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The Adjoint of a Semigroup of Linear Operators

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Preface

This lecture note is an extended version of the author's Ph.D. thesis "The adjoint of a semigroup of linear operators" (Leiden, 1992). The main difference consists of two new chapters (3 and 4) dealing with Hille-Yosida operators, extra- and interpolation and perturbation theory. Also, the sections 7.4 and 8.2 are new.

The general theory of adjoint semigroups was initiated by Phillips [Ph2], whose results are presented in somewhat more generality in the book of Hille and Phillips [HPh], and was taken up a little later by de Leeuw [dL]. Before that, Feller [Fe] had already used adjoint semigroups in the theory of partial differential equations. After these papers almost no new results on adjoint semigroups were published, although the theory of strongly continuous semigroups continued to develop rapidly. Recently the interest in adjoint semigroups revived however, due to many applications that were found to, e.g., elliptic partial differential equations [Am], population dynamics [Cea1-6], [DGT], [GH], [GW], [In], control theory [Heij], approximation theory [Ti], and delay equations [D], [DV], [HV], [V]. This stimulated also renewed interest in the abstract theory of adjoint semigroups, e.g. [Pa1-3], [GNa] and [DGH].

The aim of the present lecture note is to give a systematic exposition of the abstract theory of adjoint semigroups. Although we illustrate many results with concrete examples, we do not give applications of the theory. An exposition of the various fields where adjoint semigroups have found fruitful application would require a volume of at least comparable size. Rather, this lecture note should provide the interested reader with sufficient background material in order to make these applications easily accessible.

From the duality relation $\langle T^*(t)x^*, x\rangle = \langle x^*, T(t)x\rangle$, it follows that theorems on C_0-semigroups trivially translate into theorems on their adjoints, the difference being that the weak*-topology of X^* takes over the role of the strong topology of X. For example, $T^*(t)$ is a weak*-continuous semigroup, but not necessarily strongly continuous. From this point of view adjoint semigroups mirror, in a rather bad sense, the properties of their pre-adjoints, and no interesting new phenomena occur. This is, however, not the only way to look at adjoint semigroups. Instead, in this lecture note we try to understand the reasons why the adjoint semigroup fails to be strongly continuous and to study the extent to which it does so, and how this depends on the structure of the underlying Banach space and the properties of the pre-adjoint semigroup.

Roughly speaking, the book consists of two parts. The first part, Chapters

1-5, contains the general theory of adjoint semigroups, whereas the second part, Chapters 6-8, deals with more 'structure theoretical' topics. Let us describe in some more detail the contents of each chapter. In Chapter 1, the basic properties of the adjoint semigroup $T^*(t)$ are proved and the canonical spaces $X^{\odot}$ and $X^{\odot\odot}$ associated with the adjoint semigroup are introduced. Already at this stage we treat the adjoints of certain semigroups arising in a natural way in connection with Schauder bases. The reason is the usefulness of these semigroups for providing counter-examples to many questions in later chapters. In Chapter 2, the $\sigma(X, X^{\odot})$-topology is studied in detail. Many results show that this topology behaves rather like the weak topology, although there are also some differences. We give very simple proof of de Pagter's refinement of Phillips's characterization of $\odot$-reflexivity. In Chapter 3 we start with a systematic study of extrapolation spaces associated to a Hille-Yosida operator, having in mind that A^*, the adjoint of a generator A, is a Hille-Yosida operator. The fact that a Hille-Yosida operator on X extends to a generator of a C_0-semigroup on a suitable extrapolation space X_{-1}, provides us with a very useful tool. It allows a reduction of questions about Hille-Yosida operators and other objects associated to it to the semigroup case. This idea is at the basis of our presentation of perturbation theory in Chapter 4. Performing calculations in X_{-1} rather than in X, simplifies many arguments and reduces the proofs of the various variation-of-constants formulas to trivialities. Also in Chapter 4, we apply these ideas to the study of abstract Cauchy problems and of certain weak*-continuous semigroups on dual spaces. In Chapter 5, we take a closer look to the extent an orbit of the adjoint semigroup can fail to be strongly continuous. If $X^{\otimes}$ denotes the closed subspace of X^* consisting of those elements whose orbits are strongly continuous for $t > 0$, then we show that the quotient space $X^*/X^{\otimes}$ is either zero or non-separable. A modification of the proof is used to show that orbits in the quotient space $X^*/X^{\odot}$ are either identically zero for $t > 0$, or non-separable.

In the last three chapters, we study the relationship between the geometry of the underlying Banach space and the behaviour of the adjoint semigroup, and we take a look at several special classes of semigroups. In Chapter 6, after proving a Hahn-Banach type theorem and giving some applications, we show that there are a number of connections between continuity of the adjoint semigroup and the Banach space X or X^* having the Radon-Nikodym property or not. For example, if X^* has the RNP, then $T^*(t)$ is strongly continuous for $t > 0$. In Chapter 7, which is based on joint work with Günther Greiner, we study the rather delicate problem to describe the semigroup dual of a tensor product of two semigroups in terms of the semigroup duals of the two semigroups. The special case where $T(t)$ is translation with respect to the first coordinate on $C_0(\mathbb{R} \times K)$, is discussed in detail. Finally, in Chapter 8, which is partly based on joint work with Ben de Pagter, we study adjoints of positive semigroups. The problem of determining when the semigroup dual $X^{\odot}$ is a sublattice of X^* is discussed. Although, in general, this problem is difficult, there is detailed information on the behaviour of the adjoint semigroup in the case where X is a $C(K)$-space or $T(t)$ is a multiplication semigroup. Also, there is a section providing semigroup versions of a classical result of Wiener and Young that, with respect to the translation semigroup on $C_0(\mathbb{R})$, $T^*(t)\mu \perp \mu$ for almost all t, if μ is singular with respect to the Lebesgue measure.

At this point I would like to thank a number of persons, who have in one way or another contributed essentially to the present book. First of all, my promotor Odo Diekmann, who always encouraged me to develop my own mathematical interests. I very much appreciate the freedom I experienced in working with him. Also I thank Ben de Pagter for his constant interest and the many stimulating discussions I had with him. Not only does this book contain a number of results due to him or obtained in joint work, also many of my own results can be traced back to ideas of Ben in the form of conjectures or suggestions about what would be an interesting topic to take a closer look at. In particular, this applies to Chapters 5 and 8. I would like to thank the Tübinger school for their hospitality during my half-year stay in the Wintersemester 1990/91. Especially I thank Rainer Nagel and Günther Greiner who always showed much stimulating interest in my work and gave valuable advice. The material of Chapter 7 is joint work with Günther, which was done while he visited the CWI in 1990. Also, during my stay in Tübingen I enjoyed working with him very much. The suggestion to use extrapolation theory in matters related to Hille-Yosida operators, is due to Rainer and turned out unexpectedly useful. The warm and personal way Rainer deals with his students and co-workers is really admirable. Finally, I would like to thank Hans Heesterbeek, whose high spirits and humour made it a pleasure to share a room with him during the past four years, Adri Olde Daalhuis for his TeXnical assistance and, of course, my dear Ele.

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Appendix

Chapter 1

The adjoint semigroup

If $T(t)$ is a C_0-semigroup on a Banach space X, then elementary examples show that the adjoint semigroup $T^*(t)$ need not be a C_0-semigroup. This gives rise to the basic problem of adjoint semigroup theory: what can one say about the strong continuity of the adjoint of a C_0-semigroup?

Although we will be primarily concerned with the adjoint theory of C_0-semigroups, in some cases we have to consider semigroups which are not necessarily strongly continuous. In order to avoid constant repetition of the phrase 'Let $T(t)$ be a C_0-semigroup on a Banach space X' in almost every result, throughout we adopt the following

Convention. *The symbol $T(t)$ will always denote a C_0-semigroup with generator A on a Banach space X. Whenever we are dealing with semigroups on X which are not assumed to be C_0, the notation $U(t)$ will be used.*

In this chapter the basic concepts of adjoint semigroup theory are introduced. In Section 1.1 we recall some results on unbounded linear operators. In Section 1.2 we study the adjoint of a C_0-semigroup $T(t)$. The main result is that it is weak*-generated by the adjoint of the generator of $T(t)$. In Section 1.3 the semigroup dual space is defined and its most important properties are derived. In Section 1.4 we study the spectrum of adjoint semigroups. Finally, in Section 1.5 we compute the semigroup dual of a class of semigroups modelled on Schauder bases. Such semigroups will be used later to construct various (counter)examples.

We refer to the Appendix for some basic facts about Banach spaces and semigroups and for the terminology and notation used. Unless stated otherwise, throughout this note all vector spaces can be real or complex.

1.1. Unbounded linear operators

Let X be a Banach space. A *linear operator on* X is a pair $(A, D(A))$, where $D(A)$ is a linear subspace of X and $A : D(A) \to X$ is a linear map.

Usually we will identify $(A, D(A))$ with the map A if it is clear that A is defined on $D(A)$ only. The space $D(A)$ is called the *domain* of A.

A linear operator A is said to be *closed* if the *graph*

$$G(A) := \{(x, Ax) \in X \times X : \quad x \in D(A)\}$$

of A is closed in $X \times X$ with respect to the product topology. The operator A is *densely defined* if $D(A)$ is dense.

We will associate with a densely defined linear operator A on X a linear operator A^* on X^*, called its *adjoint*, in the following way. Define $D(A^*)$ to be the set of all $x^* \in X^*$ with the property that there is a $y^* \in X^*$ such that

$$\langle y^*, x\rangle = \langle x^*, Ax\rangle, \qquad \forall x \in D(A).$$

Since $D(A)$ is assumed to be dense, y^*, if it exists, is unique and we define

$$A^*x^* := y^*.$$

Define $R : X \times X \to X \times X$ by $R(x, y) = (-y, x)$.

Proposition 1.1.1. *If A is a densely defined linear operator on X, then A^* is a weak*-closed operator.*

Proof: Define a pairing between $X^* \times X^*$ and $X \times X$ by putting

$$\langle (x^*, y^*), (x, y)\rangle := \langle x^*, x\rangle + \langle y^*, y\rangle.$$

By means of this pairing we can identify $X^* \times X^*$ with the dual $(X \times X)^*$. By definition of A^* we have $(x^*, y^*) \in G(A^*)$ if and only if

$$\langle (x^*, y^*), (-Ax, x)\rangle = 0, \quad \forall x \in D(A).$$

In other words, $G(A^*)$ is the annihilator of $R(G(A))$. Since annihilators of linear subspaces are weak*-closed, the result follows. ////

Note that in particular A^* is (norm) closed.

Proposition 1.1.2. *If A is a closed densely defined linear operator on X, then A^* is weak*-densely defined.*

Proof: $(X^*, weak^*)$ is a locally convex topological vector space whose dual is X. Hence if A^* is not weak*-densely defined, then by the Hahn-Banach theorem there is a non-zero $x \in X$ annihilating $D(A^*)$. Since $G(A)$ (and hence $RG(A)$) is closed in $X \times X$ and $(0, x) \notin G(A)$, by the Hahn-Banach theorem there is an $(x^*, y^*) \in X^* \times X^*$ annihilating $RG(A)$ and non-zero on $R(0, x) = (-x, 0)$. In other words,

$$\langle y^*, x\rangle = \langle x^*, Ax\rangle, \quad \forall x \in D(A),$$

and

$$\langle x^*, x\rangle \neq 0.$$

But the first equality implies that $x^* \in D(A^*)$, so the second one implies that x does not annihilate $D(A^*)$, a contradiction. ////

1.2. The adjoint semigroup

Let $U(t)$ be a semigroup on a Banach space X. The *adjoint semigroup* $U^*(t)$ is the semigroup on the dual space X^* which is obtained from $U(t)$ by taking pointwise in t the adjoint operators $U^*(t) := (U(t))^*$. It is elementary to see that $U^*(t)$ is a semigroup again. If $T(t)$ is a C_0-semigroup, then

$$|\langle T^*(t)x^* - x^*, x\rangle| = |\langle x^*, T(t)x - x\rangle| \leq \|x^*\| \, \|T(t)x - x\|$$

shows that $T^*(t)$ is weak*-continuous. But $T^*(t)$ need not be strongly continuous, as is shown by several examples at the end of Section 1.3.

Recall the convention that $T(t)$ always denotes a C_0-semigroup with generator A. Since A is closed and densely defined, the adjoint A^* is a weak*-densely defined, weak*-closed operator.

Proposition 1.2.1. *$D(A^*)$ is a $T^*(t)$-invariant subspace of X^*, and for all $x^* \in D(A^*)$ we have $A^*T^*(t)x^* = T^*(t)A^*x^*$.*

Proof: Let $x^* \in D(A^*)$ and $x \in D(A)$ be arbitrary. Then for any fixed $t \geq 0$ we have

$$\begin{aligned}\langle T^*(t)x^*, Ax\rangle &= \langle x^*, T(t)Ax\rangle = \langle x^*, AT(t)x\rangle \\ &= \langle A^*x^*, T(t)x\rangle = \langle T^*(t)A^*x^*, x\rangle.\end{aligned}$$

Therefore $T^*(t)x^* \in D(A^*)$ and $A^*T^*(t)x^* = T^*(t)A^*x^*$. ////

In the next lemma we use the concept of the *weak**-*integral.* This integral, as well as some other types of integrals, is discussed in the Appendix.

Proposition 1.2.2. *weak** $\int_0^t T^*(\sigma)x^* \, d\sigma \in D(A^*)$ *for all* $t > 0$ *and* $x^* \in X^*$, *and*

$$A^*\Big(\textit{weak}^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big) = T^*(t)x^* - x^*.$$

If $x^* \in D(A^*)$, then

$$A^*\Big(\textit{weak}^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big) = \textit{weak}^* \int_0^t T^*(\sigma)A^*x^* \, d\sigma.$$

Proof: Let $x \in D(A)$ be arbitrary. Using formulas (A.2) and (A.3) of the Appendix, the identities

$$\begin{aligned}\langle \textit{weak}^* \int_0^t T^*(\sigma)x^* \, d\sigma, Ax\rangle &= \int_0^t \langle T^*(\sigma)x^*, Ax\rangle \, d\sigma = \int_0^t \langle x^*, T(\sigma)Ax\rangle \, d\sigma \\ &= \langle x^*, \int_0^t T(\sigma)Ax \, d\sigma\rangle = \langle x^*, A\int_0^t T(\sigma)x \, d\sigma\rangle \\ &= \langle x^*, T(t)x - x\rangle = \langle T^*(t)x^* - x^*, x\rangle\end{aligned}$$

show that $weak^* \int_0^t T^*(\sigma)x^* \, d\sigma \in D(A^*)$ and

$$A^*\Big(weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big) = T^*(t)x^* - x^*.$$

The second formula follows from a similar calculation: for $x \in D(A)$ we have

$$\begin{aligned}\langle A^*\Big(weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big), x\rangle &= \int_0^t \langle x^*, T(\sigma)Ax\rangle \, d\sigma \\ &= \langle weak^* \int_0^t T^*(\sigma)A^*x^* \, d\sigma, x\rangle.\end{aligned}$$

////

Let $U(t)$ be a weak*-continuous semigroup on X^*. The *weak*-generator* of $U(t)$ is the linear operator B on X^* defined by

$$D(B) := \{x^* \in X^* : \quad weak^*\text{-}\lim_{t\downarrow 0} \frac{1}{t}\big(U(t)x^* - x^*\big) \text{ exists }\};$$

$$Bx^* := weak^*\text{-}\lim_{t\downarrow 0} \frac{1}{t}\big(U(t)x^* - x^*\big), \quad x^* \in D(B).$$

In general it is not true that the weak*-generator of $T^*(t)$ uniquely determines $T^*(t)$ in the class of all weak*-continuous semigroups on X^*, cf. Section 4.4 and the notes at the end of this chapter. However, $T(t)$ is the unique C_0-semigroup on X whose adjoint is weak*-generated by A^*; this follows from Theorems 1.3.1, 1.3.3 and Corollary 1.3.7 below.

Theorem 1.2.3. *A^* is the weak*-generator of $T^*(t)$.*

Proof: Let B be the weak*-generator of $T^*(t)$ and fix $x^* \in D(A^*)$. For $x \in X$ arbitrary we have

$$\begin{aligned}\lim_{t\downarrow 0} \frac{1}{t}\langle T^*(t)x^* - x^*, x\rangle &= \lim_{t\downarrow 0} \frac{1}{t}\langle A^*\Big(weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big), x\rangle \\ &= \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \langle T^*(\sigma)A^*x^*, x\rangle \, d\sigma = \langle A^*x^*, x\rangle.\end{aligned}$$

Hence weak*-$\lim_{t\downarrow 0} \frac{1}{t}(T^*(t)x^* - x^*)$ exists and equals A^*x^*. This shows that $x^* \in D(B)$ and $Bx^* = A^*x^*$, and therefore $A^* \subset B$. To prove the converse inclusion, fix $x^* \in D(B)$. Then for any $x \in D(A)$,

$$\langle Bx^*, x\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle T^*(t)x^* - x^*, x\rangle = \langle x^*, Ax\rangle.$$

This shows that $x^* \in D(A^*)$ and $A^*x^* = Bx^*$, proving that $B \subset A^*$. ////

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1.3. The semigroup dual space

Let $T(t)$ be a C_0-semigroup on X. The *semigroup dual of X with respect to $T(t)$*, notation $X^\odot$ (usually pronounced: X-sun), is defined as the linear subspace of X^* on which $T^*(t)$ acts in a strongly continuous way:

$$X^\odot := \{x^* \in X^* : \lim_{t\downarrow 0} \|T^*(t)x^* - x^*\| = 0\}.$$

It follows trivially from this definition that $X^\odot$ is $T^*(t)$*-invariant*, which by definition means that $T^*(t)X^\odot \subset X^\odot$ holds for all $t \geq 0$. Also, since $T(t)$ is locally bounded, $X^\odot$ is a *closed* subspace of X^*.

Theorem 1.3.1. *$X^\odot$ is a closed, weak*-dense, $T^*(t)$-invariant linear subspace of X^*. Moreover $X^\odot = \overline{D(A^*)}$.*

Proof: We have already seen that $X^\odot$ is closed and $T^*(t)$-invariant. Weak*-denseness of $X^\odot$ follows from the weak*-denseness of $D(A^*)$ and $X^\odot = \overline{D(A^*)}$, which will be proved now.

Let $x^* \in D(A^*)$. Then for any $x \in X$ we have

$$\begin{aligned}|\langle T^*(t)x^* - x^*, x\rangle| &= |\langle A^*\Big(weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\Big), x\rangle| \\ &= |\int_0^t \langle T^*(\sigma)A^*x^*, x\rangle\, d\sigma| \leq t\cdot\Big(\sup_{0\leq\sigma\leq t}\|T(\sigma)\|\Big)\|A^*x^*\|\,\|x\|.\end{aligned}$$

Hence

$$\|T^*(t)x^* - x^*\| \leq t\cdot\Big(\sup_{0\leq\sigma\leq t}\|T(\sigma)\|\Big)\|A^*x^*\|$$

which shows that $D(A^*) \subset X^\odot$. Since $X^\odot$ is closed, also the norm closure $\overline{D(A^*)}$ belongs to $X^\odot$.

For the converse inclusion let $x^\odot \in X^\odot$. Then for any $x \in X$ we have

$$\begin{aligned}|\langle \frac{1}{t}\int_0^t T^*(\sigma)x^\odot\, d\sigma - x^\odot, x\rangle| &= |\frac{1}{t}\int_0^t \langle T^*(\sigma)x^\odot - x^\odot, x\rangle\, d\sigma| \\ &\leq \Big(\sup_{0\leq\sigma\leq t}\|T^*(\sigma)x^\odot - x^\odot\|\Big)\|x\|.\end{aligned}$$

Hence

$$\|\frac{1}{t}\int_0^t T^*(\sigma)x^\odot\, d\sigma - x^\odot\| \leq \sup_{0\leq\sigma\leq t}\|T^*(\sigma)x^\odot - x^\odot\| \to 0 \quad \text{as } t\downarrow 0$$

since $x^\odot \in X^\odot$. But $\frac{1}{t}\int_0^t T^*(\sigma)x^\odot\, d\sigma \in D(A^*)$, and thus we have shown that $x^\odot \in \overline{D(A^*)}$. ////

If X is reflexive, then by Theorem 1.3.1 the subspace $X^\odot$ is weakly dense, hence norm dense by the Hahn-Banach theorem. Since $X^\odot$ is also closed we obtain:

Corollary 1.3.2. *If X is reflexive, then $T^*(t)$ is strongly continuous.*

This corollary shows that adjoint semigroup theory reduces to a triviality in reflexive Banach spaces.

Let $T^\odot(t)$ denote the restriction of $T^*(t)$ to the $T^*(t)$-invariant subspace $X^\odot$. Since $X^\odot$ is closed, $X^\odot$ is a Banach space and it is clear from the definition of $X^\odot$ that $T^\odot(t)$ is a strongly continuous semigroup on $X^\odot$. We will call $T^\odot(t)$ the *strongly continuous adjoint* of $T(t)$. Let its generator be $A^\odot$. The following theorem gives a precise description of $A^\odot$ in terms of A^*.

If $(B, D(B))$ is a linear operator on a Banach space Y and Z is a linear subspace of Y containing $D(B)$, then the *part of B in Z* is the operator B_Z defined by

$$D(B_Z) := \{y \in D(B) :\ By \in Z\};$$
$$B_Z\, y := By, \quad y \in D(B_Z).$$

Theorem 1.3.3. *$A^\odot$ is the part of A^* in $X^\odot$.*

Proof: Let B be the part of A^* in $X^\odot$. If $x^\odot \in D(A^\odot)$, then

$$\lim_{t\downarrow 0} \frac{1}{t}\big(T^*(t)x^\odot - x^\odot\big) = \lim_{t\downarrow 0} \frac{1}{t}\big(T^\odot(t)x^\odot - x^\odot\big) = A^\odot x^\odot,$$

where the limits are in the strong sense. Hence these limits exist also in the weak*-sense, so by Theorem 1.2.3 it follows that $x^\odot \in D(A^*)$ and $A^*x^\odot = A^\odot x^\odot \in X^\odot$. This proves that $A^\odot \subset B$.

To prove the converse inclusion, let $x^* \in D(B)$. This means that $x^* \in D(A^*)$ and $A^*x^* \in X^\odot$. But this implies that

$$\frac{1}{t}\big(T^\odot(t)x^* - x^*\big) = \frac{1}{t}\big(T^*(t)x^* - x^*\big) = \frac{1}{t}A^*\Big(weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\Big)$$
$$= \frac{1}{t} weak^* \int_0^t T^*(\sigma)A^*x^*\, d\sigma = \frac{1}{t}\int_0^t T^*(\sigma)A^*x^*\, d\sigma.$$

The integrand of the last integral being continuous since $A^*x^* \in X^\odot$, letting $t \downarrow 0$ gives

$$\lim_{t\downarrow 0} \frac{1}{t}\big(T^\odot(t)x^* - x^*\big) = A^*x^*.$$

This shows that $x^* \in D(A^\odot)$ and $A^\odot x^* = A^*x^*$, that is, $B \subset A^\odot$. ////

Corollary 1.3.4. *A^* is the weak*-closure of $A^\odot$.*

Proof: Since A^* is a weak*-closed operator it suffices to prove that the graph of $A^\odot$ is weak*-dense in the graph of A^*. Let $x^* \in D(A^*)$. Since $D(A^*) \subset X^\odot$ we have $\frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma \in D(A^\odot)$ and $\lim_{t\downarrow 0} \frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma = x^*$. Moreover, taking the weak*-limit for $t \downarrow 0$ in

$$A^*\Big(\frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma\Big) = \frac{1}{t}\big(T^*(t)x^* - x^*\big),$$

from Theorem 1.2.3 it follows that

$$weak^*\text{-}\lim_{t\downarrow 0} A^*\Big(\frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma\Big) = A^*x^*.$$

////

Starting from the C_0-semigroup $T^\odot(t)$, the duality construction can be repeated. We define $T^{\odot *}(t)$ to be the adjoint of $T^\odot(t)$ and write $X^{\odot\odot}$ for $(X^\odot)^\odot$. Pronunciation X-sun-sun, or sometimes: X-*bosom*. $T^{\odot\odot}(t)$ and $A^{\odot\odot}$ are defined analogously. In order to relate $T(t)$ and $T^{\odot\odot}(t)$, we will now show that X can be identified with a closed subspace of $X^{\odot\odot}$. To this end, define the norm $\|\cdot\|'$ on X by

$$\|x\|' := \sup_{x^\odot \in B_{X^\odot}} |\langle x^\odot, x\rangle|,$$

where $B_{X^\odot}$ is the closed unit ball of $X^\odot$. Note that $\|x\|' \le \|x\|$ for all $x \in X$.

Theorem 1.3.5. *$\|\cdot\|'$ is an equivalent norm.*

Proof: Fix $\epsilon > 0$ and $x \in X$ arbitrary. Choose M such that $\|T(t)\| \le M$ for all t in some neighbourhood $[0, \delta)$ of 0. Choose $x^* \in B_{X^*}$ such that $|\langle x^*, x\rangle| > (1-\epsilon)\|x\|$. Choose $0 < t < \delta$ so small that $\|\frac{1}{t}\int_0^t T(\sigma)x\, d\sigma - x\| < \epsilon\|x\|$. Then

$$\begin{aligned}|\langle \frac{1}{t} weak^* \int_0^t T^*(\sigma)x^*\, d\sigma, x\rangle| &= |\langle x^*, \frac{1}{t}\int_0^t T(\sigma)x\, d\sigma\rangle| \\ &\ge |\langle x^*, x\rangle| - \epsilon\|x\| \ge (1-2\epsilon)\|x\|.\end{aligned}$$

Since $weak^* \int_0^t T^*(\sigma)x^*\, d\sigma \in X^\odot$ and $\|\frac{1}{t} weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\| \le M$ it follows that $\|x\|' \ge M^{-1}(1-2\epsilon)\ \|x\|$. Since ϵ is arbitrary it follows that $\|x\|' \ge M^{-1}\|x\|$. ////

Note that we have actually shown a little bit more, viz. $\|\cdot\|' \le \|\cdot\| \le M\|\cdot\|'$, with

$$M = \limsup_{t\downarrow 0} \|T(t)\|.$$

Define a map $j : X \to X^{\odot *}$ by $\langle jx, x^\odot\rangle := \langle x^\odot, x\rangle$. Clearly $\|j\| \le 1$ and $j(X) \subset X^{\odot\odot}$. If $j(X) = X^{\odot\odot}$ then X is said to be $\odot$-*reflexive with respect to* $T(t)$.

Corollary 1.3.6. *j is an embedding, and $M^{-1} \le \|j\| \le 1$.*

Thus we can identify X isomorphically with the closed subspace jX of $X^{\odot\odot}$. One has to be careful here, since in general this isomorphism is not *isometric*. A counterexample is given in Section 2.3. The map j will be referred to as the *natural embedding of X into $X^{\odot *}$*. The following corollary says that $T^{\odot\odot}(t)$ and $A^{\odot\odot}$ can be regarded as extensions of $T(t)$ and A respectively.

Corollary 1.3.7. *$T^{\odot\odot}(t)$ is an extension of $jT(t)$ and $A^{\odot\odot}$ is an extension of jA. Moreover, $jD(A) = D(A^{\odot\odot}) \cap jX$.*

Proof: For $x \in X$ and $x^{\odot} \in X^{\odot}$ we have

$$\langle T^{\odot\odot}(t)jx, x^{\odot}\rangle = \langle jx, T^{\odot}(t)x^{\odot}\rangle = \langle T^{\odot}(t)x^{\odot}, x\rangle = \langle x^{\odot}, T(t)x\rangle,$$

so $T^{\odot\odot}(t)jx = jT(t)x$. That $A^{\odot\odot}j$ extends jA is proved similarly. In particular $jD(A) \subset D(A^{\odot\odot}) \cap jX$. If $jx \in D(A^{\odot\odot}) \cap jX$, then

$$\begin{aligned} j \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x) &= \lim_{t\downarrow 0} \frac{1}{t}(jT(t)x - jx) \\ &= \lim_{t\downarrow 0} \frac{1}{t}(T^{\odot\odot}(t)jx - jx) = A^{\odot\odot}jx \end{aligned}$$

shows that the left hand limit exists as an element of $X^{\odot\odot}$. Since jX is closed in $X^{\odot\odot}$ the limit belongs to jX. Applying j^{-1} to the above identity shows that $x \in D(A)$. ////

We close this section with some simple examples.

Example 1.3.8. Let $T(t)$ be a uniformly continuous semigroup. From $\|T^*(t) - I\| = \|T(t) - I\|$ it is clear that also $T^*(t)$ is uniformly continuous, so in particular $T^*(t)$ is strongly continuous.

Example 1.3.9. Let $X = C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity with the sup-norm. The formula

$$T(t)f(y) := f(y+t)$$

defines a C_0-group on $C_0(\mathbb{R})$, called the *translation group*. In Chapter 7 it is shown (in more generality) that $C_0(\mathbb{R})^{\odot} = L^1(\mathbb{R})$, where, by the Radon-Nikodym theorem, we identify absolutely continuous measures in $C_0(\mathbb{R})^*$ with their density functions. Moreover, $C_0(\mathbb{R})^{\odot\odot} = BUC(\mathbb{R})$, the Banach space of bounded uniformly continuous functions on $\mathbb{R}$ with the sup-norm. Clearly, $D(A) = \{f \in C_0(\mathbb{R}) : f' \text{ exists and belongs to } C_0(\mathbb{R})\}$, and $Af = f'$. Similarly, $D(A^*)$ is the set of all absolutely continuous measures μ whose Radon-Nikodym derivative μ' exist. It can be identified with $NBV(\mathbb{R})$, the space of L^1-functions f of bounded variation, normalized so that $f(-\infty) = 0$. Also, $D(A^{\odot})$ is the subspace of $D(A^*)$ of all μ with μ' absolutely continuous again. It can be identified with $AC(\mathbb{R})$, the space of absolutely continuous L^1-functions. Finally,

$D(A^{\odot *})$ consists of all bounded Lipschitz continuous functions and $D(A^{\odot\odot})$ of those whose derivative a.e. is in $BUC(\mathbb{R})$ again.

Similarly one defines the *rotation group* $T(t)$ on $C(T)$, T the unit circle, by

$$T(t)f(e^{i\theta}) = f(e^{i(\theta+t)}).$$

Then $C(T)^{\odot} = L^1(T)$ and $C(T)^{\odot\odot} = C(T)$. In particular, $C(T)$ is $\odot$-reflexive with respect to $T(t)$.

Example 1.3.10. Let $X = c_0$ or l^p, $1 \le p < \infty$. Define $T(t)$ by

$$T(t)x_n := e^{-nt}x_n,$$

where x_n is the nth unit vector $(0, 0, .., 0, 1, 0, ..)$. This is a C_0-semigroup on X and we have $c_0^{\odot} = l^1$, $(l^1)^{\odot} = c_0$ and $(l^p)^{\odot} = l^q$ for $1 < p < \infty$, where $p^{-1} + q^{-1} = 1$.

Example 1.3.11. Let $X = C_0(\mathbb{R})$ and define $T(t) = I$, $T(t)f = P_t * f$ for $t > 0$, where

$$P_t(y) = \frac{1}{\pi}\frac{t}{t^2 + y^2}$$

is the Poisson kernel. Then $T(t)$ is a C_0-semigroup on X satisfying $T^*(t)\mu = P_t * \mu$. One verifies that $X^{\odot} = L^1(\mathbb{R})$ and $T^*(t)X^* \subset X^{\odot}$ for all $t > 0$ (cf. [Pa3]).

1.4. The spectrum of $A^{\odot}$

For a linear operator $(A, D(A))$ on a Banach space X, define

$$\varrho(A) := \{\lambda : \text{ the inverse } (\lambda - A)^{-1} \text{ exists on } X \text{ and is bounded}\},$$

where λ ranges over the scalar field. The set $\varrho(A)$ is called the *resolvent set* of A and its complement $\sigma(A)$ the *spectrum.* If A is not closed, then $\varrho(A) = \emptyset$. Indeed, suppose $\lambda \in \varrho(A)$. Then $(\lambda - A)^{-1}$ is a bounded linear operator whose inverse $\lambda - A$ is easily seen to be closed. Hence A itself must be closed.

For $\lambda \in \varrho(A)$ we write $R(\lambda, A) := (\lambda - A)^{-1}$. The bounded linear operator $R(\lambda, A)$ is called the *resolvent* of A. We have the so-called *resolvent identity*: if $\lambda, \mu \in \varrho(A)$, then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

Lemma 1.4.1. *If A is a densely defined closed operator on a Banach space X, then $\varrho(A) = \varrho(A^*)$ and for $\lambda \in \varrho(A)$ we have $R(\lambda, A)^* = R(\lambda, A^*)$.*

Proof: Suppose $\lambda \in \varrho(A)$. We will show that $\lambda \in \varrho(A^*)$. For any $x \in X$ and $x^* \in D(A^*)$ we have

$$\langle R(\lambda, A)^*(\lambda - A^*)x^*, x\rangle = \langle x^*, x\rangle$$

and consequently $R(\lambda, A^*)(\lambda - A^*)x^* = x^*$. From the definition of A^* it is easy to see that $R(\lambda, A)^*x^* \in D(A^*)$ for all $x^* \in X^*$, and for all $x \in D(A)$ we have

$$\langle(\lambda - A^*)R(\lambda, A)^*x^*, x\rangle = \langle x^*, x\rangle.$$

Since $D(A)$ is dense it follows that $(\lambda - A)R(\lambda, A)^*x^* = x^*$. We have shown that $R(\lambda, A)^*$ is a two sided inverse of $\lambda - A^*$, in other words $\lambda \in \varrho(A^*)$ and $R(\lambda, A^*) = R(\lambda, A)^*$.

Conversely, let $\lambda \in \varrho(A^*)$. We will show that $\lambda \in \varrho(A)$. Injectivity of $\lambda - A$ is proved as we did above for $\lambda - A^*$. We prove that the range of $\lambda - A$ is dense and closed. If the range were not dense, then there is a non-zero $x^* \in X^*$ such that $\langle x^*, (\lambda - A)x\rangle = 0$ for all $x \in D(A)$. Then $x^* \in D(A^*)$ and $(\lambda - A^*)x^* = 0$. From $\lambda \in \varrho(A^*)$ it follows that $x^* = 0$, a contradiction. This proves denseness. To prove closedness, let $x \in D(A)$ be arbitrary and choose $x^* \in B_{X^*}$ such that $|\langle x^*, x\rangle| \geq \frac{1}{2}\|x\|$. Let $K := \|R(\lambda, A^*)\|^{-1}$. Then

$$\begin{aligned}\|(\lambda - A)x\| &\geq K|\langle R(\lambda, A^*)x^*, (\lambda - A)x\rangle| \\ &= K|\langle(\lambda - A^*)R(\lambda, A^*)x^*, x\rangle| \geq \frac{K}{2}\|x\|.\end{aligned}$$

Now if (x_n) is a sequence such that $\lim_{n\to\infty}(\lambda - A)x_n = y$, then the above inequality implies that (x_n) is a Cauchy sequence, say with limit z. Since A is closed we have $z \in D(A)$ and $y = (\lambda - A)z$. ////

Let $R(\lambda, A)^\odot$ denote the restriction of $R(\lambda, A)^*$ to the $R(\lambda, A)^*$-invariant subspace $X^\odot$.

Theorem 1.4.2. *If A is the generator of a C_0-semigroup on X, then $\varrho(A) = \varrho(A^*) = \varrho(A^\odot)$ and $R(\lambda, A)^\odot = R(\lambda, A^\odot)$ for all $\lambda \in \varrho(A)$.*

Proof: The identity $\varrho(A) = \varrho(A^*)$ was proved in Lemma 1.4.1. Let $\lambda \in \varrho(A)$. As in the proof of 1.4.1 and by using Theorem 1.3.3 we have $R(\lambda, A)^\odot(\lambda - A^\odot)x^\odot = x^\odot$ for all $x^\odot \in D(A^\odot)$ and $(\lambda - A^\odot)R(\lambda, A)^\odot x^\odot = x^\odot$ for all $x^\odot \in X^\odot$. Hence $\lambda \in \varrho(A^\odot$ and $R(\lambda, A^\odot) = R(\lambda, A)^\odot$.

Conversely, let $\lambda \in \varrho(A^\odot)$. If $(\lambda - A)x = 0$ for some $x \in D(A)$, then for all $x^* \in D(A^*)$ we have

$$\langle(\lambda - A^*)x^*, x\rangle = \langle x^*, (\lambda - A)x\rangle = 0,$$

so x annihilates the range of $\lambda - A^*$. In particular x annihilates the range of $\lambda - A^\odot$, which equals $X^\odot$ since $\lambda \in \varrho(A^\odot)$. By the weak*-denseness of $X^\odot$ it follows that $x = 0$, so $\lambda - A$ is injective. Next, $\lambda - A$ has dense range: if not, then some non-zero $x^* \in X^*$ annihilates this range. Then $x^* \in D(A^*)$ and $(\lambda - A^*)x^* = 0$, so by Theorem 1.3.3 we have $x^* \in D(A^\odot)$ and $(\lambda - A^\odot)x^* = 0$, a contradiction to $\lambda \in \varrho(A^\odot)$. For the proof that the range of $\lambda - A$ is closed one can copy the argument in Lemma 1.4.1, the only difference being that now Theorem 1.3.5 must be invoked. ////

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