



CISM COURSES AND LECTURES NO. 292  
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

---

# **APPLICATIONS OF TENSOR FUNCTIONS IN SOLID MECHANICS**

EDITED BY

J. P. BOEHLER

---

SPRINGER - VERLAG  
WIEN - NEW YORK



INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 292



# APPLICATIONS OF TENSOR FUNCTIONS IN SOLID MECHANICS

EDITED BY

J.P. BOEHLER

UNIVERSITY OF GRENOBLE

SPRINGER - VERLAG



WIEN - NEW YORK

Le spese di stampa di questo volume sono in parte coperte da contributi  
del Consiglio Nazionale delle Ricerche.

This volume contains 36 illustrations.

This work is subject to copyright.

All rights are reserved,

whether the whole or part of the material is concerned  
specifically those of translation, reprinting, re-use of illustrations,  
broadcasting, reproduction by photocopying machine  
or similar means, and storage in data banks.

© 1987 by CISM, Udine

Printed in Italy

ISBN 3-211-81795-4 Springer Verlag Wien-New York  
ISBN 0-387-81795-4 Springer Verlag New York-Wien

## PREFACE

The mechanical behavior of materials with oriented internal structures, produced by formation processes and manufacturing procedures (crystal arrangements, stratification, fibrosity, porosity, etc.) or induced by permanent deformation (anisotropic hardening, softening, creep, internal damage growth, etc.) requires a suitable mathematical modelling. The properties of tensor valued functions of tensor variables constitute a rational basis for a consistent mathematical modelling of complex material behavior.

This book, which contains lectures presented at a CISM Advanced School, presents the principles, methods and results of applications in solid mechanics of the general laws governing tensor functions. The principles of mathematical techniques employed to derive representations of tensor functions are explained. The rules of specifying irreducible sets of tensor invariants and tensor generators for various classes of material symmetries are discussed. Representations of isotropic and anisotropic tensor functions are derived, in order to develop the general invariant forms of non-linear constitutive laws in mechanics of solids.

Within this approach, the mathematical modellization of the materials' mechanical response is explained and specific models are presented in elasticity, plasticity, hardening, internal damage and failure, for materials such as metals, composites, stratified rocks, consolidated soils and granular materials. The approach specifies a rational way to develop approximate theories and gives the necessary precision as to the number and the type of independent variables entering the mechanical laws to be used in engineering applications.

Experimental justifications as to the pertinence of the approach are given on examples of composite materials, rolled sheet-steel and stratified rocks. Information concerning proper experimental setting of tests for materials with oriented internal structures is developed.

This book is addressed to specialists in solid mechanics, both theoretical and applied, material scientists concerned with metals and composites, specialists in soil and rock mechanics and to structural engineers facing problems involving anisotropic and inelastic solids at various environments, nonlinearity and couplings.

I wish here to pay homage to the memory of my deeply missed friend Professor Antoni Sawczuk, co-coordinator of the CISM Advanced School. His untimely death did not allow him to see the final fulfilment of our shared project ; nevertheless, his help was invaluable in the preparatory phase of the Advanced School.

I would like to take this opportunity to thank CISM for having provided lecturers and participants with a chance to work together on Applications of Tensor Functions in Solid Mechanics. I am indebted to Professor Giovanni Bianchi, Secretary General of CISM, for his help with the organization of the School and to Miss Elsa Venir for her kindness and efficiency.

Jean-Paul Boehler

## CONTENTS

Chapter 1	PHYSICAL MOTIVATION <i>by J. P. Boehler</i>	
	1. Introduction	5
	2. Different domains of mechanical anisotropy	5
	3. Essential features of the anisotropic mechanical behavior of rolled sheet-steel	6
Chapter 2	INTRODUCTION TO THE INVARIANT FORMULATION OF ANISOTROPIC CONSTITUTIVE EQUATIONS <i>by J. P. Boehler</i>	
	1. Introduction	13
	2. Principle of Isotropy of Space	14
	3. Isotropic materials	16
	4. Anisotropic materials	18
	5. Orthotropic materials	21
	6. Representation of the function $\mathbf{F}$	23
	7. Conclusions	29
Chapter 3	REPRESENTATIONS FOR ISOTROPIC AND ANISOTROPIC NON-POLYNOMIAL TENSOR FUNCTIONS <i>by J. P. Boehler</i>	
	1. Introduction	31
	2. Representations for isotropic scalar and tensor functions	35

3. Representations for non-polynomial anisotropic scalar and tensor functions	
3.1. Method based on the introduction of structural tensors	40
3.2. Generalization of the Rivlin-Ericksen method	41
3.3. General anisotropy	42
3.4. Orthotropy	44
3.5. Transverse isotropy	45
3.6. Comparison with representations for polynomial anisotropic tensor functions	47
4. Representations for non-polynomial isotropic and orthotropic tensor functions in a two-dimensional space	49

Chapter 4      ANISOTROPIC LINEAR ELASTICITY

*by J. P. Boehler*

1. Introduction	55
2. Transverse isotropy ; invariant and classical formulations	57
3. Orthotropy ; invariant and classical formulations	60
4. Isotropy	64

Chapter 5      YIELDING AND FAILURE OF TRANSVERSELY ISOTROPIC SOLIDS

*by J. P. Boehler*

1. Introduction	67
2. General theory	68
3. Plastic deformations in uniaxial and triaxial tests	73
4. Failure criteria for glass/epoxy composites under confining pressure	
4.1. Introduction	80
4.2. General form of the yield criterion for triaxial tests	80

4.3. Failure modes and directional strengths	81
4.4. Proposed failure criteria	84
4.5. Comparison with experimental results	87
5. Simplified theory	
5.1. Plastic behavior	89
5.2. Yield criteria	93

Chapter 6      ON A RATIONAL FORMULATION OF ISOTROPIC AND  
ANISOTROPIC HARDENING  
*by J. P. Boehler*

1. Introduction	99
2. Classical formulation of isotropic, kinematic and anisotropic hardening	
2.1. Isotropic hardening	101
2.2. Kinematic hardening	102
2.3. Anisotropic hardening	104
2.4. Hardening phenomena which cannot be described by the classical formulations	105
2.5. Conclusions	107
3. General formulation of isotropic and anisotro- pic hardening	
3.1. Proposed general concept	108
3.2. Influence of the plastic strain on the hardening rule	108
3.3. Isotropic hardening	109
3.4. Anisotropic hardening	112
4. Examples	
4.1. Introduction	113
4.2. Proposed general isotropic hardening rule	113
4.3. Proposed general anisotropic hardening rule	116
5. Conclusions	120



Chapter 7	ANISOTROPIC HARDENING OF ROLLED SHEET-STEEL by <i>J. P. Boehler</i>	
1.	Introduction	123
2.	Constitutive relation	
2.1.	General invariant form of the constitutive relation	124
2.2.	Anisotropic hardening	126
3.	Plastic behavior	
3.1.	General invariant forms of the flow law and the yield criterion	130
3.2.	Proposed criterion	131
4.	Experimental behavior of rolled sheet-steel	
4.1.	Experimental procedure	132
4.2.	Experimental results and comparison with theoretical predictions	133
Chapter 8	ISOTROPIC POLYNOMIAL INVARIANTS AND TENSOR FUNCTIONS by <i>A. J. M. Spencer</i>	
1.	Introduction ; notations and definitions	141
2.	Results from classical theory	145
3.	Orthogonal transformation groups	146
4.	Integrity bases for vectors	149
5.	Isotropic tensors	150
6.	Isotropic invariants of vectors and second order tensors - General form	151
7.	Traces of matrix products and matrix polynomials	153
8.	Invariants of symmetric second-order tensors	157
9.	Invariants of second-order tensors and vectors ; proper orthogonal group	159
10.	Invariants of second-order tensors and vectors ; full orthogonal group	162
11.	Isotropic tensor polynomial functions of vectors and tensors	164

Chapter 9	ANISOTROPIC INVARIANTS AND ADDITIONAL RESULTS FOR INVARIANT AND TENSOR REPRESENTATIONS <i>by A. J. M. Spencer</i>	
	1. Transverse isotropy	171
	2. Orthotropic symmetry	174
	3. Crystal symmetries	175
	4. Tensors of third and higher order	176
	5. Reduction of a general tensor to a sum of traceless symmetric tensors	178
	6. Linearly independent invariants - Generating functions	181
	7. Minimality of an integrity basis	185
Chapter 10	KINEMATIC CONSTRAINTS, CONSTITUTIVE EQUATIONS AND FAILURE RULES FOR ANISOTROPIC MATERIALS <i>by A. J. M. Spencer</i>	
	1. Kinematic constraints	187
	2. Linear elasticity	191
	3. Finite elasticity	193
	4. Plasticity - Yield conditions	194
	5. Plasticity - Flow rules	197
	6. Plasticity - Hardening rules	197
Chapter 11	INVARIANTS OF FOURTH-ORDER TENSORS <i>by J. Betten</i>	
	1. Introduction	203
	2. Integrity basis for a second-order tensor	204
	3. Simplified characteristic polynomial	207
	4. The Hamilton-Cayley theorem	210
	5. Construction of simultaneous invariants	212
	6. Construction of invariants by the polarization process	214
	7. Extended characteristic polynomial	215

	8. The Lagrange multiplier method	221
Chapter 12	FORMULATION OF ANISOTROPIC CONSTITUTIVE EQUATIONS <i>by J. Betten</i>	
	1. Introduction	227
	2. The damage state in a continuum	229
	3. Stresses in a damaged continuum	236
	4. Constitutive equations involving damage and initial anisotropy	240
Chapter 13	INTERPOLATION METHODS FOR TENSOR FUNCTIONS <i>by J. Betten</i>	
	1. Introduction	251
	2. Tensor function of one variable	252
	3. Tensor function of two variables	256
	4. Interpolation at coincident points	260
	5. Polynomials of second-order and fourth-order tensors	262
	6. Simple examples	263
	7. Tensorial generalization of Norton's creep law	265
	8. Separation of tensor variables	274
Chapter 14	TENSOR FUNCTION THEORY AND CLASSICAL PLASTIC POTENTIAL <i>by J. Betten</i>	
	1. Introduction	279
	2. Isotropy	280
	3. Oriented solids	283
	4. Modification of the classical flow rule	288
	5. Anisotropy expressed through a fourth-rank tensor	295

## LIST OF AUTHORS

- J. BETTEN                      Rheinisch - Westfälische Technische Hochschule Aachen,  
Templergraben 55  
D-5100 AACHEN  
Federal Republic of Germany
- J. P. BOEHLER                Université Scientifique Technologique et Médicale de  
Grenoble  
Institut de Mécanique de Grenoble  
B.P. n° 68  
38402 SAINT MARTIN D'HERES CEDEX  
France
- A. J. M. SPENCER            The University of Nottingham  
Department of Theoretical Mechanics  
NOTTINGHAM NG7 2RD, England  
United Kingdom

**APPLICATIONS OF TENSOR FUNCTIONS**  
**IN SOLID MECHANICS**



## Chapter 1

### PHYSICAL MOTIVATION

J.P. Boehler  
University of Grenoble, France

#### 1. INTRODUCTION

Theorems of representations for tensor functions are valuable for modelling non-linear constitutive laws, particularly when the mechanical response of the material depends on more than one tensor agency. It is an approach that leads to the general invariant forms of the non-linear constitutive equations and gives the number and type of the scalar variables involved.

These representations for tensor functions have proved to be even more pertinent in attempts to model the mechanical behavior of anisotropic materials, since here invariance conditions predominate and the number of independent scalar variables cannot be found by simple arguments.

In this Chapter, we present experimental evidence of anisotropic response of materials and show the complexity of phenomena observed, which indicates the need for a rational and unified formulation of anisotropic constitutive laws.

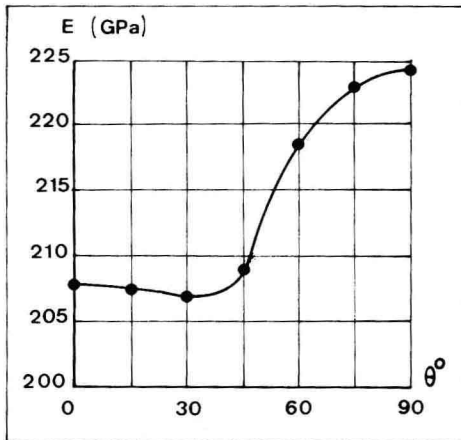


Fig.1 - Anisotropy of the elastic modulus of rolled sheet-steel (after [1]).

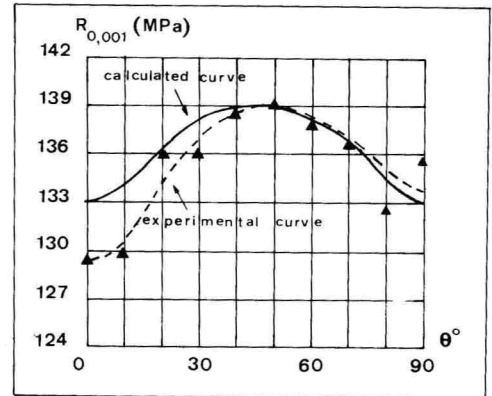


Fig.2 - Anisotropy of the elastic limit of rolled sheet-steel (after [2]).

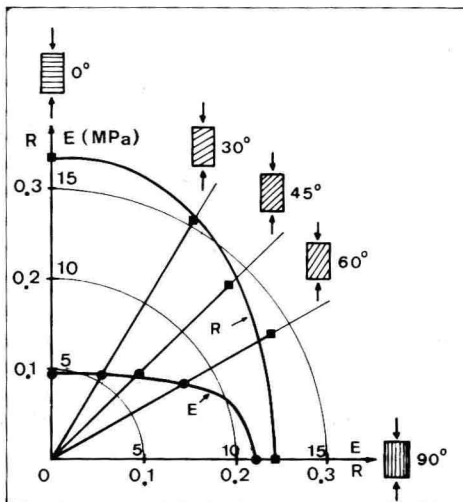


Fig.3 - Anisotropy of the elastic modulus and compressive strength of a natural clay (after [3]).

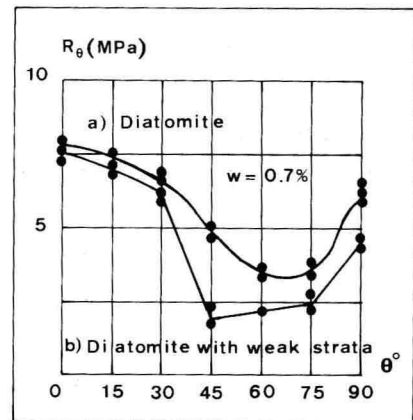


Fig.4 - Strength in simple compression for uniform and periodically non-homogeneous structure of diatomite (after [4]).



## 2. DIFFERENT DOMAINS OF MECHANICAL ANISOTROPY

Oriented internal structures of solids, such as oriented crystallographic axes, grains, particles, fissuration, cracks, cavities, etc..., result on the macroscopic level in a directional mechanical response to applied agencies. Different domains of the mechanical behavior can be influenced.

Fig.1 shows the variation of the elastic modulus  $E$  of rolled sheet-steel with respect to the angle  $\theta$  between the direction of the tensile stress and the rolling direction (after [1]), whereas the anisotropy of the elastic limit  $R$  is presented in Fig.2 (after [2]). Anisotropy of the elastic modulus  $E$  and strength  $R$  of a consolidated clay subjected to oriented compressions is given in Fig.3 (after [3]).

According to the type of the oriented internal structure, the variation of mechanical characteristics, with respect to the orientation of the material, may be continuous or discontinuous. This is shown in Fig.4 (after [4]), where the uniaxial compressive strength of diatomite, a stratified soft rock, is traced versus the orientation  $\theta$  of the specimens with respect to the normal of the strata. It is seen that the standard compressive strength possesses a minimum within the range of inclination of the strata. For a diatomite with marked layers of weaker strata, there appears a sudden drop in the strength for inclinations ranging between  $30^\circ$  and  $45^\circ$ .

In Fig.5, results regarding the strength of different consolidated clays in compression are presented (after [5, 6]) : a) range of variation for London clay ; b) Little Belt clay ; c) Vienna clay ; d) Welland clay ; e) experimental points for Grenoble clay. It is seen that the variation of strength with the orientation of the privileged direction of transverse isotropy is quite irregular. The strength either decreases, increases or passes through an extremum when the inclination of the privileged direction changes with respect to the principal stress direction. For triaxial tests on shales, Fig.6, the strength is plotted against this inclination angle for several values of the confining pressure (after [7]). Two remarks are appropriate in connection with the experimental results pre-