Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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A. Holme R. Speiser (Eds.)

Algebraic Geometry Sundance 1986

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PREFACE

This book presents some of the proceedings of the conference on Algebraic Geometry held at Sundance, in July 1986, in the mountains near Provo, Utah. Financial support, from the National Science Foundation (grant 86 – 01409) and from Brigham Young University, is gratefully acknowledged. Normally, a proceedings volume collects writups of lecures given at the conference, based on work done earlier, and the present volume, indeed, includes a number of these. We are very pleased, however, that the bulk of this volume presents research begun or carried out right at Sundance. Some of this new work may not have been done at all, had the conference not brought together the individuals involved. Beautiful surroundings, ample and contiguous spaces for lectures and discussions, meals served right beside the working areas: all contributed to an atmosphere unusually conducive to new work. But the major responsibility for the success of the conference lay, we feel, with the participants. Their enthusiasm, their interests, their eagerness, are reflected in the papers which follow. It is a pleasure to thank them here.

Audun Holme *Robert Speiser*

TABLE OF CONTENTS

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The characteristic numbers of smooth plane cubics

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March 1987

Abstract. The characteristic numbers for the family of smooth plane cubics are computed, verifying results of Maillard and Zeuthen

§1 **Introduction.** The last few years have witnessed a revived interest in the search for the 'characteristic numbers' of families, i.e. the numbers of elements in a family which are tangent to assortments of linear subspaces in general position in the ambient projective space. By the 'contact Theorem' of Fulton-Kleiman-MacPherson, these numbers determine the numbers of varieties in the family that satisfy tangency conditions to arbitrary configurations of projective varieties: this justifies the central role of the computation of the characteristic numbers in the field of enumerative geometry.

The problem received much attention in the last century, when in fact it contributed significantly to the development of algebraic geometry. Schubert's "Kalkül der abzählenden Geometrie" ([**S**]), published in 1879, is a compendium of the results obtained in a span of some decades by Schubert himself, Chasles, Halphen, Zeuthen and others. The validity of these achievements was soon questioned: in requesting rigorous foundations for algebraic geometry, Hilbert's 15th problem (1900) explicitly asked for a justification of the results in Schubert's book. Algebraic geometry found its foundations in the fifties; the challenge of justifying enumerative geometry had to wait somewhat longer to be accepted.

By now, most of the results in the "Kalkül der abzählenden Geometrie" have been verified or corrected, but the enterprise is not yet completed. While rich satisfactory theories are now available for quadrics (Van der Waerden, Vainsencher, Demazure, De Concini-Procesi, Laksov, Thorup-Kleiman, Tyrell, etc.) and triangles (Collino-Fulton, Roberts-Speiser), and much is known about twisted cubics (Kleiman-Strømme-Xambó), the families of plane curves still offer results which were 'known' in the last century and cannot be claimed such now.

The achievements of the classic school are here quite impressive. By 1864 Chasles (and others) had settled conics; already in 1871 a student of his, M.S. Maillard, computed in his thesis ([**M**]) the characteristic numbers for many families of plane cubic curves, including cuspidal, nodal, and smooth ones. One year later H.G. Zeuthen published a series of three amazingly short papers ([**Z1**]) again computing the numbers for cuspidal, nodal and smooth cubics; his results agree with Maillard's. Zeuthen finally published in 1873 a long analysis for plane curves of *any* degree ([**Z2**]), giving as an application the computation of the characteristic numbers for families of plane quartics. Apparently, noone ever tried to explicitly work out higher degree cases.

The problem for cubics or higher degree curves remained untouched - and therefore eventually unsettled- for at least one century. Then Sacchiero (1984) and Kleiman-Speiser (1985) verified Zeuthen and Maillard's results for *cuspidal* and *nodal* plane cubics. Kleiman and Speiser's approach replicates and advances Zeuthen and Maillard's, so it is expected to lead eventually to the verification of the numbers for the family of *smooth* cubics; but that program is not completed yet. Also, Sterz (1983) constructed a variety of 'complete cubics', by a sequence of 5 blow-ups over the $\mathbb{P}^9$ of plane cubics, giving some intersection relations ([**St**]).

Later, I independently constructed the same variety, by the same sequence of blow-ups. My approach was in a sense more 'geometric' than Sterz's, and I was able to use this variety to actually compute the characteristic numbers for the family of smooth plane cubics. The result once more agrees with Zeuthen and Maillard's.

There is an important difference between this approach and the classical one. Maillard and Zeuthen were computing the numbers by relating them to characteristic numbers of other more special families (e.g. cuspidal and nodal cubics); here, one aims directly to solving the specific problem for smooth cubics, and other families don't enter into play. This makes the problem more accessible in a sense, but it may on the other hand sacrifice the 'general picture' to the specific result.

In this note I describe the blow-up construction and the computation of the numbers. Full details appear, together with partial results for curves of higher degree, in my Ph.D. thesis ([**A**]), written at Brown under the supervision of W. Fulton.

Aknowledgements. I wish to thank A. Collino and W. Fulton for suggesting the problem, and for constant guidance and encouragement.

§2 The problem and the approach. Let n_p, n_ℓ be integers, with $n_p + n_\ell = 9$. The question to be answered is:

How many smooth plane cubics contain n_p points and are tangent to n_ℓ lines in general position?

The set of smooth plane cubics is given a structure of variety by identifying it with an open subvariety U of the $\mathbb{P}^9$ parametrizing all plane cubics. The conditions 'containing a point' and 'tangent to a line' determine divisors in U; call them 'point-conditions' and 'line-conditions' respectively. The question then translates into one of cardinality of intersection of n_p point-conditions and n_ℓ line-conditions in U.

One verifies that for general choice of points and lines the conditions intersect transversally in U, so that actually the cardinality of the intersection can be computed as intersection number of the divisors.

The first impulse is of course to work in the $\mathbb{P}^9$ that compactifies U: closing the conditions to divisors of $\mathbb{P}^9$ (one obtains hyperplanes from point-conditions, hypersurfaces of degree 4 from line-conditions), and using Bézout's Theorem to compute the intersection numbers. This works if $n_p \geq 5$: in this case the intersection of the divisors in $\mathbb{P}^9$ is in fact contained in U, and the result given by Bézout's Theorem is correct. If $n_p \leq 4$, non-reduced cubics appear in the intersection of the divisors in $\mathbb{P}^9$, since a curve containing a multiple component is 'tangent' to any line and clearly one can always find non-reduced cubics containing any 4 or less given points.

The conclusion is that $\mathbb{P}^9$ is not the 'right' compactification of the variety U of smooth cubics for this problem, because all line-conditions in $\mathbb{P}^9$ contain the locus of non-reduced cubics.

The intersection of all line-conditions is in fact a subscheme of $\mathbb{P}^9$ supported over the locus of non-reduced cubics. If we could blow-up $\mathbb{P}^9$ along this subscheme, this would provide us with a compactification of U in which the proper transforms of the point- and line-conditions don't intersect outside U, and taking their intersection product would answer the original question. But performing such a task requires much non-trivial information about the subscheme, and we are not able to proceed directly.

What we can perform without losing control of the situation is the blow-up of $\mathbb{P}^9$ along a certain smooth subvariety of the locus of non-reduced cubics. The blow-up creates another compactification of U, in which one can again find the support of the intersection of the 'line-conditions' (i.e., of the closure of the line-conditions of U). Again, a smooth subvariety -in fact, a component- of this locus can be chosen as a center of a new blow-up, creating a new compactification. The process can be repeated, under the heuristic principle that at each step, blowing-up the 'largest' possible non-singular subvariety/component of the intersection of all line-conditions should somehow simplify the situation.

In fact, 5 blow-ups do the job in this case: a non-singular compactification of U is produced in which 9 conditions intersect only inside U. The knowledge of the Chern classes of the normal bundles of the centers of the blow-ups is then the essential ingredient needed to compute the intersections and obtain the characteristic numbers. An intersection formula (see §4) that explicitly relates intersections under blow-ups can be used to reach the result.

Apparently, this step (the computation of the Chern classes of the normal bundles and their utilization to get the characteristic numbers) is the only one missing in Sterz's work.

Alternatively, one can use the same information to compute the Segre class of the scheme-theoretic intersection of all line-conditions in $\mathbb{P}^9$, and apply Fulton's intersection formula ([**F**, Proposition 9.1.1]). This Segre class has interesting symmetries, which may shed some light on the internal structure of this scheme.

§3 The blow-ups. In this section I will briefly describe the varieties obtained via the 5 blow-ups. Details are provided in [**A**, Chapter 2].

The diagram

$$
\begin{array}{ccccccc}
 & & \widetilde{V} = V_5 & & & & \\
 & & \downarrow & & & & \\
 & & V_4 & \longleftarrow & B_4 = \mathbb{P}(\mathcal{L}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & V_3 & \longleftarrow & B_3 = S_3 & \xleftarrow{\sim} & B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\
 & & \downarrow & & \downarrow & & \\
B_2 & \longrightarrow & V_2 & \longleftarrow & S_2 & \xleftarrow{\sim} & B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\
{\scriptstyle \mathbb{P}^3\text{-bundle}}\downarrow & & \downarrow & & \downarrow & & \\
B_1 & \longrightarrow & V_1 & \longleftarrow & S_1 & \xleftarrow{\sim} & B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\
{\scriptstyle \mathbb{P}^2\text{-bundle}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
v_3(\check{\mathbb{P}}^2) = B_0 & \longrightarrow & \mathbb{P}^9 = V_0 & \longleftarrow & S = S_0 & \longleftarrow & \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2
\end{array}
$$

contains most of the notations that will be explained in this section.

S_0 is the locus of non-reduced cubics, $B_0 = v_3(\check{\mathbb{P}}^2) \hookrightarrow \mathbb{P}^9$ is the Veronese of triple lines. B_i will be the centers of the blow-ups, V_i will be the blow-up $B\ell_{B_{i-1}} V_{i-1}$ of V_{i-1} along B_{i-1}, S_i will be the proper transforms of S_{i-1} under the i-th blow-up.

$\mathcal{L}$ is a certain sub-line bundle of the normal bundle $N_{B_3} V_3$ of B_3 in V_3. Δ is the diagonal in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

Also, E_i will be the exceptional divisor of the i-th blow-up, and 'line-conditions in V_i' will be the closure in V_i of the line-conditions of U: i.e., the line-conditions in V_i will be the proper transforms of the line-conditions in V_{i-1}.

For each blow-up I will describe the intersection of all line-conditions and indicate the choice of the center of the next blow-up. The basic strategy is to blow-up along the 'largest possible' non-singular subvariety/component of the intersection of all line-conditions. In fact, the first three blow-ups *desingularize* the support of this intersection, the last two separate the conditions.

§3.0 The $\mathbb{P}^9$ of plane cubics. We noticed already that the intersection of all line-conditions in $\mathbb{P}^9$ is supported on the locus S_0 of non-reduced cubics. This locus is the image of a map

$$\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \xrightarrow{\phi} \mathbb{P}^9$$

sending the pair of lines (λ, μ) to the cubic consisting of the line λ and of a double line supported on μ.

The map $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \xrightarrow{\phi} S_0$ is an isomorphism off the diagonal Δ in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$; therefore S_0 is non-singular off the (smooth) locus $B_0 = \phi(\Delta)$ of triple lines. In fact S_0 is singular along B_0.

B_0 is the center of the first blow-up.

§3.1 The first blow-up. Let V_1 be the blow-up of $\mathbb{P}^9$ along B_0, E_1 the exceptional divisor, S_1 the proper transform of S_0.

S_1 is isomorphic to the blow-up $B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the diagonal (call e the exceptional divisor of this blow-up); in particular, it is non-singular.

The line-conditions in V_1 intersect along the smooth 4-dimensional S_1 and along a smooth 4-dimensional subvariety of E_1.

To see this, notice that the line-condition in $\mathbb{P}^9$ corresponding to a line ℓ has multiplicity 2 along B_0, and tangent cone at a triple line λ^3 supported on the hyperplane of cubics containing $\lambda \cap \ell$. Thus, the tangent cones at λ^3 to all line-conditions in $\mathbb{P}^9$ intersect along the 5-dimensional space of cubics containing λ. It follows that the normal cones to B_0 in the line-conditions intersect in a rank-3 vector subbundle of $N_{B_0}\mathbb{P}^9$, and correspondingly that the line-conditions in V_1 intersect also along a $\mathbb{P}^2$-bundle over B_0 contained in E_1.

Call this subvariety B_1, and choose it as the center for the next blow-up. B_1 intersects $S_1 \cong B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the exceptional divisor e.

§3.2 The second blow-up. Let V_2 be the blow-up of V_1 along B_1, E_2 the exceptional divisor, $\widetilde{E}_1, S_2$ the proper transforms of E_1, S_1 respectively.

S_2 is the blow-up of S_1 along a divisor, thus it is isomorphic to S_1 and hence to $B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

A coordinate computation shows that the line-conditions in V_1 are generically smooth along B_1, and tangent to E_1. As a consequence, their proper transforms intersect in E_2 along $\widetilde{E}_1 \cap E_2$, which is a $\mathbb{P}^3$-bundle over B_1 contained in E_2.

Therefore the line-conditions in V_2 intersect along the smooth 4-dimensional S_2 and along a smooth 7-dimensional subvariety of E_2.

Choose this subvariety as the new center, call it B_2.

§3.3 The third blow-up. Let V_3 be the blow-up of V_2 along B_2, E_3 the exceptional divisor, S_3 the proper transform of S_2.

Again, S_3 is isomorphic to $B\ell_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

E_3 is a $\mathbb{P}^1$-bundle over B_2. In each fiber of this bundle there are two distinguished distinct points r_1, r_2: namely the intersections with the proper transforms of $\widetilde{E}_1$ and E_2. Now, over any point in B_2 away from $S_3 \cap E_3$, one can find line-conditions that hit the fiber precisely at r_1 or precisely at r_2. This implies that over such points the line-conditions in V_3 cannot intersect.

Thus the line-conditions in V_3 intersect only along the smooth 4-dimensional S_3.

This completes the 'desingularization of the support' of the intersection of all line-conditions, and we are ready to choose $B_3 = S_3$ as the next center.

§3.4 **The fourth blow-up.** Let V_4 be the blow-up of V_3 along B_3, E_4 the exceptional divisor.

The line-conditions in V_4 meet along a subvariety of the exceptional divisor $E_4 = \mathbb{P}(N_{B_3}V_3)$. Notice that above $B_3 - E_3 \cong S_0 - B_0$, E_4 restricts to $\mathbb{P}(N_{S_0-B_0}\mathbb{P}^9)$. Now, the tangent hyperplanes to the line-conditions in $\mathbb{P}^9$ at a non-reduced cubic $\lambda\mu^2 \in S_0 - B_0$ intersect in the 5-dimensional space of cubics containing μ. It follows that the line-conditions in V_4 meet above $B_3 - E_3$ along the projectivization of a line-subbundle of $\mathbb{P}(N_{B_3-E_3}V_3)$. This fact holds on the whole of B_3: the line-conditions in V_4 intersect along a smooth 4-dimensional subvariety of $E_4 = \mathbb{P}(N_{B_3}V_3)$, which is the projectivization $\mathbb{P}(\mathcal{L})$ of a line-subbundle of $N_{B_3}V_3$.

Choose $\mathbb{P}(\mathcal{L})$ to be the next center B_4.

§3.5 **The fifth blow-up.** Let V_5 be the blow-up of V_4 along B_4, E_5 the exceptional divisor, $\widetilde{E}_4$ the proper transform of E_4.

Finally, the intersection of all line-conditions is empty in V_5.

The verification of this fact is similar to the one in 3.3. Here, each fiber of E_5 over a point of B_4 is a 4-dimensional projective space; in this $\mathbb{P}^4$ lies a distinguished $\mathbb{P}^3$, namely the intersection of the fiber with $\widetilde{E}_4$. Now, one can produce line-conditions whose intersection is disjoint from this $\mathbb{P}^3$, and a line-condition which intersects the fiber precisely along this $\mathbb{P}^3$. Thus the intersection of the line-conditions must be empty.

V_5 is the compactification of U we were looking for.

By slightly refining the arguments, one sees that the intersection of 9 point/line-conditions in general position in V_5 must be contained in U. The characteristic numbers are then the intersection numbers of the conditions in V_5, and one can proceed with the actual computation.

§4 **The numbers.** The essential ingredients to obtain the characteristic numbers from the construction in §3 are the Chern classes of the normal bundles of the centers of the blow-ups. In fact this information would be enough to determine the whole Chow ring of the blow-ups; but we don't need that much. We have 9 divisors in $\mathbb{P}^9$, and we wish to compute the intersection numbers of their proper transforms in some blow-up of $\mathbb{P}^9$, once the Chern classes of the normal bundles of the centers are known.

This task can be accomplished directly, by repeatedly applying the

PROPOSITION. *Let V be a non-singular n-dimensional variety, $B \overset{i}{\hookrightarrow} V$ a non-singular closed subvariety of V, $X_1, \dots, X_n$ divisors on V. Let $\widetilde{V} = B\ell_B V$, and $\widetilde{X}_1, \dots \widetilde{X}_n$ the proper transforms of $X_1, \dots X_n$. Moreover, let $e_i = e_B X_i$ be the multiplicity of X_i along B. Then*

$$\int_{\widetilde{V}} \widetilde{X}_1 \cdots \widetilde{X}_n = \int_V X_1 \cdots X_n - \int_B \frac{(e_1[B] + i^*[X_1]) \cdots (e_n[B] + i^*[X_n])}{c(N_B V)}.$$

This specializes to well-known formulas when B is a point, and is itself a specialization of a more general relation among Segre classes (see [**A**, Chapter 1]). An elementary proof of the form stated here can be obtained by expanding

$$\int_V X_1 \cdots X_n = \int_{\widetilde{V}} ([\widetilde{X}_1] + e_1[E]) \cdots ([\widetilde{X}_n] + e_n[E])$$

(E is the exceptional divisor) and recalling that $\sum_{i \geq 0}[E]^i$ pushes forward to $c(N_B V)^{-1}$ by Corollary 4.2 and Proposition 4.1(a) in [**F**].

What we need to compute the intersection numbers of the conditions in V_5 is then, for each V_i:

(1) The Chern classes of $N_{B_i} V_i$;

(2) The multiplicities of the conditions in V_i along B_i;

(3) The Chow ring of B_i.

We will now indicate how this information can be obtained.

As for the multiplicities, they are obtained along the construction: the line-conditions in $\mathbb{P}^9$ have multiplicity 2 along the locus B_0 of triple lines, while line-conditions in V_i, $i > 0$, are generically smooth (hence have multiplicity 1) along B_i. Also, point-conditions never contain B_i, so their multiplicities along the centers are always 0.

The Chow rings and the normal bundles of the centers can be obtained as follows.

B_0 is the locus of cubics consisting of 'triple lines', hence it is isomorphic to $\mathbb{P}^2$; call h the hyperplane class in B_0. In fact B_0 is the third Veronese imbedding of $\mathbb{P}^2$ in $\mathbb{P}^9$: it follows that

$$c(N_{B_0}\mathbb{P}^9) = \frac{(1+3h)^{10}}{(1+h)^3}.$$

B_1 is a $\mathbb{P}^2$-bundle over B_0, thus its Chow ring is generated by the pull-back h of h from B_0 and the class ϵ of the universal line bundle $\mathcal{O}_{B_1}(-1)$. A closer analysis of the situation (see §3.1) reveals that B_1 is actually isomorphic to the projectivization of the normal bundle to the locus of double lines in the $\mathbb{P}^5$ of conics. This determines the relations between h and ϵ, and gives substantial information about the imbedding $B_1 \hookrightarrow E_1$. $N_{B_1} V_1$ is an extension of $N_{B_1} E_1$ and $N_{E_1} V_1$, and one obtaines

$$c(N_{B_1} V_1) = (1+\epsilon)\frac{(1+3h-\epsilon)^{10}}{(1+2h-\epsilon)^6}.$$

B_2 is a $\mathbb{P}^3$-bundle over B_1: its Chow ring is generated by the pull-backs h, ϵ of h, ϵ from B_1 and by the class φ of $\mathcal{O}_{B_2}(-1)$. Recall from 3.2 that $B_2 = \widetilde{E}_1 \cap E_2$: i.e., B_2 is the exceptional divisor in the blow-up of E_1 along B_1, and hence it is isomorphic to $\mathbb{P}(N_{B_1} E_1)$. This observation gives relation among h, ϵ, φ. Also, $B_2 = \widetilde{E}_1 \cap E_2$ gives at once

$$c(N_{B_2} V_2) = (1+\varphi)(1+\epsilon-\varphi).$$

$B_3 = S_3$ is isomorphic to the blow-up $B\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the diagonal. Its Chow ring is then generated by the pull-backs ℓ, m of the hyperplanes from the factors, and by the exceptional divisor e. One obtaines the relations

$$\int_{B_3} \ell^2 m^2 = 1, \quad \int_{B_3} e^2 \ell^2 = -1, \quad \int_{B_3} e^2 m^2 = -1,$$

$$\int_{B_3} e^3 \ell = -3, \quad \int_{B_3} e^3 m = -3, \quad \int_{B_3} e^4 = -6.$$

The total Chern class of $N_{B_3}V_3$ can be obtained as $\frac{c(TV_3)}{c(TB_3)}$: both $c(TV_3)$ and $c(TB_3)$ can be computed using the formula for Chern classes of blow-ups (Theorem 15.4 in [**F**]). The result is

$$c(N_{B_3}V_3) = 1 + 7\ell + 17m - 16e + 126m^2 + 99\ell m + 21\ell^2 - 315e\ell + 105e^2 + 582\ell m^2 + 237\ell^2 m - 2517e\ell^2 + 1611e^2\ell - 358e^3 + 1026\ell^2m^2 + 9174e^2\ell^2 - 3912e^3\ell + 652e^4.$$

Finally, $B_4 = \mathbb{P}(\mathcal{L})$ is also isomorphic to $\mathrm{B}\ell_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$; the Chern classes of $N_{B_4}V_4$ are easily obtained from $c_1(\mathcal{L})$, which can be computed directly as $3\ell + 3m - 4\epsilon$. One gets

$$c(N_{B_4}V_4) = 1 - 5\ell + 5m + 18m^2 - 27\ell m + 3\ell^2 + 21e\ell - 7e^2 - 30\ell m^2 + 75\ell^2 m - 225e\ell^2 + 135e^2\ell - 30e^3 + 75\ell^2 m^2.$$

Once this information is obtained, 5 applications of the proposition for each number n_p of points and n_ℓ of lines give the corresponding characteristic number. For example, the reader may now enjoy checking by hand that

numbers of smooth cubics through 4 points and tangent to 5 lines =

$$= 4^5 - 0 - 0 - 0 - 24 - 24 = 976,$$

or that

numbers of smooth cubics through 3 points and tangent to 6 lines =

$$= 4^5 - 0 - 0 - 0 - 390 - 282 = 3424.$$

The final result is the list

1	$n_p = 9, n_\ell = 0$
4	$n_p = 8, n_\ell = 1$
16	$n_p = 7, n_\ell = 2$
64	$n_p = 6, n_\ell = 3$
256	$n_p = 5, n_\ell = 4$
976	$n_p = 4, n_\ell = 5$
3424	$n_p = 3, n_\ell = 6$
9766	$n_p = 2, n_\ell = 7$
21004	$n_p = 1, n_\ell = 8$
33616	$n_p = 0, n_\ell = 9$

for the number of curves containing n_p points and tangent to n_ℓ lines, agreeing with Maillard and Zeuthen.

§5 **Concluding remarks.** It seems plausible that the same procedure worked out here for cubics could in principle be executed to get the characteristic numbers for smooth quartics or for higher degree plane curves, but the usefulness of such an endeavor is questionable at this point. Until these 'blow-up constructions' are part of a general theory, the complication of the technical details is bound to keep the work at the level of brute force computation. Part of the construction (essentially the last two blow-ups) can in fact be carried out, giving the first 'non-trivial' characteristic number for smooth plane curves of any degree (see [**A**, Chapter 3]), but this seems to be in some sense a special case. The next 'non-trivial' number can still be computed for quartics (the results agree with Zeuthen's!), but not via a straightforward generalization from the computation for cubics ([**A**, Chapter 4]).

Perhaps Kleiman and Speiser's approach, pointing in the direction of Zeuthen's monumental 'general theory', will strike more deeply into the heart of the problem.

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MULTIPLE-POINT FORMULAS AND LINE COMPLEXES

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0. Introduction

Suppose $f: X \to Y$ is a map of schemes. An *r-fold point* of f is a point x_1 of X such that there exist $x_2, \dots, x_r \in X$ with $f(x_1) = f(x_2) = \dots = f(x_r)$. Call x_1 a *strict* (or *ordinary*) r-fold point if all of the x_i's are distinct and call x_1 a *stationary* r-fold point if any of the x_i's have coalesced to become ramification points of f. (We shall say that such points lie "infinitely near" each other. See 1.7 below.) Modern multiple-point theory in algebraic geometry concerns the determination of the various loci of these singularities and the development of enumerative formulas for the intersection classes of them.

We will not attempt to give a complete recent history of the subject, but instead mention only a few of the principal figures and ideas. Contemporary multiple-point theory began with Laksov [L], who revived and refined the double-point formula of Todd. Further refinements were made by both Fulton and Laksov [F-L]. A general treatment of ordinary higher-order theory and the so-called "method of iteration" was initiated by Kleiman in [K1] and [K2]. Alternative approaches to multiple-point theory and applications thereof, using the Hilbert scheme, have also been developed, principally by Kleiman [K3] and Le Barz [LB1], [LB2]. Roberts, in the context of the iteration method, began some of the work on stationary multiple-points [Ro]. Most recently, Ran [Ra] has taken a new approach towards iteration to give, in the smooth case, a treatment of both ordinary and stationary multiple-point formulas which are valid under substantially weaker hypotheses than those given by Kleiman. However, Ran does not give any mechanical procedures for generating formulas in $A^{\bullet}(X)$. It is important to note that none of the techniques mentioned above yields a satisfactory general treatment of multiple-points of maps which have $\bar{S}_2$-singularities.

A wide variety of enumerative problems may be tackled by casting the problems in terms of the determination of appropriate multiple-point singularities of suitable maps. Let us cite some examples. One can recover Clebsch's formula $\delta = 1/2(d-1)(d-2) - g$ for the number δ of nodes of a plane curve of degree d and genus g having only nodes for singularities from the double-point formula applied to the normalization map of the curve. The Riemann-Hurwitz formula is nothing more than a special case of a general stationary double-point formula (see the formula for $n_{(2)}$ in §3). Finally, other formulas, both classical and new, for lines having prescribed contact with varieties in $\mathbb{P}^n$ may also be deduced (see [C1], [LB1], [LB2]).

This article consists of a sketch of an adaptation of Kleiman's iteration method in [K2] to generate stationary multiple-point formulas, and also a sketch of a new application of the theory to the problem of computing certain coincidence formulas for line complexes (see [Sch], §36). We outline the major steps: detailed proofs of the results in §§1-3 appear in [C2]. The general set-up, the necessary notation, and the definition of the stationary multiple-point classes are given in §1. In §2 we describe how these classes should be interpreted and under what conditions the resulting formulas are valid. §3 consists of the formulas themselves and a description of the main ingredients of the computations. Finally, in §4 we give an outline of the application of stationary multiple-point formulas to the line complex problem.

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1. Set-Up and Notation (see [K2], §4 and [C2], §§1-2)

Let $f: X \to Y$ be a separated map of schemes. Consider the following inductive construction: set $X_0 := Y$, $X_1 := X$, $f_0 := f$ and, for $r \geq 1$, define new spaces X_{r+1} and maps $f_r: X_{r+1} \to X_r$ from the diagram below.

$$\begin{array}{ccccc}
 & & X_{r+1} := \mathbb{P}(I_r) & \xleftarrow{\;j_{r+1}\;} & E_{r+1} := p^{-1}(\Delta_r) \\
 & & \downarrow p & & \downarrow \\
X_r & \xleftarrow{\;p_1\;} & X_r \times_{X_{r-1}} X_r & \longleftarrow & \Delta_r, \quad \text{ideal sheaf } I_r \\
\downarrow f_{r-1} & \square & \downarrow p_2 & & \\
X_{r-1} & \xleftarrow[\;f_{r-1}\;]{} & X_r & & f_r := p_2 p.
\end{array} \tag{1.1}$$

This construction defines X_{r+1} as the *residual scheme* of the diagonal Δ_r in $X_r \times_{X_{r-1}} X_r$. Note that when Δ_r is regularly embedded in the fibred product (as happens, for example, if f is a smooth morphism), then X_{r+1} is the same as the blowup of the diagonal in the fibred product ([K2], 2.2, p. 28). The exceptional set E_{r+1} equals $\mathbb{P}(I_r/I_r^2)$ and has $\mathcal{O}_{X_{r+1}}(1)$ for ideal sheaf. However, E_{r+1} need not, in general, be a divisor in X_{r+1}.

For $r \geq 1$ define the *switch involution* $i_r\colon X_{r+1} \to X_{r+1}$ to be the natural covering of the self-map on $X_r \times_{X_{r-1}} X_r$ that reverses coordinates. Then i_r has the following properties:

$$i_r \mid E_{r+1} = \mathrm{id} \tag{1.2}$$

$$i_r^{*}\mathcal{O}_{X_{r+1}}(1) = \mathcal{O}_{X_{r+1}}(1), \tag{1.3}$$

$$f_r i_r = p_1 p \text{ in } (1.1), \tag{1.4}$$

if f is proper, Y noetherian, then for $r \geq 2$, $s \geq 1$

$$i_{s*}(f_{s+1} \cdots f_{r+s-1})_*[X_{r+s}] = (f_{s+1} \cdots f_{r+s-1})_*[X_{r+s}]. \tag{1.5}$$

(Note that f proper implies that f_s is, too, for $s \geq 1$.)

The scheme X_r may be seen to parametrize ordered r-tuples of points in the fibres of f, including those points which lie "infinitely near" one another. In order to identify r-tuples of points in particular infinitely near configurations, we need the following.

Definition 1.6. Let $\underline{a} = (a_1, \ldots, a_k)$ be a nondecreasing partition of r. Set $b_s = \sum_{\zeta=s}^{k} a_\zeta$ and define $T_{\underline{a}} \subset X_r$ by

$$T_{\underline{a}} := (f_{r-1}j_r)^{-1} \cdots (f_{b_2+1}j_{b_2+2})^{-1} f_{b_2}^{-1} (f_{b_2-1}j_{b_2})^{-1} \cdots (f_{b_3+1}j_{b_3+2})^{-1} f_{b_3}^{-1}$$
$$\cdots f_{a_k}^{-1}(f_{a_k-1}j_{a_k})^{-1} \cdots (f_1 j_2)^{-1}(X).$$

Note that if $\underline{a} = (1,1,\ldots,1)$, then $T_{\underline{a}} = f_{r-1}^{-1} \cdots f_1^{-1}(X) = X_r$.

Definition/Proposition 1.7 ([C2], 2.3). A geometric point of $N_{\underline{a}} := f_1 i_1 \cdots f_{r-1} i_{r-1}(T_{\underline{a}})$ is a point $x \in X$ such that there exist $r - 1$ other points of the fibre through x and also such that

a_1 of the points (including x itself) are infinitely near each other,

a_2 of the remaining $r - a_1$ points are infinitely near each other,