde{\sigma}) \models \varphi[\bar{a},\bar{U},\bar{V}] .$$

We only treat the case $\varphi = \exists X \ni t\, \psi$. Set $a_o = t^{\mathfrak{A}}[\bar{a}]$. Assume $(\mathfrak{A},\sigma) \models \varphi[\bar{a},\bar{U},\bar{V}]$. Choose $V \in \sigma$ such that $a_o \in V$ and $(\mathfrak{A},\sigma) \models \psi[\bar{a},\bar{U},\bar{V},V]$ By induction hypothesis, $(\mathfrak{A},\tilde{\sigma}) \models \psi[\bar{a},\bar{U},\bar{V},V]$. Hence, $(\mathfrak{A},\tilde{\sigma}) \models \varphi[\bar{a},\bar{U},\bar{V}]$. – Now suppose $(\mathfrak{A},\tilde{\sigma}) \models \varphi[\bar{a},\bar{U},\bar{V}]$. Let $V \in \tilde{\sigma}$ be such that $a_o \in V$ and $(\mathfrak{A},\tilde{\sigma}) \models \psi[\bar{a},\bar{U},\bar{V},V]$. By induction hypothesis, $(\mathfrak{A},\sigma) \models \psi[\bar{a},\bar{U},\bar{V},V]$: Since $V \in \tilde{\sigma}$, there is a $V' \in \sigma$ such that $a_o \in V' \subset V$. ψ is negative in X because $\exists X \ni t\, \psi \in L_t$. Thus by 2.1, $(\mathfrak{A},\sigma) \models \psi[\bar{a},\bar{U},\bar{V},V']$, hence $(\mathfrak{A},\sigma) \models \varphi[\bar{a},\bar{U},\bar{V}]$.

2.4 Corollary. Suppose that σ_1 and σ_2 are bases of the same topology on A, $\tilde{\sigma}_1 = \tilde{\sigma}_2$. Let φ be an L_t-sentence. Then

$$(\mathfrak{A},\sigma_1) \models \varphi \quad \text{iff} \quad (\mathfrak{A},\sigma_2) \models \varphi .$$

The properties "Hausdorff", "regular", "discrete" and "trivial" of topologies may be expressed by L_t-sentences (though the sentences φ_{haus} and φ_{disc} of the last section are not in L_t):

$$\underline{\text{haus}} = \forall x\, \forall y\, (x = y \vee \exists X \ni x\, \exists Y \ni y\, \forall z \neg (z \in X \wedge z \in Y))$$

$$\underline{\text{reg}} = \forall x\, \forall X \ni x\, \exists Y \ni x\, \forall y\, (y \in X \vee \exists W \ni y\, \forall z\, (\neg z \in W \vee \neg z \in Y))$$

$$\underline{\text{disc}} = \forall x\, \exists X \ni x\, \forall y\, (y \in X \to y = x)$$

$$\underline{\text{triv}} = \forall x\, \forall X \ni x\, \forall y\, y \in X .$$

For an n-ary function symbol $f \in L$ the continuity of f is expressed in L_t by

$$\varphi = \forall x_1 \ldots \forall x_n \; \forall Y \ni fx_1 \ldots x_n \; \exists X_1 \ni x_1 \ldots \exists X_n \ni x_n$$
$$\forall y_1 \ldots \forall y_n (y_1 \in X_1 \wedge \ldots \wedge y_n \in X_n \to fy_1 \ldots y_n \in Y),$$

i.e. one has for all topological structures $(\mathfrak{A},\sigma)$

$(\mathfrak{A},\sigma) \models \varphi$ iff f^A is a continuous map from A^n to A (where A^n carries the product topology).

The class of topological groups and the class of topological fields are axiomatizable in L_t; for example, if $L = \{\cdot, ^{-1}, e\}$ then the topological groups are just the structures which are models of the group axioms and the sentences " $\cdot$ is continuous", and " $^{-1}$ is continuous".

By topological model theory (or topological logic) we understand the study of topological structures using the formal language L_t (and variants of L_t).

2.5 Exercise. (a) Show that for unary $f \in L$, "f is an open map" may be expressed in L_t (compare 1.3 (b)).
(b) Show that for unary $P \in L$, "P is open" may be expressed in L_t (compare 1.3 (c)).
(c) Show that for $\varphi \in L_t$ there is a $\psi \in L_{\omega\omega}$ such that for all topological structures $(\mathfrak{A},\sigma)$ with $(\mathfrak{A},\sigma) \models$ disc one has:

$$(\mathfrak{A},\sigma) \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \psi .$$

Similarly for models of triv.

§ 3 Beginning topological model theory

Using the invariance of the sentences of L_t one can derive many theorems for topological logic from its classical analogues. This section contains some examples.

Given $\Phi \cup \{\varphi\} \subset L_2$ we write $\Phi \models \varphi$ resp. $\Phi \models_t \varphi$ if each weak structure resp. topological structure that is a model of Φ is a model of φ.

3.1 Lemma. Suppose $\Phi \cup \{\varphi\} \subset L_t$.
(a) Φ has a topological model iff $\Phi \cup \{$bas$\}$ has a weak model.

(b) $\Phi \models_t \varphi$ iff $\Phi \cup \{$bas$\} \models \varphi$.