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Contents.

Introduction

The main object of this book is the study of homotopy equivalences between 3-manifolds. Here a 3-manifold M is always compact, orientable and irreducible. Moreover, we suppose that M is boundary-irreducible, that is, for any component G of ∂M, $\pi_1 G \to \pi_1 M$ is injective. However, we do not always insist that the boundary is non-empty. Examples of such 3-manifolds are the knot spaces (of non-trivial knots).

If M_1 and M_2 are such 3-manifolds with non-empty boundaries, and $f\colon M_1 \to M_2$ a homotopy equivalence, one knows by [Wa 4] that f can be deformed into a homeomorphism, provided it preserves the peripheral structure, that is, $f|\partial M_1$ can be deformed into ∂M_2.

We are interested here in homotopy equivalences whose restrictions to the boundary cannot be deformed into the boundary. Such homotopy equivalences will be called exotic. Our main result is a classification theorem for exotic homotopy equivalences.

Before describing this theorem let us briefly recall the situation in the 2-dimensional case. Many exotic homotopy equivalences can be found between surfaces F_1, F_2 with boundaries. For example, there is of course at least one such homotopy equivalence between the torus with one hole and the 2-sphere with three holes. Since a surface with boundary is a $K(\pi,1)$-space whose fundamental group is free, the exotic homotopy equivalences of surfaces can be analyzed by using the presentations of the outer automorphism groups of free groups [Ni 1]. In particular, it follows that they are (finitely) generated by Dehn flips (along arcs). Here a Dehn flip means an exotic homotopy equivalence $f\colon F_1 \to F_2$ for which there is an arc k in F_2, $k \cap \partial F_2 = \partial k$, such that $f^{-1}k$ is again an arc and that f is the identity outside of regular neighborhoods of these arcs. If, on the other hand, a homotopy equivalence $f\colon F_1 \to F_2$ is boundary preserving, i.e. not exotic, it can be deformed into a homeomorphism (Nielsen's theorem). Furthermore, one knows that there is a finite set of Dehn twists which generate a normal subgroup of index two in the whole mapping class group of a surface (orientable or not) (see [De 1] [Li 1, 2, 4].) Here a Dehn twist is a homeomorphism which is the identity outside of a regular neighborhood of a closed curve. Hence, altogether, this shows that the homotopy equivalences of surfaces are built up of locally defined maps (neglecting orientation-phenomena). However, one does not know in general the relations between homotopy equivalences (but see the recent work of Hatcher and Thurston on the mapping class group).

In order to switch to dimension three, take the product of a surface with the 1-sphere. In this way we get our first examples of exotic homotopy equivalences of 3-manifolds. Indeed these homotopy equivalences are generated by Dehn flips along annuli. Similarly, S^1-bundles over a surface with boundary contain a lot of exotic homotopy equivalences. Still more can be found in Seifert fibre spaces with boundaries. Here the exceptional fibres give rise to additional phenomena. Although every homotopy equivalence of a sufficiently large Seifert fibre space can be deformed into a fibre preserving map which maps exceptional fibres to exceptional fibres (see 28.4), the restriction to the complement of the exceptional fibres is in general not a homotopy equivalence. Hence exotic homotopy equivalences of Seifert fibre spaces might be rather complicated, and they are, in fact, not yet completely understood.

The examples considered so far are found in very special 3-manifolds and one might expect in other 3-manifolds still more freedom for constructing exotic homotopy equivalences. In contrast to that, it is very difficult to find them. Hence one turns hindsight into foresight and conjectures that the above examples are the only ones (up to modifications). This idea turns out to be correct and therefrom it emerges what we call the characteristic submanifold. Since the concept of the characteristic submanifold plays a crucial role throughout the whole book we present here an explicit definition of it--at least in the absolute case:

Definition. Let M be a 3-manifold (with or without boundary). A codim zero submanifold V of M is called a characteristic submanifold if the following holds:

1. Each component X of V admits a structure as Seifert fibre space, with fibre projection, $p: X \to B$, such that

$$X \cap \partial M = p^{-1}p(X \cap \partial M),$$

or as I-bundle, with fibre projection, $p: X \to B$, such that

$$X \cap \partial M = (\partial X - p^{-1}\partial B)^{-}.$$

2. If W is a non-empty codim zero submanifold of M which consists of components of $(M - V)^{-}$, then $V \cup W$ is not a submanifold satisfying 1.
3. If W' is a submanifold of M satisfying 1 and 2 , then W' can be deformed into V, by using a proper isotopy.

A codim zero submanifold W of M is called an essential F-manifold (F = fibered) if 1 holds and if every component of $(\partial W - \partial M)^{-}$ is incompressible (a surface G in M, $G \cap \partial M = \partial G$, is called incompressible if it is not a 2-sphere and $\pi_1 G \to \pi_1 M$ is injective).

Some work is required to show that the characteristic submanifold for sufficiently large 3-manifolds (in the sense of [Wa 4], e.g. if $\partial M \neq \emptyset$) indeed exists (i.e. that it is well-defined--of course we do not assert that it is always non-empty) and that it is unique, up to ambient isotopy.

Example. The characteristic submanifold of a Seifert fibre space M is equal to M.

It turns out that the characteristic submanifold is a very useful geometric structure in M. In particular, our classification of exotic homotopy equivalences can be given within this concept. Indeed, we shall prove the following (see 24.2).

Classification theorem. Let M_1, M_2 be 3-manifold (irreducible etc.) with non-empty boundaries, and V_1, V_2 resp. their characteristic submanifolds. Let $f: M_1 \to M_2$ be any homotopy equivalence. Then f can be deformed so that afterwards

1. $f(V_1) \subset V_2$ and $f(\overline{M_1 - V_1}) \subset \overline{M_2 - V_2}$,

2. $f|V_1\colon V_1 \to V_2$ is a homotopy equivalence,

3. $f|\overline{M_1 - V_1}\colon \overline{M_1 - V_1} \to \overline{M_2 - V_2}$ is a homeomorphism.

The proof of this theorem takes up a large part of these notes. An outline of the proof will be given below. But first we would like to mention some of its consequences, and also we will describe some results, obtained in the course of its proof, which are of interest in their own right.

To begin, recall from [Wa 4] that a homotopy equivalence between 3-manifolds which is a homeomorphism on the boundaries can be deformed into a homeomorphism by using a homotopy which is constant on the boundary. That means--in our context--that in the classification theorem above, V_i, $i = 1,2$, can be replaced by the submanifold W_i which consists of all those components of V_i which do meet ∂M_i. This in turn implies

Corollary. If there are no essential annuli, there are no exotic homotopy equivalences.

Here an essential annulus A in M, $A \cap \partial M = \partial A$, means an annulus which is incompressible and not boundary-parallel.

To consider a concrete example, let M be the knot space of a non-trivial knot. If the submanifold W is not trivial (i.e. not a regular neighborhood of ∂M) one knows a priori that the knot is either non-prime, or a cable knot, or a torus knot (see 14.8). In any other case, the classification theorem says that the homotopy type of M contains just one 3-manifold, up to homeomorphism.

Recall that, in general, 3-manifolds can be homotopic without being homeomorphic. However, with the help of the classification theorem, we shall see that the homotopy type of a 3-manifold (irreducible, sufficiently large, etc.) contains only finitely many 3-manifolds (§29).

The last remark leads us to the isomorphism problem for the fundamental groups of sufficiently large 3-manifolds. This problem asks for an algorithm for deciding whether or not two such fundamental groups are isomorphic. Since a sufficiently large 3-manifold is a $K(\pi,1)$-space (recall the restrictions in the beginning) one knows that every isomorphism between their fundamental groups is induced by a homotopy equivalence. Using this fact, together with the classification theorem, the above isomorphism problem can be reduced to the homeomorphism problem for sufficiently large 3-manifolds (§29). But the latter problem was completely solved recently [Ha 2] [He 1] [Th 1]. Hence, in particular, the isomorphism problem for knot groups is solved.

Having established the classification theorem we can push the study of homotopy equivalences a bit further still. Two directions of this study are conceivable. One is to describe one given exotic homotopy equivalence more fully, the other is to study the (exotic) homotopy equivalences all at once.

To describe a result towards the first direction, let a homotopy equivalence $f: M_1 \to M_2$ be given. Furthermore let us assume that the 3-manifolds M_1, M_2 contain no Klein bottles and no essential annuli which separate a solid torus (e.g. no exceptional fibre). Then we find an essential F-manifold O_f of M_2 which is unique, up to ambient isotopy, and which has the following properties:

1. f can be deformed such that afterwards

   $f|f^{-1}O_f: f^{-1}O_f \to O_f$ is a homotopy equivalence, and

   $f|(M_1 - f^{-1}O_f)^-: (M_1 - f^{-1}O_f)^- \to (M_2 - O_f)^-$ is a homeomorphism.
2. O_f can be properly isotoped into all essential F-manifolds which satisfy 1.

O_f will be called an <u>obstruction</u> <u>submanifold</u> <u>for</u> f, because f is exotic if and only if $O_f \neq \emptyset$. Some work is required to establish the obstruction submanifold for homotopy equivalences of surfaces (see 30.15). After this the forementioned result can be deduced with the help of the classification theorem (see §28).

As a first attack in the second direction we investigate the <u>mapping</u> <u>class</u> <u>group</u> $H(M)$ of sufficiently large 3-manifolds. Our approach to this is the following. First observe that, by the uniqueness of the characteristic submanifold, V, the computation of H(M) can be split into that of $H_{\overline{M-V}}(V)$ (= isotopy classes of homeomorphisms h: V → V which extend to M) and that of $H_V(\overline{M - V})$ (= isotopy classes of homeomorphisms h: $\overline{M - V} \to \overline{M - V}$ which extend to M). For $H_{\overline{M-V}}(V)$ one can give a fairly explicit computation (see §25), using the recent presentation of the mapping class group of surfaces [HT 1]. Furthermore, we shall prove (§27) that the mapping class group of a simple 3-manifold is finite (and so $H_V(\overline{M-V})$). Here a simple 3-manifold means a sufficiently large 3-manifold whose characteristic submanifold is trivial. To prove this, we use the theory of characteristic submanifolds and Haken's finiteness theorem for surfaces [Ha 1] in order to reduce the problem to the conjugacy problem for the mapping class group of surfaces. The latter was recently solved [He 1][Th 1]. Altogether, we obtain in any sufficiently large 3-manifold a finite set of Dehn twists (along annuli and tori) which generate a normal subgroup in the mapping class group of finite index (see §27, and cf. the 2-dimensional case mentioned in the beginning). As a consequence of this property of the mapping class group we obtain infinitely many examples of surface-homeomorphisms which cannot be extended to any 3-manifold (see 27.10).

Observe that the definition of the characteristic submanifold does not depend on the presence of any homotopy equivalence. In fact, the characteristic submanifold has still other very nice properties besides the above relationship with exotic homotopy equivalences. The most important one is that one can prove a certain enclosing theorem. To describe this we first have to give a definition of an essential singular annulus and torus.

Definition. Let T be an annulus or torus. Then a map f: (T,∂T) → (M,∂M) will be called an essential singular annulus or torus if f induces an injection of the fundamental groups and if it cannot be deformed into ∂M.

Note that by the above definition of the characteristic submanifold, the characteristic submanifold contains all essential (non-singular) annuli or tori of a sufficiently large 3-manifold,up to proper isotopy. In addition to this we shall prove (see §12):

Enclosing theorem. If M is a sufficiently large 3-manifold (with or without boundary), then every essential singular annulus or torus in M can be deformed into the characteristic submanifold of M.

Now recall that the characteristic submanifold consists of I-bundles and Seifert fibre spaces only. Of course, one finds many essential non-singular annuli and tori in such 3-manifolds. Hence, as an immediate consequence of the enclosing theorem, we obtain the "annulus" and "torus theorems" [Wa 6]: the existence of an essential singular annulus implies the existence of an essential non-singular one. The same is true for the torus, except in the very special case of a Seifert fibre space over the 2-sphere with holes, where the sum of boundary components and exceptional fibres is at most three. Furthermore, essential singular annuli and tori can be fairly explicitely classified in I-bundles and Seifert fibre spaces, and so it follows from the enclosing theorem that any such map can be deformed into the composition of a covering map and an immersion without triple points.

Working in a suitable relative framework and using the notion of an essential map, we also obtain a version of the enclosing theorem for essential maps of I-bundles and Seifert fibre spaces (see §13). As an immediate corollary we get: if M has a finite covering which is a Seifert fibre space, then M must be a Seifert-fibre space itself (see 12.11).

In an appendix, we finally apply the enclosing theorem to questions about the fundamental groups of sufficiently large 3-manifolds. There we give a geometric characterization of sufficiently large 3-manifolds whose fundamental group is an R-group (an R-group is a group, where $x^n = y^n$ implies $x = y$), and we apply this to give another proof of Shalen's result [Sh 1] that no element of a 3-manifold group is infinitely divisible. An easy consequence of this and the enclosing theorem is that the centralizer of any element of a sufficiently large 3-manifold group is always carried by an embedded

Seifert fibre space (or a 2-sheeted covering of such a submanifold).

We now give a more detailed description of parts I - IV of this paper in the form of a Leitfaden.

Part I: The concepts of characteristic submanifolds and manifolds with boundary-patterns.

Part 1 consists of three chapters.

Chapter I. As a first motivation (others will become obvious later) for this chapter the reader should keep in mind the following observation. On the one hand, the "ball" and the "cube" are topologically the same, but on the other hand, the notion of a "cube" does involve much more information. For example, we may distinguish corners, edges, and faces of the cube. That is, we have more information about the boundary. Since we would like to obtain results about manifolds whose boundaries are non-empty, it would be wise to preserve as much information about the boundary as possible. From the effort to present this information in a more organized way the concept of "manifolds with boundary-patterns" emerges.

A boundary-pattern of an n-manifold M, $n \geq 1$, is a collection of (n-1)-manifolds in the boundary of M which meet nicely (see Def. 1.1). An admissible map is a map which preserves this structure (see Def. 1.2)--the same with admissible deformations and admissible homotopy equivalences.

It appears that the ensueing formalism is of some interest in its own right, e.g. there is a relative version of the loop-theorem for manifolds with boundary-patterns (see 2.1), and this is equivalent (or at least implies) the main technical result of[Wa 5]. Furthermore, we introduce the notions of "useful boundary-patterns" and "essential maps" (see Def. 2.2 and Def. 3.1). These notions are relativized versions of "boundary irreducible" and "maps which induce an injection of the fundamental groups". A first advantage of these new notions is immediate from their definitions: While, in general, 3-manifolds do not stay boundary-irreducible after splitting at incompressible surfaces, this is true for 3-manifolds with useful boundary-patterns after splitting at essential surfaces.

After having reproduced the proof of Waldhausen's theorem [Wa 4] for 3-manifolds with non-empty boundaries (respectively with boundary-patterns) (3.4) we finally conclude chapter I by establishing some general position theorems (see e.g. 4.4 and 4.5).

Note. Throughout the whole book we have to work entirely within the framework of manifolds with boundary-patterns. However, in this introduction we mostly ignore the boundary-patterns, for convenience. Here the reader should keep in mind that "deformation", "homotopy equivalence" etc. always mean "admissible deformation", "admissible homotopy equivalence" etc.

Chapter II. In this chapter we study singular essential annuli and tori in I-bundles, Seifert fibre spaces (§5), generalized Stallings fibrations (§6) and generalized Seifert fibre spaces (§7). In particular, the proof of the annulus- and torus-theorem for Stallings fibrations is contained here (§6). The 3-manifolds considered in this chapter are fairly special. But on the other hand, information about them is very important for us, because many questions on 3-manifolds can be reduced, via characteristic submanifolds, to questions on these special 3-manifolds.

Chapter III. Here the concept of characteristic submanifolds will be developed. Several definitions of characteristic submanifolds will be given--the most convenient one was already mentioned in the beginning (at least in the absolute case). Using the finiteness theorem for surfaces [Kn 1], [Ha 3], we prove the existence of a characteristic submanifold for sufficiently large 3-manifolds with useful boundary-patterns (§9). Later on (§10) we prove some useful facts about these submanifolds (including the equivalence of all their definitions (10.1)), and we end up with the proof of the uniqueness of characteristic submanifolds, up to ambient isotopy.

Part II: The enclosing theorems.

Part II consists of two chapters.

Chapter IV. In this chapter the first enclosing theorem will be proved, asserting that every essential singular torus, annulus, or square in a sufficiently large 3-manifold M with useful boundary-pattern can be admissibly deformed into the characteristic submanifold of M.

Here we give a short indication of the proof. It uses induction on a hierarchy. Recall from [Wa 4] that a hierarchy is a sequence $M = M_0, M_1, \dots, M_n$ of 3-manifolds, where M_i is given by splitting M_{i-1} along an incompressible surface. The boundary-pattern of each M_i is the "trace" of the previous splittings.

Since the first step of this induction is trivial, we turn at once to the inductive step from M_{i+1} to M_i. For this denote by S the surface which splits M_i to M_{i+1}, and identify a regular neighborhood of S with $S \times I$.

Let f_i be any essential singular torus, annulus, or square in M_i. Then with the help of chapter I, we may assume that f_i is deformed so that firstly $f_{i+1} = f_i | f_i^{-1} M_{i+1}$ consists also of such singular surfaces in M_{i+1}, and so, by our induction assumption, that secondly f_{i+1} is contained in the characteristic submanifold V_{i+1} of M_{i+1}. Observe that, by the very definition, every essential F-manifold in M_i can be isotoped into the characteristic submanifold V_i of M_i. Hence it suffices to prove that f_i can be deformed into an essential F-manifold. This is comparatively easy if S is either a torus, annulus, or square. So let us assume that S cannot be chosen to be one of these surfaces. Then, in general, the surfaces $V_{i+1} \cap (S \times 0)$ and $V_{i+1} \cap (S \times 1)$ do not correspond via $S \times I$.

In this situation we use a combing process. Similar sorts of this process will be used also in chapter VIII and in the proof of the finiteness of the mapping class group. Generally speaking, a combing process is an organized way (by means of the characteristic submanifold) to extend results which are true for a 2-dimensional submanifold to the whole 3-manifold. In the case at hand, the corresponding 2-manifold result is the following

Lemma (see §11). Let F be a surface which is not a torus, annulus or square. Let F_0, F_1 be two essential surfaces in F which are in a "very good position" (this position can always be obtained by an isotopy of F_0, see §11). Then every essential singular curve (closed or not) which can be deformed into both F_0 and F_1 can be deformed into $F_0 \cap F_1$.

By inductive application of this lemma, we find in V_{i+1} an essential F-manifold W_{i+1} with the properties that f_{i+1} can be deformed into W_{i+1} and that, moreover, $W_{i+1} \cap (S \times 0)$ and $W_{i+1} \cap (S \times 1)$ correspond via $S \times I$. Thus the components of W_{i+1} can now be fitted together, across $S \times I$, and the outcome is a submanifold in M_i such that f_i can be deformed into one of its components, X.

If X is an essential I-bundle or Seifert fibre space, we are done. If not, it is a generalized Stallings fibration. Then, applying the results of Chapter II, the existence of f implies the existence of an essential non-singular torus, annulus or square in M_i. But this contradicts our choice of S.

Chapter V. As a corollary of the above enclosing theorem on essential singular tori, annuli, and squares, we prove in §13 an enclosing theorem for essential maps of I-bundles and Seifert fibre spaces.

With the help of these enclosing theorems we are in the position to present the classification theorem in the special case of homotopy equivalences $f: M_1 \to M_2$, where M_2 is a 3-manifold whose boundary consists of tori. This will be proved in §15: first we observe that ∂M_1 necessarily consists of tori as well. Hence $f|\partial M_1$ is a system of essential singular tori in M_2. Recalling Waldhausen's theorem [Wa 4] it now follows immediately from the first enclosing theorem that f is homotopic to a homeomorphism, provided the characteristic submanifold of M_2 is a regular neighborhood of ∂M_2. If it is not, a bit more work is required, using also the second enclosing theorem.

However, in the general case, i.e. if the boundaries are arbitrary, the proof of the classification theorem is much more complicated. The idea is not to consider the restriction $f|\partial M_1$ as suggested by the Waldhausen theorem, but instead to use induction on a great hierarchy--a concept which will be described later. To make this idea work we first have to prove certain splitting theorems.

Part III: The splitting theorems.

Part III consists of two chapters, in which two splitting theorems will be proved. The first of these splitting theorems (see 18.3) says that every homotopy equivalence $f: M_1 \to M_2$ can be deformed so that afterwards $f|V_1: V_1 \to V_2$ and $f|\overline{M_1 - V_1}: \overline{M_1 - V_1} \to \overline{M_2 - V_2}$ are homotopy equivalences. Here V_i, $i = 1,2$, denotes the characteristic submanifold of M_i. This theorem will be proved in chapter VI (the use of boundary-patterns is here crucial, both in the proof and in the correct formulation of the theorem, see 18.3).

Chapter VI. In §15 and §16, we prove the existence of two homotopies, one which deforms f so that afterwards $f^{-1}V_2$ is an essential F-manifold, and one which deforms f so that afterwards $f(V_1) \subset V_2$. In §17 we see that these homotopies can be chosen independently from each other. Indeed, we find a homotopy f_t such that $f_1^{-1}V_2$ is an essential F-manifold and such that every component of V_1 is a component of $f_1^{-1}V_2$. The properties of the characteristic submanifolds then ensure that V_1 is necessarily equal to $f_1^{-1}V_2$.

In §18 it will be shown that the characteristic submanifold is very rigid with respect to homotopies. More precisely, we prove that any given homotopy h_t of a 3-manifold M with $h_i^{-1}V = V$, $i = 0,1$, can be deformed (relative the ends) into a homotopy g_t with $g_t^{-1}V = V$, for all $t \in I$. This completes the proof of the first splitting theorem.

Chapter VII. Having established the above splitting theorem we are led to consider the behavior of homotopy equivalences in the complement of the characteristic submanifolds. The proper setting for this problem is to study homotopy equivalences between simple 3-manifolds. This will be done in chapter VII:

A simple 3-manifold is a sufficiently large 3-manifold whose