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Introduction to Symplectic Dirac Operators

$$(X \cdot Y - Y \cdot X) \cdot \varphi = -\mathrm{i}\omega(X,Y)\varphi$$

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Preface

This book aims to give a systematic and self-contained introduction to the theory of symplectic Dirac operators and to reflect the current state of the subject. At the same time, it is intended to establish the idea that symplectic spin geometry and symplectic Dirac operators may give valuable tools in symplectic geometry and symplectic topology, which have become important fields and very active areas of mathematical research.

The basic idea of symplectic spin geometry goes back to the early 1970s, when Bertram Kostant introduced symplectic spinors in order to give the construction of the half-form bundle and the half-form pairings in the context of geometric quantization [37]. During the next two decades, however, almost no attention has been given to a closer study of symplectic spin geometry itself.

In 1995, the first author introduced symplectic Dirac operators [24] and started a systematical investigation [25, 26, 27]. These *symplectic Dirac operators* are called *Dirac operators*, since they are defined in an analogous way as the classical Riemannian Dirac operator known from Riemannian spin geometry (cf. e.g. [21]). They are called *symplectic* because they are constructed by use of the symplectic setting of the underlying symplectic manifold. All tools which were necessary for that construction have already been known and accepted, mainly in mathematical physics. These are the symplectic Clifford algebra (also known as Weyl algebra), the metaplectic group, the metaplectic representation (Segal–Shale–Weil representation) acting on $L^2(\mathbb{R}^n)$, metaplectic structures, and symplectic connections.

One of the basic ideas in differential geometry is that the study of analytic properties of certain differential operators acting on sections of vector bundles yields geometric and topological properties of the underlying base manifold. There are several classical results in that direction. An example is Hodge–de Rham theory. Here, one considers the Hodge–Laplace–Beltrami operator Δ

acting on differential forms. This operator is one of the most studied operators in global Riemannian geometry and his spectrum gives important topological invariants. In particular, the dimension of the kernel of Δ on p-forms over a closed Riemannian manifold is the p-th Betti number. Other well known and well studied operators are the Kodaira–Hodge–Laplace operator on differential forms with values in a holomorphic vector bundle or the classical Dirac operator on Riemannian manifolds. Now, symplectic spinor fields are sections in an $L^2(\mathbb{R}^n)$-Hilbert space bundle over a symplectic manifold and symplectic Dirac operators, acting on symplectic spinor fields, are associated to the symplectic manifold in a very natural way. Hence they may be expected to give interesting applications in symplectic geometry and symplectic topology.

It is our opinion that, besides the already stated, there are further close relations to mathematical physics. Some steps towards this direction have been made by the first author and Andreas Klein in [28, 29, 30].

Another perspective could be the extension of Clifford analysis and spin geometry to super differential geometry. According to the most used version of super geometry developed by Bertram Kostant, geometrical structures over a supermanifold consist of $\mathbb{Z}_2$-graded objects and thus have an even as well as an odd part. Then one can imagine that a metric has to satisfy some kind of graded symmetry which, roughly speaking, corresponds to a symmetric object on the even part as well as to a skew symmetric object on the odd part of the supermanifold. For the first one, we have the classical Riemannian spin geometry, whereas the second one is basically given by symplectic spin geometry. The main aspects of that idea are treated in a paper by Frank Sommen [45].

Although the construction of symplectic Dirac operators follows the same procedure as for the classical Riemannian Dirac operator, using the symplectic structure of the underlying manifold instead of the Riemannian metric, there are essential differences to the Riemannian case. These are caused by the fact that the algebraic structure of the symplectic Clifford algebra is completely different from that of Riemannian spin geometry. For the classical Clifford algebra, we have the relation $\mathbf{v}^2 = -\|\mathbf{v}\|^2$, whereas the algebraic structure of the symplectic Clifford algebra is given by $\mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} = -\omega_0(\mathbf{v}, \mathbf{w})$. This implies essentially different properties for the Clifford multiplications, which enter into the definition of the Dirac operators.

Moreover, the non-compactness of the symplectic group leads to analytic difficulties. Namely, since the typical fiber of the symplectic spinor bundle is the Hilbert space $L^2(\mathbb{R}^n)$, we deal with operators acting on sections of a vector bundle of infinite rank. For elliptic formally self-adjoint pseudo-differential operators with positive definite leading symbol acting on sections in a vector bundle of finite rank, one has a completely developed theory. So, in order to be able to apply these techniques, we are interested in equivariance properties of our operators with respect to a certain decomposition of the symplectic spinor bundle into a series of subbundles of finite rank. It turns out that an associated second order operator respects this decomposition provided that a technical assumption, which always can be realized, holds true.

Let us now briefly describe the content of each chapter of this book. The first chapter is introductory. It contains preliminaries and basic material needed for our considerations. Chapter 2 is devoted to symplectic connections. In particular, we introduce a further Ricci tensor, which we will call symplectic Ricci tensor. To our knowledge, no attention has been given to this tensor in previous studies. In fact, this symplectic Ricci tensor is a new object in the case of non-vanishing torsion. To date, mostly only torsion-free symplectic connections have been considered. It turns out that, in our context, it is convenient to work also with symplectic connections with torsion and that the symplectic Ricci tensor is more suitable for our purposes. The next chapter introduces the symplectic spinor bundle and the spinor derivative and analyzes a splitting property of the spinor bundle. In Chapter 4, we give the definition of symplectic Dirac operators and describe in detail how these operators depend on the objects from which they are built. Chapter 5 is concerned with an associated second order operator of Laplace type and addresses properties of this operator. The objective of Chapter 6 is the situation for a special class of symplectic manifolds, namely Kähler manifolds. Here, we also investigate the example of $\mathbb{C}P^1$. The aim of Chapter 7 is to construct a Fourier transform for symplectic spinor fields and to derive consequences for the symplectic Dirac operators. The last chapter focuses on relations to mathematical physics, in particular to quantization. This closes the circle to the beginnings by Bertram Kostant.

The present text is originated in research ideas of the first author and provides an extended version of her "Habilitationsschrift", which was never published separately. Starting with helpful discussions from the beginning of the investigations and proposing many improvements in the selection and presentation of the material, the second author became more and more involved into the subject. We decided to write this book tree years ago and, for this book, he made a thorough revision of the material, in particular, to improve the strictness of the presentation. Then it took time to compose it in a natural and organic way. Furthermore, our working places are no longer as close to each other as before and it became difficult to keep our discussions at its intensive level. Now we consider the text ready for publication and hope to present the reader a mature work.

During the work on the book, we received financial support from DFG, the German Research Foundation, contract HA 3056/1-1,2. Many thanks are due to our students Paul Rosenthal and Steffen Rudnick for proof-reading the LaTeX-type-written manuscript.

We dedicate this book to our daughter Karen. She accepts our mathematical family circle and enriches it with her interest. We are grateful to being able to give her an understanding of the fascination of mathematics. Karen, you are a wonderful teenager which makes us enjoying the challenge of teaching mathematics.

Göttingen and Greifswald, November 2004

Katharina & Lutz Habermann

Contents

1

Background on Symplectic Spinors

This chapter serves two purposes. First it gives a survey on fundamental relations between the symplectic Clifford algebra, the metaplectic group, and its Lie algebra. The second is to provide several elementary facts used in the later computations. Most of it is well known, but we have summarized the material in a form that makes it applicable for our considerations.

1.1 Symplectic Group and Clifford Algebra

We consider the $2n$-dimensional real vector space $\mathbb{R}^{2n}$ equipped with its standard symplectic form ω_0. We write any $\mathbf{v} \in \mathbb{R}^{2n}$ as

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}' \\ \mathbf{v}'' \end{pmatrix}$$

with vectors $\mathbf{v}', \mathbf{v}'' \in \mathbb{R}^n$ and any $2n \times 2n$-matrix A as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $n \times n$ matrices a, b, c, d. Let $\mathbf{I}$ be the unit element of the linear group $\mathrm{GL}(n, \mathbb{R})$ and set

$$J_0 = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} .$$

Then

$$\omega_0(\mathbf{v}, \mathbf{w}) = \langle J_0\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}', \mathbf{w}'' \rangle - \langle \mathbf{v}'', \mathbf{w}' \rangle$$

for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2n}$, where $\langle\ ,\ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^{2n}$ as well as on $\mathbb{R}^n$. Hence the standard basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^{2n}$ forms a symplectic basis, which means that

$$\omega_0(\mathbf{a}_j, \mathbf{a}_k) = \omega_0(\mathbf{b}_j, \mathbf{b}_k) = 0 \quad \text{and} \quad \omega_0(\mathbf{a}_j, \mathbf{b}_k) = \delta_{jk} \tag{1.1.1}$$

for $j, k = 1, \ldots, n$.

The symplectic group $\mathrm{Sp}(n, \mathbb{R})$ is the group of all automorphisms of $\mathbb{R}^{2n}$ which preserve the symplectic form ω_0. That is, $\mathrm{Sp}(n, \mathbb{R})$ is the group of those $A \in \mathrm{GL}(2n, \mathbb{R})$ that satisfy

$$\omega_0(A\mathbf{v}, A\mathbf{w}) = \omega_0(\mathbf{v}, \mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2n}$, which is equivalent to

$$A^{\mathrm{T}} J_0 A = J_0$$

as well as

$$A J_0 A^{\mathrm{T}} = J_0 \ .$$

In particular, $J_0 \in \mathrm{Sp}(n, \mathbb{R})$.

We denote the space of all symmetric real $n \times n$-matrices by $S(n)$ and set

$$D(n) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \left(a^{\mathrm{T}}\right)^{-1} \end{pmatrix} : a \in \mathrm{GL}(n, \mathbb{R}) \right\}$$

and

$$N(n) = \left\{ \begin{pmatrix} \mathbf{I} & b \\ 0 & \mathbf{I} \end{pmatrix} : b \in S(n) \right\} \ .$$

Obviously, $D(n)$ and $N(n)$ are subgroups of $\mathrm{Sp}(n, \mathbb{R})$. Moreover, we have (cf. [16])

Proposition 1.1.1 $\mathrm{Sp}(n, \mathbb{R})$ *is generated by* $D(n) \cup N(n) \cup \{J_0\}$. □

Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $\mathbf{v} \mapsto \mathbf{v}' + \mathrm{i}\mathbf{v}''$, the Hermitian inner product of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2n}$ is

$$\langle \mathbf{v}, \mathbf{w} \rangle - \mathrm{i}\omega_0(\mathbf{v}, \mathbf{w}) \ .$$

Therefore, the intersection of $\mathrm{Sp}(n, \mathbb{R})$ with the orthogonal group $\mathrm{O}(2n)$ is the unitary group $\mathrm{U}(n)$. One can prove (cf. [16])

Proposition 1.1.2 (1) $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(n, \mathbb{R})$.

(2) $\mathrm{Sp}(n, \mathbb{R})$ *is homeomorphic to the product* $\mathrm{U}(n) \times \mathbb{R}^{n^2+n}$. □

Corollary 1.1.3 $\mathrm{Sp}(n, \mathbb{R})$ *is connected and its fundamental group is* $\mathbb{Z}$.

Proof. This follows from Proposition 1.1.2 and the corresponding properties of $\mathrm{U}(n)$. □

The symplectic Lie algebra, i.e. the Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ is given by the space of all endomorphisms X of $\mathbb{R}^{2n}$ satisfying

$$\omega_0(X\mathbf{v},\mathbf{w})+\omega_0(\mathbf{v},X\mathbf{w})=0$$

for all $\mathbf{v},\mathbf{w}\in\mathbb{R}^{2n}$. We identify the space $S^2(\mathbb{R}^{2n})$ of symmetric 2-tensors of $\mathbb{R}^{2n}$ with $\mathfrak{sp}(n,\mathbb{R})$ by assigning to $\mathbf{v}_1\odot\mathbf{v}_2\in S^2(\mathbb{R}^{2n})$ the endomorphism

$$\mathbf{v}\in\mathbb{R}^{2n}\mapsto\omega_0(\mathbf{v},\mathbf{v}_1)\mathbf{v}_2+\omega_0(\mathbf{v},\mathbf{v}_2)\mathbf{v}_1\in\mathbb{R}^{2n}\,.$$

Then we have

Lemma 1.1.4 *For every $X\in\mathfrak{sp}(n,\mathbb{R})$,*

$$X=\frac{1}{2}\sum_{j=1}^{n}(X\mathbf{a}_j\odot\mathbf{b}_j-\mathbf{a}_j\odot X\mathbf{b}_j)\,.$$

Proof. The assertion follows from

$$\begin{aligned}\sum_{j=1}^{n}&(\omega_0(\mathbf{v},X\mathbf{a}_j)\mathbf{b}_j+\omega_0(\mathbf{v},\mathbf{b}_j)X\mathbf{a}_j-\omega_0(\mathbf{v},\mathbf{a}_j)X\mathbf{b}_j-\omega_0(\mathbf{v},X\mathbf{b}_j)\mathbf{a}_j)\\ &=\sum_{j=1}^{n}(\omega_0(X\mathbf{v},\mathbf{b}_j)\mathbf{a}_j-\omega_0(X\mathbf{v},\mathbf{a}_j)\mathbf{b}_j)\\ &\quad+X\left(\sum_{j=1}^{n}(\omega_0(\mathbf{v},\mathbf{b}_j)\mathbf{a}_j-\omega_0(\mathbf{v},\mathbf{a}_j)\mathbf{b}_j)\right)\\ &=2\,X\mathbf{v}\end{aligned}$$

for any $\mathbf{v}\in\mathbb{R}^{2n}$. □

The Lie bracket of $\mathfrak{sp}(n,\mathbb{R})$ now writes as

$$\begin{aligned}[\mathbf{v}_1\odot\mathbf{v}_2,\mathbf{v}_3\odot\mathbf{v}_4]=&-\omega_0(\mathbf{v}_2,\mathbf{v}_4)\mathbf{v}_1\odot\mathbf{v}_3-\omega_0(\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_1\odot\mathbf{v}_4\\ &-\omega_0(\mathbf{v}_1,\mathbf{v}_4)\mathbf{v}_2\odot\mathbf{v}_3-\omega_0(\mathbf{v}_1,\mathbf{v}_3)\mathbf{v}_2\odot\mathbf{v}_4\,.\end{aligned}$$

In matrix notation, the above identification is given by

$$\mathbf{v}_1\odot\mathbf{v}_2=\begin{pmatrix}\mathbf{v}_1''\otimes\mathbf{v}_2'+\mathbf{v}_2''\otimes\mathbf{v}_1' & -\mathbf{v}_1'\otimes\mathbf{v}_2'-\mathbf{v}_2'\otimes\mathbf{v}_1'\\ \mathbf{v}_1''\otimes\mathbf{v}_2''+\mathbf{v}_2''\otimes\mathbf{v}_1'' & -\mathbf{v}_1'\otimes\mathbf{v}_2''-\mathbf{v}_2'\otimes\mathbf{v}_1''\end{pmatrix}\,. \tag{1.1.2}$$

Here, $x\otimes y$ for $x,y\in\mathbb{R}^n$ means the endomorphism

$$z \in \mathbb{R}^n \mapsto \langle x, z\rangle y \in \mathbb{R}^n .$$

Clearly, the endomorphism adjoint to $x \otimes y$ is then

$$(x \otimes y)^{\mathrm{T}} = y \otimes x . \tag{1.1.3}$$

Next we define the symplectic Clifford algebra, which is also referred to as Weyl algebra. In contrast to the Riemannian case, this algebra is infinite dimensional.

Definition 1.1.5 *The symplectic Clifford algebra* $\mathbf{Cl}(n)$ *is the associative unital algebra over* $\mathbb{R}$ *generated by* $\mathbb{R}^{2n}$ *with the relations*

$$\mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} = -\omega_0(\mathbf{v},\mathbf{w})$$

for $\mathbf{v},\mathbf{w} \in \mathbb{R}^{2n}$.

A basis of $\mathbf{Cl}(n)$ is formed by

$$\mathbf{a}_1^{\alpha_1}\cdot\mathbf{a}_2^{\alpha_2}\cdot\ldots\cdot\mathbf{a}_n^{\alpha_n}\cdot\mathbf{b}_1^{\beta_1}\cdot\mathbf{b}_2^{\beta_2}\cdot\ldots\cdot\mathbf{b}_n^{\beta_n} ,$$

where α_j, β_j are non-negative integers. According to Equation (1.1.1),

$$\mathbf{a}_j\cdot\mathbf{a}_k = \mathbf{a}_k\cdot\mathbf{a}_j , \quad \mathbf{b}_j\cdot\mathbf{b}_k = \mathbf{b}_k\cdot\mathbf{b}_j , \quad \text{and} \quad \mathbf{a}_j\cdot\mathbf{b}_k - \mathbf{b}_k\cdot\mathbf{a}_j = -\delta_{jk}$$

for $j,k = 1,\ldots,n$.

As usual, we set

$$[v,w] = v\cdot w - w\cdot v$$

for $v,w \in \mathbf{Cl}(n)$, which gives $\mathbf{Cl}(n)$ the structure of a real Lie algebra. Let $\mathfrak{a}(n)$ denote the subspace of $\mathbf{Cl}(n)$ which is spanned by $\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v}$ for $\mathbf{v},\mathbf{w} \in \mathbb{R}^{2n}$.

Lemma 1.1.6 *The space* $\mathfrak{a}(n)$ *is a Lie subalgebra of* $\mathbf{Cl}(n)$ *which is isomorphic to the symplectic Lie algebra* $\mathfrak{sp}(n,\mathbb{R})$.

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v} \in \mathbb{R}^{2n}$ and $v \in \mathfrak{a}(n)$. First we observe that

$$\begin{aligned}
[\mathbf{v}_1\cdot\mathbf{v}_2, \mathbf{v}] &= \mathbf{v}_1\cdot\mathbf{v}_2\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v}_1\cdot\mathbf{v}_2 \\
&= \mathbf{v}_1\cdot\mathbf{v}\cdot\mathbf{v}_2 + \omega_0(\mathbf{v},\mathbf{v}_2)\mathbf{v}_1 - \mathbf{v}\cdot\mathbf{v}_1\cdot\mathbf{v}_2 \\
&= \omega_0(\mathbf{v},\mathbf{v}_1)\mathbf{v}_2 + \omega_0(\mathbf{v},\mathbf{v}_2)\mathbf{v}_1 \\
&= (\mathbf{v}_1 \odot \mathbf{v}_2)\mathbf{v} .
\end{aligned}$$

Thus

$$[\mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{v}_1, \mathbf{v}] = 2(\mathbf{v}_1 \odot \mathbf{v}_2)\mathbf{v} \tag{1.1.4}$$

and

$$[v, \mathbf{v}] \in \mathbb{R}^{2n} . \tag{1.1.5}$$

Further we have

$$[\mathbf{v}_1 \cdot \mathbf{v}_2, v] = \mathbf{v}_1 \cdot [\mathbf{v}_2, v] + [\mathbf{v}_1, v] \cdot \mathbf{v}_2$$

and hence

$$\begin{aligned}[\mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_2 \cdot \mathbf{v}_1, v] = \mathbf{v}_1 \cdot [\mathbf{v}_2, v] + [\mathbf{v}_2, v] \cdot \mathbf{v}_1 \\ +\mathbf{v}_2 \cdot [\mathbf{v}_1, v] + [\mathbf{v}_1, v] \cdot \mathbf{v}_2 .\end{aligned} \tag{1.1.6}$$

Equations (1.1.5) and (1.1.6) imply that $\mathfrak{a}(n)$ is a Lie subalgebra of $\mathbf{Cl}(n)$. From Equation (1.1.4) and the Jacobi identity, we get the second part of the assertion. □

1.2 The Stone–von Neumann Theorem

In this section, we want to recall some facts of representation theory and formulate the Stone–von Neumann theorem. This fundamental theorem will be used in the next section to construct the metaplectic representation. For details and proofs, we refer to [1, 16, 49].

Let G be a connected Lie group and let $\mathfrak{H}$ be a separable complex Hilbert space. We endow the group $\mathrm{GL}(\mathfrak{H})$ of invertible bounded linear operators on $\mathfrak{H}$ with the strong topology. That means that a sequence (T_k) in $\mathrm{GL}(\mathfrak{H})$ converges to T_0 if and only if the sequence $(T_k h)$ converges to $T_0 h$ for all $h \in \mathfrak{H}$.

Definition 1.2.1 *A representation of G on $\mathfrak{H}$ is a continuous group homomorphism $\mathfrak{r} : G \to \mathrm{GL}(\mathfrak{H})$. It is called unitary if it maps into the unitary group $\mathrm{U}(\mathfrak{H})$ of $\mathfrak{H}$.*

Definition 1.2.2 *Let $\mathfrak{r}$ be a representation of G on $\mathfrak{H}$.*

(1) *A subspace $\mathfrak{W} \subset \mathfrak{H}$ is said to be $\mathfrak{r}$-invariant if*

$$\mathfrak{r}(a)\mathfrak{W} \subset \mathfrak{W}$$

for all $a \in G$.

(2) *$\mathfrak{r}$ is called irreducible if the only $\mathfrak{r}$-invariant closed subspaces of $\mathfrak{H}$ are $\{0\}$ and $\mathfrak{H}$.*

The following proposition is a version of Schur's lemma.

Proposition 1.2.3 *A unitary representation* $\mathfrak{r}$ *of* G *on* $\mathfrak{H}$ *is irreducible if and only if the only bounded linear operators* $T : \mathfrak{H} \to \mathfrak{H}$ *such that*

$$T \circ \mathfrak{r}(a) = \mathfrak{r}(a) \circ T$$

for all $a \in G$ *are scalar multiples of the identity.* □

Let $\mathfrak{r}$ be a fixed representation of G on $\mathfrak{H}$ and let $\mathfrak{H}^\infty$ denote the space of smooth vectors of $\mathfrak{r}$, i.e. the set of all $h \in \mathfrak{H}$ such that the map

$$a \in G \mapsto \mathfrak{r}(a)h \in \mathfrak{H}$$

is smooth.

Theorem 1.2.4 (Gårding) $\mathfrak{H}^\infty$ *is a dense invariant subspace of* $\mathfrak{H}$. □

The differential of $\mathfrak{r}$ is the homomorphism $\mathfrak{r}_* : \mathfrak{g} \to \mathrm{End}(\mathfrak{H}^\infty)$ from the Lie algebra $\mathfrak{g}$ of G into the endomorphism algebra $\mathrm{End}(\mathfrak{H}^\infty)$ of $\mathfrak{H}^\infty$ given by

$$\mathfrak{r}_*(X)h = \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{r}(\exp(tX))h\right|_{t=0}$$

for $X \in \mathfrak{g}$ and $h \in \mathfrak{H}^\infty$. By Theorem 1.2.4, each operator $\mathfrak{r}_*(X)$ can be considered as an unbounded operator on $\mathfrak{H}$.

Definition 1.2.5 *Let* $\mathfrak{r}_1 : G \to \mathrm{U}(\mathfrak{H}_1)$ *and* $\mathfrak{r}_2 : G \to \mathrm{U}(\mathfrak{H}_2)$ *be representations of* G *on Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$. *Then* $\mathfrak{r}_1$ *is said to be equivalent to* $\mathfrak{r}_2$ *if there exists a bounded linear bijection* $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ *such that*

$$T \circ \mathfrak{r}_1(a) = \mathfrak{r}_2(a) \circ T$$

for all $a \in G$. *If, in addition,* T *can be chosen unitary,* $\mathfrak{r}_1$ *is called unitary equivalent to* $\mathfrak{r}_2$.

Now we consider the Heisenberg group $\mathrm{H}(n)$ of $\mathbb{R}^{2n}$, i.e. $\mathrm{H}(n) = \mathbb{R}^{2n} \times \mathbb{R}$ with group multiplication given by

$$(\mathbf{v}, s) \cdot (\mathbf{w}, t) = \left(\mathbf{v} + \mathbf{w}, s + t + \frac{1}{2}\omega_0(\mathbf{v}, \mathbf{w})\right) .$$

Setting

$$(\mathfrak{r}_S(\mathbf{v}, s)f)(x) = \mathrm{e}^{-\mathrm{i}(s+\langle \mathbf{v}', x\rangle - \langle \mathbf{v}', \mathbf{v}''\rangle/2)} f(x - \mathbf{v}'')$$

for $(\mathbf{v}, s) \in \mathrm{H}(n)$, $f : \mathbb{R}^n \to \mathbb{C}$ and $x \in \mathbb{R}^n$, we obtain a unitary representation $\mathfrak{r}_S$ of $\mathrm{H}(n)$ on the Hilbert space $L^2(\mathbb{R}^n)$ of square integrable functions on $\mathbb{R}^n$. This representation, which is called Schrödinger representation, is irreducible. Furthermore,

$$\mathfrak{r}_S(0, s)f = \mathrm{e}^{-\mathrm{i}s} f$$

for any $s \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$. These properties turn out to be characteristic for $\mathfrak{r}_S$.

Theorem 1.2.6 (Stone–von Neumann) *Let* $\mathfrak{r}$ *be an irreducible unitary representation of the Heisenberg group* $\mathrm{H}(n)$ *on a separable complex Hilbert space* $\mathfrak{H}$ *such that*

$$\mathfrak{r}(0,s)h = \mathrm{e}^{-\mathrm{i}s}h$$

for all $s \in \mathbb{R}$ *and* $h \in \mathfrak{H}$. *Then* $\mathfrak{r}$ *is unitary equivalent to the Schrödinger representation* $\mathfrak{r}_{\mathrm{S}}$. □

1.3 Metaplectic Representation

In view of Corollary 1.1.3, the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ has a unique connected double cover $\mathrm{Mp}(n,\mathbb{R})$. This covering group is called metaplectic group. Let $\rho : \mathrm{Mp}(n,\mathbb{R}) \to \mathrm{Sp}(n,\mathbb{R})$ denote the covering homomorphism. By Lemma 1.1.6, we may identify the Lie algebra $\mathfrak{mp}(n,\mathbb{R})$ of $\mathrm{Mp}(n,\mathbb{R})$ with the subalgebra $\mathfrak{a}(n)$ of $\mathbf{Cl}(n)$ and think the differential $\rho_* : \mathfrak{mp}(n,\mathbb{R}) \to \mathfrak{sp}(n,\mathbb{R})$ of ρ to be realized by the homomorphism given in the proof of this lemma. So

$$\rho_*(v)\mathbf{v} = [v, \mathbf{v}] \tag{1.3.1}$$

and

$$\rho_*(\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v}) = 2\,\mathbf{v}\odot\mathbf{w} \tag{1.3.2}$$

for $v \in \mathfrak{mp}(n,\mathbb{R})$ and $\mathbf{v},\mathbf{w} \in \mathbb{R}^{2n}$. The inverse of ρ_* is explicitly described by

Lemma 1.3.1 *For every* $X \in \mathfrak{sp}(n,\mathbb{R})$,

$$\rho_*^{-1}(X) = \frac{1}{2}\sum_{j=1}^{n}(\mathbf{b}_j\cdot X\mathbf{a}_j - \mathbf{a}_j\cdot X\mathbf{b}_j)\,.$$

Proof. By

$$\begin{aligned}\mathbf{b}_j\cdot X\mathbf{a}_j &= X\mathbf{a}_j\cdot\mathbf{b}_j - \omega_0(\mathbf{b}_j, X\mathbf{a}_j)\,,\\ \mathbf{a}_j\cdot X\mathbf{b}_j &= X\mathbf{b}_j\cdot\mathbf{a}_j - \omega_0(\mathbf{a}_j, X\mathbf{b}_j)\end{aligned}$$

and

$$\omega_0(\mathbf{b}_j, X\mathbf{a}_j) = \omega_0(\mathbf{a}_j, X\mathbf{b}_j)\,,$$

one has

$$\begin{aligned}&\sum_{j=1}^{n}(\mathbf{b}_j\cdot X\mathbf{a}_j - \mathbf{a}_j\cdot X\mathbf{b}_j)\\ &= \frac{1}{2}\sum_{j=1}^{n}(X\mathbf{a}_j\cdot\mathbf{b}_j + \mathbf{b}_j\cdot X\mathbf{a}_j - \mathbf{a}_j\cdot X\mathbf{b}_j - X\mathbf{b}_j\cdot\mathbf{a}_j)\,.\end{aligned}$$

Applying Lemma 1.1.4 and Equation (1.3.2), one gets the assertion. □

In the following, we outline the construction of a unitary representation of $\mathrm{Mp}(n,\mathbb{R})$ on $L^2(\mathbb{R}^n)$. Let $A \in \mathrm{Sp}(n,\mathbb{R})$. Then

$$\tau_A(\mathbf{v}, s) = (A\mathbf{v}, s)$$

defines an automorphism τ_A of the Heisenberg group $\mathrm{H}(n)$. Thus, composing the Schrödinger representation $\mathbf{r}_{\mathrm{S}}$ with τ_A, we obtain an irreducible unitary representation $\mathbf{r}_{\mathrm{S}}^A = \mathbf{r}_{\mathrm{S}} \circ \tau_A$ of $\mathrm{H}(n)$. Obviously,

$$\mathbf{r}_{\mathrm{S}}^A(0, s)f = \mathrm{e}^{-\mathrm{i}s} f$$

for all $s \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$. Therefore, by the Stone–von Neumann theorem, there exists an operator $U(A) \in \mathrm{U}(L^2(\mathbb{R}^n))$ such that

$$U(A) \circ \mathbf{r}_{\mathrm{S}}(\mathbf{v}, s) = \mathbf{r}_{\mathrm{S}}^A(\mathbf{v}, s) \circ U(A) \tag{1.3.3}$$

for all $(\mathbf{v}, s) \in \mathrm{H}(n)$. Due to Proposition 1.2.3, $U(A)$ is determined up to a scalar factor of modulus one. Since

$$A \in \mathrm{Sp}(n,\mathbb{R}) \mapsto \tau_A \in \mathrm{Aut}(\mathrm{H}(n))$$

is a group homomorphism, it follows that the operators $U(A)$ give rise to a projective unitary representation of $\mathrm{Sp}(n,\mathbb{R})$, which means that

$$U(AB) = \mathbf{c}(A, B)\, U(A) \circ U(B)$$

for any $A, B \in \mathrm{Sp}(n,\mathbb{R})$ with some function $\mathbf{c} : \mathrm{Sp}(n,\mathbb{R}) \times \mathrm{Sp}(n,\mathbb{R}) \to S^1$.

Let us compute $U(A)$ in the cases that A is one of the generators of $\mathrm{Sp}(n,\mathbb{R})$ according to Proposition 1.1.1. Let $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the Fourier transform, i.e. that unitary operator on $L^2(\mathbb{R}^n)$ which is given by

$$(\mathcal{F}f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle x, y\rangle} f(y)\, \mathrm{d}y$$

for any f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions on $\mathbb{R}^n$.

Lemma 1.3.2 *Set $U(A) = \mathcal{F}^{-1}$ for $A = J_0$. Then Equation (1.3.3) is fulfilled.*

Proof. Let $(\mathbf{v}, s) \in \mathrm{H}(n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Substituting $y = z - \mathbf{v}'$, we conclude

$$\begin{aligned}
&((\mathbf{r}_{\mathrm{S}}(\mathbf{v}, s) \circ \mathcal{F})f)(x) \\
&= (2\pi)^{-n/2} \mathrm{e}^{-\mathrm{i}(s + \langle \mathbf{v}', x\rangle - \langle \mathbf{v}', \mathbf{v}''\rangle/2)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle x - \mathbf{v}'', y\rangle} f(y)\, \mathrm{d}y \\
&= (2\pi)^{-n/2} \mathrm{e}^{-\mathrm{i}(s + \langle \mathbf{v}', x\rangle - \langle \mathbf{v}', \mathbf{v}''\rangle/2)} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle x - \mathbf{v}'', z - \mathbf{v}'\rangle} f(z - \mathbf{v}')\, \mathrm{d}z \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle x, z\rangle} \mathrm{e}^{-\mathrm{i}(s - \langle \mathbf{v}'', z\rangle + \langle \mathbf{v}', \mathbf{v}''\rangle/2)} f(z - \mathbf{v}')\, \mathrm{d}z \\
&= ((\mathcal{F} \circ \mathbf{r}_{\mathrm{S}}(J_0\mathbf{v}, s))f)(x)\,.
\end{aligned}$$