PRINCIPLES AND APPLICATIONS OF ELECTROMAGNETIC FIELDS

PLONSEY
COLLIN

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ROBERT PLONSEY

ROBERT E. COLLIN

Associate Professors of Electrical Engineering
Case Institute of Technology

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PREFACE

In view of the number of texts published in the area of electromagnetic fields for undergraduate students in recent years, another book in this area needs some explanation. Generally, we find that the available books suffer from one or more of the following deficiencies:

1. Inadequate coverage of topics
   a. Very little or no material beyond Maxwell's equations
   b. Omission of important derivations
   c. Lack of depth in treatment
2. Lack of separation or distinction between purely mathematical and physical concepts
3. *Ad hoc* treatment of the subject of fields in the presence of material bodies
4. Insufficient number of examples worked out
5. A lack of problems designed to give the student ability and confidence in developing analytical solutions and extensions to the theory in contrast with routine drill problems

The current text represents the authors' efforts to overcome these shortcomings. We have endeavored to produce a book in electromagnetic theory suitable for undergraduate use that is sophisticated enough to establish a firm basis for advanced study in this area. Furthermore, a sufficient number of applications treated in adequate depth are included to illustrate the basic concepts and also provide a nontrivial background in a diverse number of areas of current interest.

We feel that an adequate text should be sufficiently complete and have enough scope to warrant a place on a personal bookshelf of the undergraduate student after he leaves school. In many cases it is desirable to avoid presentation of a lengthy derivation of certain formulas in class. On the other hand, many students wish to know how a given result is obtained, and in this case an outline of the steps to be followed suffices, provided a detailed derivation is included in the text. We have therefore included some material in the text of a more advanced and sophisticated nature for the advanced or curious student and as a supplement to the main course. The chapters have been organized so that this material is reserved to the last sections. These are set in smaller type and may be omitted by the instructor without detriment to the main continuity of the text.

The first nine chapters of the book constitute the basic principles of electromagnetic theory and are more than ample for a junior- or senior-level course of one-semester duration. The level can be varied somewhat by inclusion or elimination of certain of the topics. The organization of these chapters is fairly standard and includes vector analysis, electrostatics, mathematical techniques in the solution of Laplace's equation, current fields, magnetostatics, and time-varying fields (Maxwell's equations), in that order. The material on vector analysis gives greater emphasis to the relationship between fields and their sources. The climax of this development is in the presentation of the Helmholtz theorem. The flux concept is introduced in the same chapter so that the student will understand its general usefulness in representing any type of vector field. Vector-analysis techniques are freely utilized in the main body of the text.

The theory of electrostatics is developed in Chapter 2 for free-space conditions. In this way the electric field and its relationship to charge sources are developed as a basic formulation. The atomic properties of dielectrics are considered in some detail in Chapter 3. The effect of the presence of dielectrics in an electric field is next explained in terms of the induced dipole sources, from which the equivalent charge source is then determined. This procedure is also followed in Chapters 6 and 7, where the subject of magnetostatics is first developed for currents under free-space conditions. The effect of magnetic materials is then introduced in terms of the induced current sources.

Chapter 4 discusses the method of separation of variables and conformal-mapping techniques for obtaining solutions to Laplace's equation with specified boundary conditions. This chapter includes a discussion of cylindrical and spherical functions and sufficient elementary material on functions of a complex variable to be essentially self-contained. Since this chapter presents no new physical concepts, it may be omitted without detriment to the continuity of the text. Simple solutions to Laplace's equation, including image theory, are included in Chapter 3. The uniqueness theorem is developed in Chapter 2.

Chapter 5 discusses currents and Ohm's law. It presents a formal solution to the computation of the resistance of an arbitrary conducting body. The graphical techniques of flux plotting and the use of the electrolytic tank as auxiliary techniques for solution of Laplace's equation are included here.

Energy storage in electric and magnetic fields is discussed in Chapters 3 and 8, respectively. The calculation of the electric or magnetic force by the principle of virtual work is developed. Finally, the Maxwell stress tensor is introduced, and its significance for the field concept discussed.

In Chapter 9 the displacement current is postulated and Maxwell's equations are formulated. The development of vector and scalar potentials, their relationship to the sources, and retardation effects are then discussed. The chapter concludes with a fairly complete description of the relationship between circuit theory and field theory.

Many fields books written for the undergraduate terminate after introducing Maxwell's equations. Since Maxwell's equations form a climax and the bulk of the practical applications involve time-varying fields, it can hardly be considered a wise choice to terminate the book just when the door to a large variety of interesting and important applications has been opened. We feel that the student can proceed to the application of Maxwell's equations most expediently when he does not have to change texts, since this eliminates the necessity of getting acquainted with another author's notation and way of doing things. For this reason we have included three rather complete chapters on applications. These are on wave guiding, radiation, and interaction of fields with charged particles.

The material on wave guiding is included in Chapter 10. This chapter also discusses plane waves, refraction at a plane interface, and cavity resonators. The technique for accounting for losses at good conductors is considered so that attenuation in waveguides and Q of cavities may be computed. The chapter includes a general treatment of the resolution of fields into TEM, TM, and TE modes. The rectangular and circular waveguides are considered.

The subject of radiation is considered in Chapter 11. The simple linear antenna is described, and the fundamental properties of arrays developed. A full description of the receiving antenna and reciprocity is also included.

The final chapter discusses the interaction of fields with charged particles. This includes the subject of electron ballistics and parallel-plane vacuum-tube theory, including the effect of space charge and transit time. Space-charge-wave theory is developed and applied to the klystron and the traveling-wave tube. In the latter case the helix as a slow-wave structure is described. Propagation in gyrotropic media is considered, with specific application to the ionosphere and ferrites. Finally, an introduction to magnetohydrodynamics is given.

The greatest utility of the book is seen as a full-year course which would substantially cover the entire text. This would provide a very firm foundation in electromagnetic theory and also a good insight into a variety of applications. Chapters 9 to 12 may also be used in a one-semester course on applications of electromagnetic fields if the students have a sufficient background in the fundamentals. As with the full course, this would serve as an introduction to advanced graduate study in special topics. The level of this material may in some instances be deemed suitable for an introductory graduate course.

Objections are at times raised as regards presenting field theory in a layered package, that is, electrostatics, stationary currents, magnetostatics, etc. It certainly is feasible to begin with Maxwell's equations and specialize to static fields and then return to time-varying fields. This is essentially a direct-analysis point of view. By beginning with the experimental laws for static fields and building up and generalizing to time-varying fields the whole approach becomes more of a synthesis procedure. This we feel makes the subject material more acceptable from the student's point of view. It also permits the concept of a vector field to be firmly developed in connection with fields that have a relatively simple behavior, for example, electrostatic field vs. time-varying electromagnetic fields.

We do not particularly feel that relativistic electrodynamics should be introduced at the undergraduate level. The practical applications are few in number, and usually insufficient time is available to give anything more than a brief introduction, which probably leaves the student confused rather than informed in the subject. We have adopted the conventional formulation of considering **E** and **B** as the fundamental force vectors and treat magnetization on the Amperian-current basis. This is in contrast to the view adopted by Professor Chu of the Massachusetts Institute of Technology. Although Professor Chu's formulation has certain features to recommend it when considering the four-vector formulation of electrodynamics, we feel it wise to adhere to the conventional formulation. Not only does this keep our presentation similar to that in most other reference books the student will consult; it is also in keeping with the scope of our presentation.

Since this book is based, to a large extent, on the combined work of many earlier contributors, it is impossible to acknowledge this on an individual basis. We should like, however, to express our thanks to those who assisted us during the preparation of the manuscript. We are greatly appreciative of the help received from colleagues at Case Institute of Technology and in particular from Professors Forest E. Brammer and Robert D. Chenoweth. Many fruitful ideas were also received from students who made use of a preliminary version of this book. We are also grateful to Professor John R. Martin, Acting Chairman of the Department of Electrical Engineering, for making available facilities for the preparation of the manuscript. And finally, we are

indebted to Misses Florence Alaimo, Ruth Hudak, and Janet Leonard, who expertly deciphered the original handwritten manuscript and typed the final copy.

It may be worth noting that a toss of a coin determined the order of the authors' names for the book.

Robert Plonsey
Robert E. Collin

CONTENTS

CHAPTER 1

VECTOR ANALYSIS

This chapter develops the mathematics of vector analysis that will be needed in the succeeding chapters of the book. Based on the work of this chapter, it is possible to considerably simplify the formulation of the physical laws of electromagnetic theory. Furthermore, manipulation of the equations with the goal of solving physical problems is also greatly facilitated. One of the purposes of this chapter is to lay the necessary groundwork in the use of vector algebra and vector calculus.

Another purpose is also sought in this chapter. For along with the mathematical simplifications in the use of vector analysis, there go certain concomitant philosophical concepts. This chapter, consequently, contains a discussion of fields, the flux representation of vector fields, and some general remarks concerning sources of fields. The definition of the divergence and curl of a field can then be understood as measures of the strength of the sources and vortices of a field. When in the succeeding chapters the specific nature of the electric and magnetic fields is considered, the student will have an appropriate framework into which to fit them.

Although much effort has been directed to the development of a physical basis for the mathematical definitions of this chapter, they may still seem somewhat artificial. The full justification of their utility, and a deepening of their meaning, will become apparent when the physical laws of electromagnetics are considered in the later chapters.

1.1. Scalars and Vectors

In this book we deal with physical quantities that can be measured. The measurement tells how many times a given unit is contained in the quantity measured. The simplest physical quantities are those that are completely specified by a singe numlber, along with a known unit. Such quantities are called scalars. Volume, density, and mass are examples of scalars.

Another group of physical quantities are called vectors. We may see how the vector arises if we consider as an example a linear displacement of a point from a given initial position. It is true that the final position of the point could be described in terms of three scalars, e.g., the cartesian coordinates of the final point with respect to axes chosen through the initial point. But this obscures the fact that the concept of displacement is a single idea and does not depend on a coordinate system. Consequently, we introduce displacements as quantities of a new type and establish a system of rules for their use. All physical quantities which can be represented by such displacements and which obey their respective rules are called vectors.

The vector can be represented graphically by a straight line drawn in the direction of the vector, the sense being indicated by an arrowhead and its length made proportional to the magnitude of the vector. Examples of vector quantities include displacement, acceleration, and force. In this book all vector quantities are designated by boldface type, while their magnitudes only are indicated through the use of italics.

1.2. Addition and Subtraction of Vectors

From the definition of a vector, just given, it is possible to deduce the rule for addition of vectors. Thus, consider two vectors **A** and **B** as illustrated in Fig. 1.1. Vector **A** represents the displacement of a movable point from point 1 to point 2. Vector **B** represents a displacement from point 2 to point 3. The result is equivalent to a total displacement of a point from 1 to 3. This linear displacement from 1 to 3 is called the resultant, or geometric sum of the two displacements (1,2) and (2,3). It is represented by the vector **C**, which we call the sum of the vectors **A** and **B**:

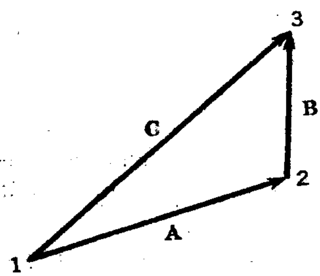

FIG. 1.1. Vector addition.

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.1}$$

Note that vectors **A** and **B** are of the same dimensions and type and that the geometric construction of Fig. 1.1 requires that the origin of one be placed at the head of the other. We may inquire whether the order of addition is of significance.

Consider that the displacement **B** is made first and then the displacement **A**. In this case the movable point describes the path 143 as in Fig. 1.2 and consequently produces the same resultant. Vector addition thus obeys the commutative law; i.e., the geometric sum of two vectors is independent of the order of addition so that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.2}$$

The path 143 and 123 together make up a parallelogram whose diagonal is the resultant of the displacement represented by the two adjacent sides.

Accordingly, the law of vector addition is often referred to as the parallelogram law. This law of addition is characteristic of the quantities called vectors. Thus it is proved in statics that forces acting on a rigid body follow the parallelogram law of addition; consequently, such forces are vectors.

It is easy to show that vectors satisfy the associative law of addition, which states that the order of adding any number of vectors is immaterial. Thus the sum of three vectors **A**, **B**, **C** can be expressed as

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1.3}$$

The proof can be established by considering Fig. 1.3, in which the same resultant (1,4) is arrived at by carrying out the summation indicated by

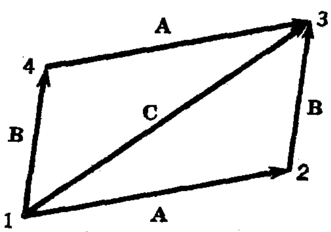

FIG. 1.2. Illustration of parallelogram of vector addition.

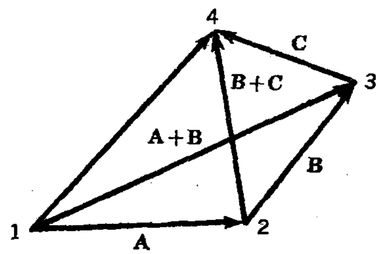

FIG. 1.3. Illustration of associative law of vector addition.

either the left- or right-hand side of Eq. (1.3). The former path is (1,3,4); the latter is (1,2,4).

To obtain the difference of two vectors **A** − **B**, it becomes necessary to define the negative of a vector. This is taken to mean a vector of the same magnitude but of opposite direction to the original vector. Thus

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{1.4}$$

We may therefore define vector subtraction as follows: A vector **B** is subtracted from a vector **A** by adding to **A** a vector of the same magnitude as **B** but in the opposite direction. In the parallelogram of Fig. 1.2 a diagonal from 4 to 2 would represent the geometric difference **A** − **B**.

1.3. Unit Vectors and Vector Components

The result of multiplying a vector **A** by a positive scalar m is to produce a new vector in the same direction as **A** but whose magnitude is that of **A** times m. The resultant **P** is thus related to **A** and m by the following:

$$\mathbf{P} = m\mathbf{A} \tag{1.5}$$

$$|\mathbf{P}| = m|\mathbf{A}| \qquad \text{or} \qquad P = mA \tag{1.6}$$

A unit vector is one whose magnitude is unity. It is often convenient to express a vector as the product of its magnitude and a unit vector having the same direction. Thus if $\mathbf{a}$ is a unit vector having the direction of $\mathbf{A}$, then $\mathbf{A} = A\mathbf{a}$. The result expressed by (1.5) follows immediately, since $m\mathbf{A} = mA\mathbf{a}$. The three unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ parallel to the right-hand rectangular axes x, y, z, respectively, are of particular importance.

The components of a vector are any vectors whose sum is the given vector. We shall often find it convenient to choose as components the three rectangular components of cartesian coordinates. Thus, if A_x, A_y, A_z are the magnitude of the projections of vector $\mathbf{A}$ on the x, y, z axes, its rectangular components are $\mathbf{a}_xA_x$, $\mathbf{a}_yA_y$, $\mathbf{a}_zA_z$. The vector $\mathbf{A}$ is completely determined by its components since the magnitude is given by

$$A^2 = A_x^2 + A_y^2 + A_z^2 \tag{1.7}$$

and the direction cosines l, m, n are given by

$$l = \frac{A_x}{A} \qquad m = \frac{A_y}{A} \qquad n = \frac{A_z}{A} \tag{1.8}$$

For brevity, we shall usually designate A_x, A_y, A_z, without the associated unit vectors, as the components of $\mathbf{A}$.

Equal vectors have the same magnitude and direction; consequently, their respective rectangular components are equal. Therefore a vector equation can always be reduced, in general, to three scalar equations. For example, $\mathbf{A} + \mathbf{B} = \mathbf{C}$ can be expressed as

$$\mathbf{a}_x(A_x + B_x) + \mathbf{a}_y(A_y + B_y) + \mathbf{a}_z(A_z + B_z) = \mathbf{a}_xC_x + \mathbf{a}_yC_y + \mathbf{a}_zC_z \tag{1.9}$$

i.e., addition is commutative and associative. Since the vector represented by the left-hand side of (1.9) equals that of the right-hand side, we are led to the result

$$A_x + B_x = C_x \qquad A_y + B_y = C_y \qquad A_z + B_z = C_z \tag{1.10}$$

1.4. Vector Representation of Surfaces

Figure 1.4 illustrates a plane surface of arbitrary shape. We may represent this surface by a vector $\mathbf{S}$ whose length corresponds to the magnitude of the surface area and whose direction is specified by the normal to the surface. To avoid ambiguity, however, some convention must be adopted which establishes the positive sense of the normal.

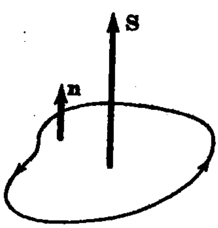

FIG. 1.4. Vector surface area. $\mathbf{n}$ is a unit surface normal.

When the surface forms part of a closed surface, the positive normal is usually taken as directed outward. For an open surface the positive normal can be associated with the positive sense of describing the periphery. This relationship is defined by taking the positive normal in the direction that a right-hand screw would advance when turned so as to describe the positive periphery. This definition actually arises out of a mathematical description of certain physical phenomena which will be discussed in later chapters. One can choose either positive periphery or positive normal arbitrarily.

If the surface is not plane, it is subdivided into elements which are sufficiently small so that they may be considered plane. The vector representing the surface is then found by vector addition of these components. This means that an infinite number of surfaces correspond to a given surface vector. The unit surface normal is designated by **n**.

1.5. The Vector Product of Two Vectors

Certain rules have been set up governing multiplication of vectors. The vector or cross product **A** × **B** of two vectors **A** and **B** is, by definition,

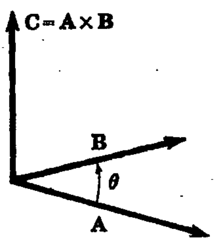

FIG. 1.5. Vector cross product.

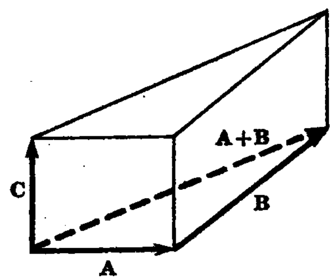

FIG. 1.6. Illustration for the distributive law of vector multiplication.

a vector of magnitude $AB \sin \theta$ in the direction of the normal to the plane determined by **A** and **B**. Its sense is that of advance of a right-hand screw rotated from the first vector to the second through the angle θ between them, as in Fig. 1.5. Since the direction reverses if the order of multiplication is interchanged, the commutative law of multiplication does not hold. Actually, we have

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.11}$$

This definition of vector product was chosen because it corresponds to a class of physically related quantities. Geometrically, the magnitude $|\mathbf{A} \times \mathbf{B}|$ is the area of a parallelogram formed by **A** and **B** as the sides. If we think of the periphery of the parallelogram as described from the origin to head of **A** followed by the origin to head of **B**, then, in accordance with the definitions in the last section, **A** × **B** represents the vector area of the parallelogram.

The preceding geometric interpretation is the basis for a proof that vector multiplication follows the distributive law. Thus, consider the prism described in Fig. 1.6, whose sides are **A**, **B**, **A** + **B**, and **C**. Since the total surface is closed, the vector representing the total surface of the prism is zero (see Prob. 1.6). Consequently, taking the positive normal as directed outward, the sum of the component surface areas may be set equal to zero, giving

$$\tfrac{1}{2}(\mathbf{A} \times \mathbf{B}) + \tfrac{1}{2}(\mathbf{B} \times \mathbf{A}) + \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times (\mathbf{A} + \mathbf{B}) = 0 \qquad (1.12)$$

from which we obtain

$$\mathbf{C} \times (\mathbf{A} + \mathbf{B}) = \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B} \qquad (1.13)$$

Equation (1.13) expresses the distributive law of multiplication.

The vector product of two vectors can be expressed in terms of the rectangular components of each vector. Since the distributive law holds,

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) \times (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z) \\ &= \mathbf{a}_x \times \mathbf{a}_x A_x B_x + \mathbf{a}_x \times \mathbf{a}_y A_x B_y + \mathbf{a}_x \times \mathbf{a}_z A_x B_z \\ &\quad + \mathbf{a}_y \times \mathbf{a}_x A_y B_x + \mathbf{a}_y \times \mathbf{a}_y A_y B_y + \mathbf{a}_y \times \mathbf{a}_z A_y B_z \\ &\quad + \mathbf{a}_z \times \mathbf{a}_x A_z B_x + \mathbf{a}_z \times \mathbf{a}_y A_z B_y + \mathbf{a}_z \times \mathbf{a}_z A_z B_z \end{aligned} \qquad (1.14)$$

The sine of the angle between two vectors is zero when they are in the same or opposite directions and is ± 1 when they are orthogonal. It is thus easy to verify that

$$\begin{gathered} \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \qquad \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \qquad \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \\ \mathbf{a}_x \times \mathbf{a}_x = \mathbf{a}_y \times \mathbf{a}_y = \mathbf{a}_z \times \mathbf{a}_z = 0 \end{gathered} \qquad (1.15)$$

so that (1.14) simplifies to

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x) \qquad (1.16)$$

A convenient way of remembering the formula given by (1.16) is to note that it is obtained from the formal expansion of the following determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (1.17)$$

Once one term of the expansion is found, the remaining can be obtained by cyclical permutation; that is, replace x by y, y by z, and z by x. For example, the first term in (1.17) is $\mathbf{a}_x(A_y B_z - A_z B_y)$, from which the second term is found to be $\mathbf{a}_y(A_z B_x - A_x B_z)$ by replacing x, y, z by y, z, x, respectively.

1.6. The Scalar Product of Two Vectors

As mentioned, vector multiplication is useful in mathematically describing the relationship between vectors that arise out of a class of physical problems. In handling another class of physically related quantities, it will be desirable to define a scalar product of two vectors.