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Advanced Engineering Mathematics

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This book is dedicated,
with appreciation, to my students

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SELECTED FORMULAS

CARTESIAN COORDINATES: $u = u(x, y, z)$, $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$

$$\nabla u = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad d\mathbf{R} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$

$$dA = \begin{cases} dy\,dz \text{ (constant-}x\text{ surface)} \\ dx\,dz \text{ (constant-}y\text{ surface)}, \\ dx\,dy \text{ (constant-}z\text{ surface)} \end{cases} \quad dV = dx\,dy\,dz$$

CYLINDRICAL COORDINATES: $u = u(r, \theta, z)$, $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z$

$$\nabla u = \left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial u}{\partial z}\hat{\mathbf{e}}_z$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}v_\theta + \frac{\partial}{\partial z}v_z$$

$$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{e}}_z$$

$$= \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & r v_\theta & v_z \end{vmatrix}$$

$$\mathbf{R} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z, \quad d\mathbf{R} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta + dz\,\hat{\mathbf{e}}_z$$

$$dA = \begin{cases} r\,d\theta\,dz \text{ (constant-}r\text{ surface)} \\ dr\,dz \quad \text{(constant-}\theta\text{ surface)}, \\ r\,dr\,d\theta \text{ (constant-}z\text{ surface)} \end{cases} \quad dV = r\,dr\,d\theta\,dz$$

SPHERICAL COORDINATES: $u = u(\rho, \phi, \theta)$, $\mathbf{v} = v_\rho \hat{\mathbf{e}}_\rho + v_\phi \hat{\mathbf{e}}_\phi + v_\theta \hat{\mathbf{e}}_\theta$

$$\nabla u = \left(\hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_\theta \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \theta}\right)u$$

$$= \frac{\partial u}{\partial \rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \phi}\hat{\mathbf{e}}_\phi + \frac{1}{\rho \sin\phi}\frac{\partial u}{\partial \theta}\hat{\mathbf{e}}_\theta$$

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$\nabla \cdot \mathbf{v} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 v_\rho) + \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \phi}(v_\phi \sin\phi) + \frac{1}{\rho \sin\phi}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \mathbf{v} = \frac{1}{\rho \sin\phi}\left[\frac{\partial}{\partial \phi}(v_\theta \sin\phi) - \frac{\partial v_\phi}{\partial \theta}\right]\hat{\mathbf{e}}_\rho$$

$$+ \frac{1}{\rho}\left[\frac{1}{\sin\phi}\frac{\partial v_\rho}{\partial \theta} - \frac{\partial(\rho v_\theta)}{\partial \rho}\right]\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho v_\phi)}{\partial \rho} - \frac{\partial v_\rho}{\partial \phi}\right]\hat{\mathbf{e}}_\theta$$

$$= \frac{1}{\rho^2 \sin\phi}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \rho\sin\phi\,\hat{\mathbf{e}}_\theta \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ v_\rho & \rho v_\phi & \rho \sin\phi\, v_\theta \end{vmatrix}$$

$$\mathbf{R} = \rho\hat{\mathbf{e}}_\rho, \quad d\mathbf{R} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\, d\phi\,\hat{\mathbf{e}}_\phi + \rho \sin\phi\, d\theta\,\hat{\mathbf{e}}_\theta$$

$$dA = \begin{cases} \rho^2|\sin\phi|\,d\phi\,d\theta & \text{(constant-}\rho\text{ surface)} \\ \rho|\sin\phi|\,d\rho\,d\theta & \text{(constant-}\phi\text{ surface)}, \\ \rho\,d\rho\,d\phi & \text{(constant-}\theta\text{ surface)} \end{cases} \quad dV = \rho^2\,|\sin\phi|\,d\rho\,d\phi\,d\theta$$

GAUSS DIVERGENCE THEOREM: $\int_{\mathscr{V}} \operatorname{div} \mathbf{v}\, dV = \int_{\mathscr{S}} \hat{\mathbf{n}} \cdot \mathbf{v}\, dA$

GREEN'S FIRST IDENTITY: $\int_{\mathscr{V}} (\nabla u \cdot \nabla v + u \nabla^2 v)\, dV = \int_{\mathscr{S}} u \frac{\partial v}{\partial n}\, dA$

GREEN'S SECOND IDENTITY: $\int_{\mathscr{V}} (u \nabla^2 v - v \nabla^2 u)\, dV = \int_{\mathscr{S}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dA$

STOKES' THEOREM: $\int_{\mathscr{S}} \hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v}\, dA = \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{R}$

GREEN'S THEOREM: $\int_{\mathscr{S}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\mathscr{C}} (P\, dx + Q\, dy)$

SELECTED FORMULAS

FOURIER SERIES

(a) $f(x)$ $2l$-periodic on $-\infty < x < \infty$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx, \quad n \geq 1$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx, \quad n \geq 1$$

(b) $f(x)$ defined only on $0 < x < L$:

Half-range sine: $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ (HS1)

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad \text{(HS2)}$$

Half-range cosine: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$ (HC1)

$$a_0 = \frac{1}{L} \int_{0}^{L} f(x)\, dx \qquad \text{(HC2)}$$

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad \text{(HC3)}$$

Quarter-range sine: $f(x) = \sum_{n=1,3,\ldots}^{\infty} b_n \sin \frac{n\pi x}{2L}$ (QS1)

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{2L}\, dx \qquad \text{(QS2)}$$

Quarter-range cosine: $f(x) = \sum_{n=1,3,\ldots}^{\infty} a_n \cos \frac{n\pi x}{2L}$ (QC1)

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{2L}\, dx \qquad \text{(QC2)}$$

FOURIER INTEGRAL: $f(x) = \int_{0}^{\infty} [a(\omega) \cos \omega x + b(\omega) \sin \omega x]\, d\omega$

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x\, dx, \quad b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x\, dx$$

GAMMA FUNCTION: $\Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x > 0)$

$$\Gamma(x) = (x-1)\Gamma(x-1) \quad (x > 1)$$

$$\Gamma(n+1) = n!, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

ERROR FUNCTION: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \quad \operatorname{erf}(\infty) = 1$

BESSEL EQUATION OF ORDER ν: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

$$y(x) = \begin{cases} AJ_\nu(x) + BJ_{-\nu}(x), & \nu \neq \text{integer} \\ CJ_n(x) + DY_n(x), & \nu = n = \text{integer} \end{cases}$$

MODIFIED BESSEL EQUATION OF ORDER ν: $x^2y'' + xy' + (-x^2 - \nu^2)y = 0$

$$y(x) = \begin{cases} AI_\nu(x) + BI_{-\nu}(x), & \nu \neq \text{integer} \\ CI_n(x) + DK_n(x), & \nu = n = \text{integer} \end{cases}$$

A RELATED ODE: $\frac{d}{dx}\left(x^a\frac{dy}{dx}\right) + bx^cy = 0$

$$y(x) = x^{\nu/\alpha}Z_\nu(\alpha\sqrt{b}x^{1/\alpha}),$$

where Z_ν is any Bessel function of order ν, and

$$\alpha = \frac{2}{c-a+2}, \quad \nu = \frac{1-a}{c-a+2}$$

Preface

Purpose.

This book is intended primarily as a text for a single- or multi-semester course in applied mathematics for students in engineering or science. Beyond such a course, or courses, it is intended that the book be useful for reference and for self-study. Thus, explanations are sometimes more detailed than they might have been, common difficulties are anticipated and discussed, and the writing is somewhat conversational.

It should be emphasized that by "self-study" we do not necessarily mean outside the context of a formal course. Rather, we are interested in possibilities even within course settings. For if the text can be understood through self-study, then the instructor has more instructional options available. For instance, in a given lecture one might answer questions on the reading, highlight a few points, and then take a detailed look at some pedagogically-interesting exercises, rather than present a complete discussion of the assigned reading.

Topic Coverage.

Topic coverage is traditional, with one exception. Most texts of this type assume only a knowledge of the calculus, and begin with a detailed treatment of ordinary differential equations (ODEs). Here, we begin, instead, with linear algebra. Thus, prerequisites for this book include the usual calculus sequence, together with a course, or at least a part of a course, on ordinary differential equations. Nevertheless, there is ODE material included: The needed elementary theory and solution techniques are reviewed, for convenient reference, in Appendix A, matrix methods are employed for systems of coupled equations in Chapter 4, the Fourier and Laplace transforms are used to solve ODEs in Chapters 11 and 12, and series solutions of the Bessel and Legendre equations are studied in Chapter 17.

Some important topics have *not* been included: qualitative (phase plane and existence/uniqueness) and quantitative (numerical integration) discussions of nonlinear ODEs, linear programming, nondeterministic methods, variational methods and optimization, fast Fourier transform, numerical integration, and the finite-element method. A number of these topics could be grouped under the heading "numerical methods." Numerical methods were not omitted due to any lack of importance. On the contrary, it was felt that they are now so extensive and pervasive that they demand more attention than could be accommodated in the present text. It is true that we do include sections, in Part IV, on the finite-difference solution of partial differential equations (PDEs), but we do so for the following reasons. First, that discussion provides an important and convincing application of matrix theory, which was the subject of Part I. Second, there is a "hands-on" flavor to the numerical solution methods which nicely complements the more theoretical looking methods of separation of variables and integral transforms. Finally, the finite-difference methods discussed are at least a modest representative of the larger domain of numerical methods.

Organization.

Since there are more than 20 chapters, the book is organized into five parts in order to provide a logical framework. These parts are as follows:

Part I:	Linear Algebra
Part II:	Multivariable Calculus and Field Theory
Part III:	Fourier Series; Fourier and Laplace Transforms
Part IV:	Partial Differential Equations
Part V:	Complex Variable Theory

Roughly speaking, Parts I, II, III, and V are rather independent, whereas Part IV draws heavily on Parts II and III and, in the finite-difference sections, also on Part I.

A number of sections are listed as OPTIONAL. The point here is not that these sections *should* be omitted, but rather that they *can* be omitted in a shorter course. For example, whereas non-Cartesian coordinates are of great importance in engineering and science, it is possible that in a briefer discussion of field theory an instructor might wish to limit discussion to the Cartesian case. Thus plane polar, cylindrical, and spherical polar coordinates are split off as optional "parallel" sections in both Parts II and IV. Further, Chapter 9 can be studied without having first studied the optional sections in Chapter 8 on surfaces and volumes. As one more example, Sections 3.7 (Change of Basis) and 3.8 (Vector Transformation) are listed as optional simply because they are not drawn on subsequently in Part I.

In my own course, a junior-level course for mechanical and electrical engineers, which follows three semesters of calculus and one semester of differential equations, the twenty-five 1-hour-and-20-minute classes (excluding examinations) are allotted as follows:

Linear Algebra:	6 classes, covering Sections 1.1 to 4.3, with Sections 2.6, 3.7, and 3.8 omitted
Field Theory:	9 classes, covering Sections 7.1 to 7.6, 8.2, and Chapter 9

Fourier Series and PDEs: 10 classes, covering Sections 10.1 to 10.3, 13.1 to 13.3, 14.1, 14.2, 14.5, 15.1 to 15.3, 15.5, 16.1, and 16.2

Finally, it should be mentioned that special functions—such as the delta, gamma, Bessel, and Legendre functions—are introduced when needed, rather than in a separate chapter.

Exercises.

Exercises listed at the end of a section cover material presented in that section. These exercises serve a number of purposes. First, it is occasionally stated in the text that "it can be shown," with the details left to the exercises. Second, there are numerous "drill"-type exercises. These are complemented by exercises which are more thought provoking and/or deal with applications. Finally, in most sections there is important supplementary information introduced in the exercises. The idea here was to keep the text part relatively lean and to include supplementary material as part of the exercises. This material is also intended to increase the value of the text in terms of subsequent reference. To illustrate, consider Sections 17.2 and 17.3, on Bessel and Legendre functions. For a text of this type, the Bessel function discussion is a bit long (as is felt to be justified not only because of the importance of Bessel functions but also because consideration of the Bessel equation is somewhat representative of other special functions as well). Thus it is important to have the follow-up section on Legendre functions be especially crisp and to the point. To accomplish that goal, considerable information about Legendre functions is relegated to the exercises, including Rodriques' formula, the generating function, integral representation, Legendre functions of the second kind, associated Legendre functions, and applications to electrostatics.

As mentioned above, it is recommended that the instructor consider the option of lecturing on text material rather selectively, and balancing that against self-study. In that event, it is hoped that the exercises may be used as a source of interesting material for lecture and class discussion.

Finally, answers are given at the end of the book for exercises indicated by asterisks. In addition, a Solutions Manual is available to instructors, and can be obtained either through the Prentice-Hall sales representative or by contacting the College Book Division, Prentice-Hall, Englewood Cliffs, NJ 07632.

References.

References to other books appear in three different ways. First, there are one or two suggested references for each of the five parts, as follows:

PART I: HOWARD ANTON, *Elementary Linear Algebra,* 2nd ed. New York: Wiley, 1977.

BEN NOBLE, *Applications of Undergraduate Mathematics in Engineering.* New York: Macmillan, 1967.

PART II: J. E. MARSDEN AND A. J. TROMBA, *Vector Calculus,* San Francisco: W. H. Freeman, 1976.

H. M. SCHEY, *Div, Grad, Curl, and All That.* New York: W. W. Norton, 1973.

PART III: W. E. BOYCE AND R. C. DIPRIMA, *Elementary Differential Equations and Boundary Value Problems,* 3rd ed. New York: Wiley, 1977.

	R. V. CHURCHILL AND J. W. BROWN, *Fourier Series and Boundary Value Problems*, 3rd ed. New York: McGraw-Hill, 1978.
PART IV:	R. V. CHURCHILL AND J. W. BROWN, *Fourier Series and Boundary Value Problems*, 3rd ed. New York: McGraw-Hill, 1978.
	DAVID L. POWERS, *Boundary Value Problems*, 2nd ed. New York: Academic Press, 1977.
PART V:	E. B. SAFF AND A. D. SNIDER, *Fundamentals of Complex Analysis*, Englewood Cliffs, NJ: Prentice-Hall, 1976.

References to these texts are capitalized: for instance, ANTON, and SAFF AND SNIDER. The instructor may wish to place these suggested references on library reserve for the class. I generally also include Morris Kline's *Mathematical Thought From Ancient to Modern Times* (New York, Oxford University Press: 1972) among the reserve reading as an excellent historical source. Second, rather specific references are footnoted. Finally, additional general references are listed at the end of each part.

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Closure.

In spite of all precautions, errors will no doubt be found in using this book, and I encourage you to let me know about them, so they can be corrected in subsequent printings. Beyond that, any comments, suggestions for improvement, or ideas that you wish to share would be most appreciated, especially regarding the choice of topics.

Michael D. Greenberg

Contents

PART III FOURIER SERIES; FOURIER AND LAPLACE TRANSFORMS

PART IV PARTIAL DIFFERENTIAL EQUATIONS