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Hyperbolic Systems of Balance Laws

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Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, July 14–21, 2003

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Preface

This volume includes the lecture notes delivered at the CIME Course "Hyperbolic Systems of Balance Laws" held July 14-21, 2003 in Cetraro (Cosenza, Italy). The present volume includes lectures notes by A. Bressan, D. Serre, K. Zumbrun and M. Williams and an appendix by A. Bressan on the center manifold theorem. These are among the "hot topics" in this field and can be of great interest, not only to professional mathematicians, but also for physicists and engineers.

The concept of hyperbolic systems of balance laws was introduced by the works of natural philosophers of the eighteenth century, predominantly L. Euler (1755), and has over the past one hundred and fifty years become the natural framework for the study of gas dynamics and, more broadly, of continuum physics. During this period of time great personalities like Stokes, Challis, Riemann, Rankine, Hugoniot, Lord Rayleigh and later Prandtl, Hadamard, H. Lewy, G.I. Taylor and many others wrote several fundamental papers, thus laying the groundwork for the further development of the mathematical theory. However the first part of the past century did not see much activity on the part of mathematicians in this field and it was only during the Second World War, in connection with the Manhattan Project, that associated research received a great impetus.

Many important scientists like J. Von Neumann, R. Courant, K.O. Friedrichs, H. Bethe and Ya. Zeldowich became interested in this field and proposed many new key concepts, the influence of which remains very great to the present day.

Immediately after the Second World War there was a considerable development in mathematical theory, with key results being obtained by a new generation of great mathematicians like S.K. Godunov, P. Lax, F. John, C. Morawetz and O. Oleinik, who led the field until the mid 1960s, when J. Glimm published an outstanding paper which marked the most important breakthrough in the history of this field. Glimm was able to prove the global existence of general systems in one space dimension, with small BV data. This result introduced a new approach to nonlinear wave interaction, but the proof was not fully deterministic. Tai-Ping Liu was later able to remove the probabilistic part of the proof, thus making it completely deterministic.

The relation between hyperbolic balance laws and continuum physics is not covered in any of the lectures in the present volume, but was the core topic of a series of lectures delivered in the Cetraro School by C. Dafermos entitled "Conservation Laws on Continuum Mechanics." In his wonderful monograph, published by Springer-Verlag in the Grundlehren der Mathematischen Wissenschaften, vol. 325, Dafermos provides an extremely thorough account of the most relevant aspects of the theory of hyperbolic conservation laws and systematically develops their ties to classical mechanics.

The notes by Alberto Bressan in this volume are intended to provide a self-contained presentation of recent results on hyperbolic conservation laws, based on the vanishing viscosity approach. Glimm's aforementioned theory was based on the construction of partially smooth approximating solutions with a locally self-similar structure. In order to get a uniform bound in BV norm, interaction potential was a crucial tool, an idea Glimm borrowed from physics. This potential, though a nonlinear functional, displays quadratic behaviour and decreases with time, provided the initial data have a small total variation.

In the 1990s Bressan and Tai-Ping Liu, together with various collaborators, completed this theory by proving the continuous dependence on initial data. The relations with the theory of compressible fluids have raised, since the very beginning of the theory, the question whether the inviscid solutions are in practice the same as the solutions with low viscosity. Although this fact had been established for various specific situations, it was only very recently that Bianchini and Bressan discovered a way to prove that, if the total variation of initial data remains sufficiently small, then the solutions of a viscous system of conservation laws converge to the solutions of the inviscid system, as long as the viscosity tends to zero. This approach allows the stability results obtained using the previous theories to be generalized.

The results are based on various technical steps, which in the present lecture notes Bressan describes in great detail, making a remarkable effort to make this difficult subject also accessible to non-specialists and young doctoral students. The notes of D. Serre cover the existence and stability of discrete shock profiles, another very exciting topic which, since the 1940s, has greatly interested applied mathematicians, including Von Neumann, Godounov and Lax, who were motivated by the need for efficient numerical codes to approximate the solutions of compressible fluid systems, including situations where shocks are present. It was immediately clear to them that a number of challenging and difficult mathematical problems needed to be solved. Partial differential equations are often approximated by finite difference schemes. The consistency and stability of a given scheme are usually studied through a linearization along elementary solutions such as constants, travelling waves, and shocks.

The study of the existence and stability of travelling waves faces significant difficulties; for example, existence may fail in rather natural situations because of small divisors problems. Ties to many branches of mathematics, ranging from dynamical systems to arithmetic number theory, prove to be relevant in this field. Serre's notes greatly emphasize this interdisciplinary aspect. His lecture notes not only provide a very useful and comprehensive introduction to this specific topic, but moreover propose a class of truly interesting and challenging problems in modern spectral theory. The analysis of the vanishing viscosity limit is far from being fully understood in the multidimensional setting. It is, in any case, important to understand the presence of stable and unstable modes along boundaries and shock profiles, where the most relevant linear and nonlinear phenomena take place. As such, the stability of viscous shock waves was the main focus of the lectures delivered by M. Williams and K. Zumbrun. This topic started, for the inviscid case, with the pioneering papers of Kreiss, Osher, Rauch and Majda. Later Metivier brought into the field a number of far-reaching ideas from microlocal analysis, in particular the paradifferential calculus introduced by Bony. The stability condition is expressed in terms of the so-called Kreiss-Lopatinskiĭ determinant. The viscous case can benefit from many of these ideas, but new tools are also needed.

Linearizing the system about a given profile (made stationary by Galilean invariance), and taking the Laplace transform in time and the Fourier transform in the hyperplane orthogonal to the direction of propagation, allows the formulation of an eigenvalue equation for a differential operator with variable coefficients. A necessary condition for the viscous profile to be stable is that these eigenvalue equations do not have (nontrivial) solutions. The Evans function technique provides a means to quantify this criterion.

But some rather subtle issues, in particular regarding regular dependence on parameters, call for a cautious approach. This was understood in a celebrated paper by J. Alexander, Gardner and C. Jones. Necessary stability conditions are expressed in terms of:
(1) the transversality of the connection in the travelling wave ODE, and
(2) the Kreiss-Lopatinskiĭ condition, which is known to ensure weak inviscid stability. The argument relies on the low frequency behaviour of the Evans function. Unlike the Kreiss-Lopatinskiĭ determinant Δ, encoding the linearized stability of the inviscid shock, the Evans function is not explicitly computable. But Zumbrun and Serre's result shows that the Evans function is tangent to Δ in the low frequency limit. Kevin Zumbrun's lectures focused on the planar stability for viscous shock waves in systems with real viscosity. His course provided an extensive overview of the technical tools and central concepts involved. He took great care to make such a difficult matter comparatively simple and approachable to an audience of young mathematicians.

M. Williams' course focused on the short time existence of curved multidimensional viscous shocks and the related small viscosity limit. It provided an accessible account of the main ideas and methods, trying to avoid most of the technical difficulties connected with the use of paradifferential calculus. His final lecture introduced the analysis of long time stability for planar viscous shocks. A fairly complete list of references to books and research articles is included at the bottom of all of the lecture notes.

The course was attended by several young mathematicians from various European countries, who worked very hard during the whole period of the Summer School. However they were also able to enjoy the beautiful hosting facility provided by CIME, in the paradise-like sea resort of Cetraro, under the Calabrian sun.

This course was organized with the collaboration and financial support of the European Network on Hyperbolic and Kinetic Equations (HyKE).

I would like to express my gratitude to the CIME Foundation, to the CIME Director Prof. Pietro Zecca and to the CIME Board Secretary Prof. Elvira Mascolo, for their invaluable help and support, and for the tremendous efforts they have invested to return the CIME Courses to their traditional greatness.

Pierangelo Marcati

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BV Solutions to Hyperbolic Systems by Vanishing Viscosity

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1 Introduction

The aim of these notes is to provide a self-contained presentation of recent results on hyperbolic systems of conservation laws, based on the vanishing viscosity approach.

A system of conservation laws in one space dimension takes the form

$$u_t + f(u)_x = 0\,. \tag{1.1}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of **conserved quantities** while the components of $f = (f_1, \ldots, f_n)$ are called the **fluxes**. Integrating (1.1) over a fixed interval $[a, b]$ we find

$$\begin{aligned}\frac{d}{dt}\int_a^b u(t,x)\,dx \;&=\; \int_a^b u_t(t,x)\,dx \;=\; -\int_a^b f\big(u(t,x)\big)_x\,dx\\ = f\big(u(t,a)\big) - f\big(u(t,b)\big) \;&=\; [\text{inflow at } a] - [\text{outflow at } b]\,.\end{aligned}$$

Each component of the vector u thus represents a quantity which is neither created nor destroyed: its total amount inside any given interval $[a, b]$ can change only because of the flow across boundary points.

Systems of the form (1.1) are commonly used to express the fundamental balance laws of continuum physics, when small viscosity or dissipation effects are neglected. For a comprehensive discussion of conservation laws and their derivation from basic principles of physics we refer to the book of Dafermos [D1].

Smooth solutions of (1.1) satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0\,, \tag{1.2}$$

where $A(u) \doteq Df(u)$ is the Jacobian matrix of first order partial derivatives of f. We notice, however, that if u has a jump at some point x_0, then the left hand side of (1.2) contains a product of the discontinuous function $x \mapsto A(u(x))$

with the distributional derivative u_x, which in this case contains a Dirac mass at the point x_0. In general, such a product is not well defined. The quasilinear system (1.2) is thus meaningful only within a class of continuous functions. On the other hand, working with the equation in divergence form (1.1) allows us to consider discontinuous solutions as well, interpreted in distributional sense. We say that a locally integrable function $u = u(t,x)$ is a **weak solution** of (1.1) if $t \mapsto u(t,\cdot)$ is continuous as a map with values in $\mathbf{L}^1_{loc}$ and moreover

$$\iint \{u\phi_t + f(u)\phi_x\}\, dx\, dt \;=\; 0 \tag{1.3}$$

for every differentiable function with compact support $\phi \in \mathcal{C}^1_c$.

The above system is called **strictly hyperbolic** if each matrix $A(u) \doteq Df(u)$ has n real, distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. One can then find dual bases of left and right eigenvectors of $A(u)$, denoted by $l_1(u), \ldots, l_n(u)$ and $r_1(u), \ldots, r_n(u)$, normalized according to

$$|r_i| = 1\,, \qquad l_i \cdot r_j = \begin{cases} 1 & if \quad i = j, \\ 0 & if \quad i \neq j. \end{cases} \tag{1.4}$$

To appreciate the effect of the non-linearity, consider first the case of a linear system with constant coefficients

$$u_t + Au_x = 0. \tag{1.5}$$

Call $\lambda_1 < \cdots < \lambda_n$ the eigenvalues of the matrix A, and let l_i, r_i be the corresponding left and right eigenvectors as in (1.4). The general solution of (1.5) can be written as a superposition of independent linear waves:

$$u(t,x) = \sum_i \phi_i(x - \lambda_i t) r_i\,, \qquad \phi_i(y) \doteq l_i \cdot u(0,y)\,.$$

Notice that here the solution is completely decoupled along the eigenspaces of A, and each component travels with constant speed, given by the corresponding eigenvalue of A.

In the nonlinear case (1.2) where the matrix A depends on the state u, new features will appear in the solutions.

(i) Since the eigenvalues λ_i now depend on u, the shape of the various components in the solution will vary in time (fig. 1). Rarefaction waves will decay, and compression waves will become steeper, possibly leading to shock formation in finite time.

(ii) Since the eigenvectors r_i also depend on u, nontrivial interactions between different waves will occur (fig. 2). The strength of the interacting waves may change, and new waves of different families can be created, as a result of the interaction.

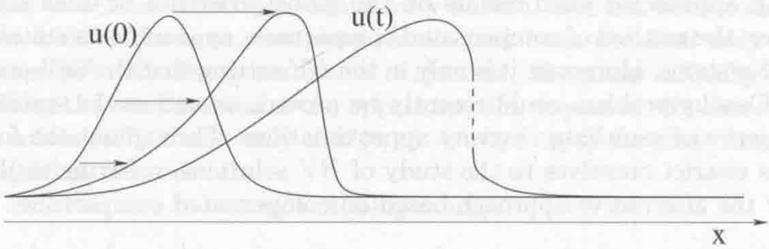

Fig. 1.

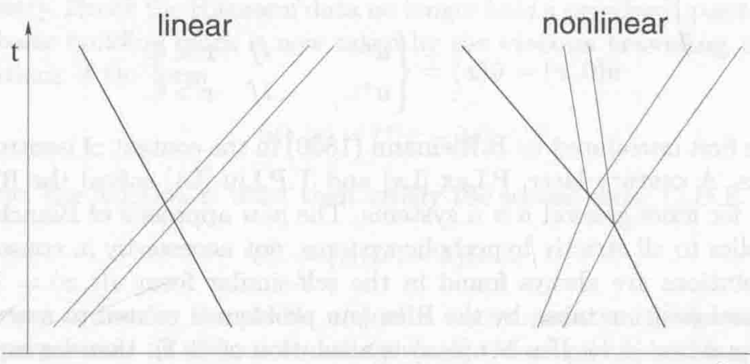

Fig. 2.

The strong nonlinearity of the equations and the lack of regularity of solutions, also due to the absence of second order terms that could provide a smoothing effect, account for most of the difficulties encountered in a rigorous mathematical analysis of the system (1.1). It is well known that the main techniques of abstract functional analysis do not apply in this context. Solutions cannot be represented as fixed points of continuous transformations, or in variational form, as critical points of suitable functionals. Dealing with vector valued functions, comparison principles based on upper or lower solutions cannot be used. Moreover, the theory of accretive operators and contractive nonlinear semigroups works well in the scalar case [C], but does not apply to systems. For the above reasons, the theory of hyperbolic conservation laws has largely developed by *ad hoc* methods, along two main lines.

1. The BV setting, considered by Glimm [G]. Solutions are here constructed within a space of functions with bounded variation, controlling the BV norm by a wave interaction functional.

2. The $\mathbf{L}^\infty$ setting, considered by Tartar and DiPerna [DP2], based on weak convergence and a compensated compactness argument.

Both approaches yield results on the global existence of weak solutions. However, the method of compensated compactness appears to be suitable only for 2×2 systems. Moreover, it is only in the BV setting that the well-posedness of the Cauchy problem could recently be proved, as well as the stability and convergence of vanishing viscosity approximations. Throughout the following we thus restrict ourselves to the study of BV solutions, referring to [DP2] or [Se] for the alternative approach based on compensated compactness.

Since the pioneering work of Glimm, the basic building block toward the construction and the analysis of more general solutions has been provided by the **Riemann problem**, i.e. the initial value problem with piecewise constant data

$$u(0,x) = \bar{u}(x) = \begin{cases} u^- & if \quad x < 0, \\ u^+ & if \quad x > 0. \end{cases} \tag{1.6}$$

This was first introduced by B.Riemann (1860) in the context of isentropic gas dynamics. A century later, P.Lax [Lx] and T.P.Liu [L1] solved the Riemann problem for more general $n \times n$ systems. The new approach of Bianchini [Bi] now applies to all strictly hyperbolic systems, not necessarily in conservation form. Solutions are always found in the self-similar form $u(t,x) = U(x/t)$. The central position taken by the Riemann problem is related to a symmetry of the equations (1.1). If $u = u(t,x)$ is a solution of (1.1), then for any $\theta > 0$ the function

$$u^\theta(t,x) \doteq u(\theta t, \theta x)$$

provides another solution. The solutions which are invariant under these rescalings of the independent variables are precisely those which correspond to some Riemann data (1.6).

For a general Cauchy problem, both the Glimm scheme [G] and the method of front tracking [D1], [DP1], [B1], [BaJ], [HR] yield approximate solutions of a general Cauchy problem by piecing together a large number of Riemann solutions. For initial data with small total variation, this approach is successful because one can provide a uniform a priori bound on the amount of new waves produced by nonlinear interactions, and hence on the total variation of the solution. It is safe to say that, in the context of weak solutions with small total variation, nearly all results on the existence, uniqueness, continuous dependence and qualitative behavior have relied on a careful analysis of the Riemann problem.

In [BiB] a substantially different perspective has emerged from the study of vanishing viscosity approximations. Solutions of (1.1) are here obtained as limits for $\varepsilon \to 0$ of solutions to the parabolic problems

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon \tag{1.7}$$

with $A(u) \doteq Df(u)$. This approach is very natural and has been considered since the 1950's. However, complete results had been obtained only in the scalar case [O], [K]. For general $n \times n$ systems, the main difficulty lies in establishing the compactness of the approximating sequence. We observe that $u^\varepsilon(t,x)$ solves (1.7) if and only if $u^\varepsilon(t,x) = u(t/\varepsilon, x/\varepsilon)$ for some function u which satisfies

$$u_t + A(u)u_x = u_{xx}\,. \tag{1.8}$$

In the analysis of vanishing viscosity approximations, the key step is to derive a priori estimates on the total variation and on the stability of solutions of (1.8). For this parabolic system, the rescaling $(t,x) \mapsto (\theta t, \theta x)$ no longer determines a symmetry. Hence the Riemann data no longer hold a privileged position. The role of basic building block is now taken by the **viscous traveling profiles**, i.e. solutions of the form

$$u(t,x) = U(x - \lambda t).$$

Of course, the function U must then satisfy the second order O.D.E.

$$U'' = \big(A(U) - \lambda\big)U'.$$

In this new approach, the profile $u(\cdot)$ of a viscous solution is viewed locally as a superposition of viscous traveling waves. More precisely, let a smooth function $u : \mathbb{R} \mapsto \mathbb{R}^n$ be given. At each point x, looking at the second order jet (u, u_x, u_{xx}) we seek traveling profiles $U_1, \ldots, U_n$ such that

$$U_i'' = \big(A(U_i) - \sigma_i\big)U_i' \tag{1.9}$$

for some speed σ_i close to the characteristic speed λ_i, and moreover

$$U_i(x) = u(x) \qquad i = 1, \ldots, n\,, \tag{1.10}$$

$$\sum_i U_i'(x) = u_x(x)\,, \qquad \sum_i U_i''(x) = u_{xx}(x)\,. \tag{1.11}$$

It turns out that this decomposition is unique provided that the traveling profiles are chosen within suitable center manifolds. We let $\tilde{r}_i$ be the unit vector parallel to U_i', so that $U_i' = v_i \tilde{r}_i$ for some scalar v_i. One can show that $\tilde{r}_i$ remains close to the eigenvector $r_i(u)$ of the Jacobian matrix $A(u) \doteq Df(u)$, but $\tilde{r}_i \neq r_i(u)$ in general. The first equation in (1.11) now yields the decomposition

$$u_x = \sum_i v_i \tilde{r}_i\,. \tag{1.12}$$

If $u = u(t,x)$ is a solution of (1.8), we can think of v_i as the density of i-waves in u. The remarkable fact is that these components satisfy a system of evolution equations

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = \phi_i \qquad i = 1, \ldots, n\,, \tag{1.13}$$

where the source terms ϕ_i on the right hand side are INTEGRABLE over the whole domain $\{x \in \mathbb{R},\ t > 0\}$. Indeed, we can think of the sources ϕ_i as new waves produced by interactions between viscous waves. Their total strength is controlled by means of viscous interaction functionals, somewhat similar to the one introduced by Glimm in [G] to study the hyperbolic case. Since the left hand side of (1.13) is in conservation form and the vectors $\tilde{r}_i$ have unit length, for an arbitrarily large time t we obtain the bound

$$\big\|u_x(t)\big\|_{\mathbf{L}^1} \leq \sum_i \big\|v_i(t)\big\|_{\mathbf{L}^1} \leq \sum_i \left(\big\|v_i(t_0)\big\| + \int_{t_0}^{t}\int \big|\phi_i(s,x)\big|\,dxds \right). \quad (1.14)$$

This argument yields global BV bounds and stability estimates for viscous solutions. In turn, letting $\varepsilon \to 0$ in (1.7), a standard compactness argument yields the convergence of u^ε to a weak solution u of (1.1).

The plan of these notes is as follows. In Section 2 we briefly review the basic theory of hyperbolic systems of conservation laws: shock and rarefaction waves, entropies, the Liu admissibility conditions and the Riemann problem. For initial data with small total variation, we also recall the main results concerning the existence, uniqueness and stability of solutions to the general Cauchy problem. Section 3 contains the statement of the main new results on vanishing viscosity limits and an outline of the proof. In Section 4 we derive some preliminary estimates which can be obtained by standard parabolic techniques, representing a viscous solution in terms of convolutions with a heat kernel. Section 5 discusses in detail the local decomposition of a solution as superposition of viscous traveling waves. The evolution equations (1.13) for the components v_i and the strength of the source terms ϕ_i are then studied in Section 6. This will provide the crucial estimate on the total variation of viscous solutions, uniformly in time. In Section 7 we briefly examine the stability and some other properties of viscous solutions. The existence, uniqueness and stability of vanishing viscosity limits are then discussed in Section 8.

2 Review of Hyperbolic Conservation Laws

In most of this section, we shall consider a strictly hyperbolic system of conservation laws satisfying the additional hypothesis

(H) For each $i = 1, \ldots, n$, the i-th field is either **genuinely nonlinear**, so that $D\lambda_i(u)\cdot r_i(u) > 0$ for all u, or **linearly degenerate**, with $D\lambda_i(u)\cdot r_i(u) = 0$ for all u.

We observe that the i-th characteristic field is genuinely nonlinear iff the eigenvalue λ_i is strictly increasing along each integral curve of the corresponding field of eigenvectors r_i. It is linearly degenerate when λ_i is constant along each such curve.