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BY

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FOURIER SERIES

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PREFACE

This tract is based on lectures which each of us has given in Cambridge or elsewhere. There are already a good many books on the subject; but we think that there is still room for one which is written in a modern spirit, concise enough to be included in this series, yet full enough to serve as an introduction to Zygmund's standard treatise.

We have not written for physicists or for beginners, but for mathematicians interested primarily in the theory and with a certain foundation of knowledge. In particular we assume acquaintance with the elements of Lebesgue's theory of integration: it is impossible to understand the theory of Fourier series properly without it, and experience shows that it is well within the powers of any clever undergraduate. The actual knowledge needed here can be acquired quite easily from chapters x–xii of Titchmarsh's *Theory of Functions*. As regards the theory of trigonometrical series, the book is 'officially' complete in itself; but we recognize unofficially that practically all our readers will have some knowledge of the subject (such as the substance of Titchmarsh's chapter xiii) already.

We have naturally been forced to omit much which we should have liked to include. In particular we have no space for the inequalities of Young and Hausdorff, Marcel Riesz's theorem concerning conjugate series, theorems concerning Cesàro summation of general order, or uniqueness theorems involving summable series. And we give no results about special series except a few which we require to illustrate the general theory.

The notes at the end are not systematic; we have inserted only such references and comments as we could make shortly and seemed to us likely to be useful. In particular we make no attempt to give any adequate idea of the history of the subject: Euler, Fourier himself, Poisson and Dirichlet are hardly mentioned. It is quite impossible in an account like this to do any justice to the great mathematicians who founded the theory.

We have to thank Miss S. M Edmonds, Dr W. H. J. Fuchs, Dr A. J. Macintyre, and Dr A. C. Offord for assistance with the proof-sheets and many valuable criticisms.

G. H. H.
W. W. R.

Cambridge and Aberdeen, Sept. 1943

PREFACE TO SECOND EDITION

G. H. Hardy is no more. British Mathematics has lost its undisputed leader; my Refugee colleagues in this country mourn in him the sincere humanitarian who offered understanding sympathy, advice, and assistance in difficult times; and I myself, if I may claim it, miss a real friend.

No doubt the reader of this Tract will notice in its style the hand of the master: the final draft was written by Hardy. I have not interfered with it in the new edition. A few misprints and mistakes were pointed out to us by various colleagues; the worst are the 'slip' in the proof (ii) of Theorem 31, and the false form of Theorem 59. These are now corrected.

W. W. R.

Newcastle upon Tyne, June 1949

PREFACE TO THIRD EDITION

A few remaining mistakes are now corrected, amongst them the wrong numerical value of the Gibbs constant. An unusual form of Egoroff's theorem has been employed in the proof of Theorem 89. This form is now explicitly stated in § 1.4 and its proof indicated in the Notes.

W. W. R.

Newcastle upon Tyne, September 1955

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We use the abbreviations t.s. (1), c.s. (3), F.s., F.c. (4), p.p. (5), o.s., n.o.s. (11) in senses defined on the pages indicated in brackets.

The O, o notation is used in the manner now customary: see, e.g., Hardy, *Pure Mathematics* (ed. 8, Cambridge, 1941), p. 164. The symbol $\sim$ is used occasionally (e.g. p. 49) for asymptotic equality, but more generally to express the relation of a function to its F.s.

$\bar{z}$ is the conjugate complex of z. $[x]$ is the integral part of x. Min (x, y) and Max (x, y) are the lesser and greater of x and y. For Max $|f|$ see p. 7.

$\sum_{\alpha}^{\beta} u_n$ denotes a sum over $\alpha \leqq n \leqq \beta$, whether α and β are integral or not. If $\beta < \alpha$, it is 0. We sometimes omit limits in sums and integrals, when it is clear what they are.

$\langle a, b \rangle$ is the closed interval $a \leqq x \leqq b$. We use this symbol only when the closure of the interval is important, using (a, b) when the interval is open or when the distinction is irrelevant.

H is a 'constant', i.e. a number independent of the parameters of the argument, whose precise value is immaterial. We sometimes use ϵ for 'any positive number' (with the emphasis on its possible smallness) without special explanation (e.g. pp. 7, 14).

A new symbol occurring for the first time in a formula without explanation is *defined by* the formula. Thus $C_n(\theta)$ is defined as $c_n e^{ni\theta}$ by (1.1.8), p. 1.

CONTENTS

I. GENERALITIES

1.1. Trigonometrical series. A trigonometrical series (t.s.) is a series of the form

$$\tfrac{1}{2}A_0(\theta)+\sum_1^{\infty} A_n(\theta), \tag{1.1.1}$$

where

$$A_0(\theta)=a_0, \quad A_n(\theta)=a_n\cos n\theta+b_n\sin n\theta \quad (n>0). \tag{1.1.2}$$

We call this series $T(\theta)$ or simply T.

The partial sum of $T(\theta)$, of rank n, is

$$s_n(\theta)=\tfrac{1}{2}A_0(\theta)+\sum_1^{n} A_m(\theta). \tag{1.1.3}$$

The coefficients a_n and b_n are given, in the first instance, for $n \geqq 0$ and $n \geqq 1$ respectively. We define a_n and b_n, for other integral values of n, by

$$a_{-n}=a_n \ (n>0), \quad b_0=0, \quad b_{-n}=-b_n \ (n>0), \tag{1.1.4}$$

and c_n by

$$c_n=\tfrac{1}{2}(a_n-ib_n); \tag{1.1.5}$$

so that

$$a_n=c_n+c_{-n}, \quad b_n=i(c_n-c_{-n}). \tag{1.1.6}$$

Conversely, if the c_n are given, we may define a_n and b_n by (1.1.6). Then

(1.1.7)

$$s_n(\theta)=c_0+\sum_1^{n}\{(c_m+c_{-m})\cos m\theta+i(c_m-c_{-m})\sin m\theta\}=\sum_{-n}^{n} c_m e^{mi\theta}.$$

We may therefore also define $T(\theta)$ as

$$\sum_{-\infty}^{\infty} C_n(\theta)=\sum_{-\infty}^{\infty} c_n e^{ni\theta}, \tag{1.1.8}$$

and $s_n(\theta)$ by (1.1.7).

We shall call (1.1.1) a 'real' and (1.1.7) a 'complex' t.s. The adjectives refer to the trigonometrical or exponential functions which occur in the series. The coefficients a_n and b_n in (1.1.2) may be complex; but we may suppose them real, when this is convenient, by considering the real and imaginary parts of $T(\theta)$ separately. The series are formal series: there is no implication of their convergence for all or any θ. But (1.1.8) should be thought of as a limiting form of (1.1.7), i.e. as a series in some sense 'equally extended' in the positive and negative directions.

In the simplest cases the series have a sum function $f(\theta)$, and their coefficients can be expressed simply in terms of $f(\theta)$. Suppose, for example, that the series are uniformly convergent. Then, multiplying by $\cos m\theta$ and $\sin m\theta$ or, in the complex case, $e^{-mi\theta}$, integrating term by term over $(-\pi, \pi)$, using the familiar formulae

$$(1.1.9)\quad \begin{cases} \displaystyle\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{matrix} 0 \ (m \neq n) \\ \pi \ (m = n \neq 0), \\ 2\pi \ (m = n = 0) \end{matrix} \\ \displaystyle\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = \begin{matrix} 0 \ (m \neq n) \\ \pi \ (m = n \neq 0), \\ 0 \ (m = n = 0) \end{matrix} \\ \displaystyle\int_{-\pi}^{\pi} \cos m\theta \sin n\theta \, d\theta = 0, \\ \displaystyle\int_{-\pi}^{\pi} e^{(n-m)i\theta} \, d\theta = \begin{matrix} 0 \ (m \neq n) \\ 2\pi \ (m = n) \end{matrix}, \end{cases}$$

and finally replacing m by n, we find that

$$(1.1.10)\quad \begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta, \quad b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta, \\ c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta) \, e^{-ni\theta} \, d\theta. \end{cases}$$

If f is real, a_n and b_n are real, c_n and c_{-n} conjugate. If f is even, $b_n = 0$ and

$$(1.1.11)\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta) \cos n\theta \, d\theta.$$

If f is odd, $a_n = 0$ and b_n may be reduced similarly.

1.2. Trigonometrical series and harmonic functions. It will be useful to begin by indicating the formal connections between the theory of t.s. and the general theory of harmonic and analytic functions, of which in a sense it is a part. In what follows $z = x + iy = re^{i\theta}$ is a complex variable and $\bar{z} = x - iy = re^{-i\theta}$ its conjugate. To fix our ideas, we suppose a_n and b_n *real* (so that

c_n and c_{-n} are conjugate). We also suppose a_n and b_n *bounded* (as they will be in nearly all our work). The power series in r which we write down will then be convergent for $r<1$ and, for a fixed r, uniformly in θ.

If

$$(1.2.1)\qquad u(r,\theta)=\sum_{-\infty}^{\infty}c_n r^{|n|}e^{ni\theta}=c_0+\sum_1^{\infty}c_n r^n e^{ni\theta}+\sum_1^{\infty}c_{-n}r^n e^{-ni\theta},$$

then u is a harmonic function, a solution of either of the equations

$$(1.2.2)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad \left(r\frac{\partial}{\partial r}\right)^2 u+\frac{\partial^2 u}{\partial\theta^2}=0.$$

It is real and regular for $r<1$. We can also write

$$(1.2.3)\qquad u(r,\theta)=\tfrac{1}{2}a_0+\sum_1^{\infty}A_n(\theta)\,r^n,$$

and $T(\theta)$ is the result of writing $r=1$ in either form of u. Now

$$(1.2.4)\qquad u(r,\theta)=\tfrac{1}{2}\{F(z)+\overline{F(z)}\},$$

where

$$(1.2.5)\qquad F(z)=c_0+2\sum_1^{\infty}c_n z^n.$$

Thus is the real part of $F(z)$. Also, if we write

$$(1.2.6)\qquad B_n(\theta)=b_n\cos n\theta-a_n\sin n\theta,$$

we have

$$(1.2.7)\qquad F(z)=u(r,\theta)-iv(r,\theta),$$

where

$$(1.2.8)\qquad v(r,\theta)=\sum_1^{\infty}B_n(\theta)\,r^n.$$

We shall call u and v *conjugate harmonic* functions, and the series

$$(1.2.9)\qquad \tilde{T}(\theta)=\sum_1^{\infty}B_n(\theta),$$

obtained by writing $r=1$ in (1.2.8), the *conjugate series* (c.s.) of $T(\theta)$.

It will be convenient to prove here two formulae needed in Ch. III. If $C_0=c_0$, $C_n=2c_n$ for $n>0$, so that $F(z)=\Sigma C_n z^n$, and $r<1$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(u-iv)\,e^{-ni\theta}\,d\theta=C_n r^n\quad(n\geqq 0),\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}(u-iv)\,e^{ni\theta}\,d\theta=0\quad(n>0).$$

Hence (combining the first equation with the conjugate of the second)

$$(1.2.10)\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} u\,d\theta = \mathbf{R}(C_0),\quad \frac{1}{\pi}\int_{-\pi}^{\pi} ue^{-ni\theta}\,d\theta = \frac{1}{i\pi}\int_{-\pi}^{\pi} ve^{-ni\theta}\,d\theta = C_n r^n$$

for $n > 0$. Actually, our C_0 here is real.

1.3. Trigonometrical Fourier series. Our proof of the formulae (1.1.10) depended on the hypothesis that $T(\theta)$ was uniformly convergent, a drastic assumption unlikely to be satisfied by a t.s. chosen at random. The formulae themselves suggest that we should look at the series from a quite different point of view.

We start from a (real or complex) function $f(\theta)$ integrable (in the sense of Lebesgue) in the interval $(-\pi, \pi)$. It is then convenient to define $f(\theta)$, for all real θ, as a function with period 2π, so that $f(\theta+2\pi) = f(\theta)$, and in particular $f(\pi) = f(-\pi)$, whenever $f(\theta)$ is defined for one of the values of θ in question.

We now define a_n, b_n, and c_n by (1.1.10). We call a_n and b_n the 'real', c_n the complex, *Fourier constants* (F.c.) of $f(\theta)$, and (1.1.1) or (1.1.8) its *Fourier series* (F.s.). We express the fact that a_n and b_n are the F.c. of $f(\theta)$, and (1.1.1) its F.s., by writing $f \sim (a_n, b_n)$ or

$$(1.3.1)\qquad f(\theta) \sim \tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Similarly we write $f \sim (c_n)$ or $f(\theta) \sim \Sigma c_n e^{ni\theta}$, and call (1.1.1) the 'real', (1.1.8) the 'complex', F.s. of $f(\theta)$. We shall sometimes write $T(f)$ for the F.s. of f, and $\bar{T}(f)$ for the conjugate t.s.

Since all the functions in (1.1.10) are periodic, we can replace the range of integration by any range $(\xi, \xi+2\pi)$. In particular, it is often convenient to regard $(0, 2\pi)$, rather than $(-\pi, \pi)$, as the fundamental interval.

To say that a t.s. is a F.s. is to say that its coefficients a_n, b_n, or c_n are expressible in the form (1.1.10), that is to say that a certain system of integral equations has a solution. It is plain that the meaning of this statement depends on the definition of integration which we are using. We have adopted Lebesgue's: any restriction or enlargement of the definition of integration would lead to a corresponding change in the class of F.s.

We shall see, for example, that the series

$$(1.3.2)\quad \tfrac{1}{2} + \cos\theta + \cos 2\theta + \dots,\qquad (1.3.3)\quad \frac{\sin 2\theta}{\log 2} + \frac{\sin 3\theta}{\log 3} + \dots$$

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are not, in our sense, F.s.; but the coefficients of each series can be expressed in 'Fourier form' by an appropriate generalization of the notion of an integral. Those of (1.3.2) can be expressed in the form

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta\, d\phi(\theta),$$

a 'Stieltjes integral' in which $\phi(\theta)$ is $-\frac{1}{2}\pi$, 0, and $\frac{1}{2}\pi$ for negative, zero, and positive θ; and those of (1.3.3) in the form (1.1.10), $f(\theta)$ being the sum of the series and the integral for a_n (which is 0) a 'principal value' in the sense of Cauchy.

A t.s. may or may not converge, and it may or may not be a F.s.; and there is no obvious correlation between the two properties (though the simplest series may be expected to have both). The series (1.3.3) converges for all θ, but is not a F.s.; on the other hand there are F.s. which do not converge for any θ. It is not even plain *a priori* that a t.s., known to converge and to be a F.s., is necessarily the F.s. of its sum.

Trigonometrical series are a special class of *orthogonal* series; and there is a considerable part of their theory which is best regarded as part of the theory of these more general series, and which we shall develop from this point of view in Ch. II. But we must begin by a short résumé of certain parts of the theory of functions of a real variable with which we shall assume the reader to be acquainted.

1.4. Measure and integration. We take as known the elements of Lebesgue's theory of measure and integration. We denote by $L(a, b)$, or simply L, the class of functions $f(x)$ integrable, in Lebesgue's sense, over (a, b). The interval of integration will always be finite. We shall sometimes say 'f is L' for 'f belongs to L'. We regard the integral of a *non-negative* function f as defined whenever f is measurable, and having a finite or infinite value according as f is or is not L.

We call a set of measure 0 a *null set*: null sets are irrelevant in the theory of integration. If f and g differ only in a null set, we say that they are *equivalent*, and write $f \equiv g$. We also say that $f = g$ for almost all x, or almost always (or almost everywhere), or 'p.p.' (*presque partout*). If $f \equiv 0$, we say that f is *null*. We write mE for the measure of a set E.

We shall sometimes use letters for other classes of functions besides L; in particular we shall denote by B, C, C_k, and V the classes of bounded functions, continuous functions, functions with k continuous derivatives, and functions of bounded variation*.

We take for granted the classical theorems concerning integration and differentiation, the theorems of partial integration and substitution, the first and second mean value theorems, and the two most familiar theorems concerning passage to the limit under the integral sign. These last are: (i) if $f_n(x) \to f(x)$ p.p. and $|f_n(x)| \leqq \phi(x)$, where $\phi(x)$ is L and independent of n, then

$$\int f_n(x)\,dx \to \int f(x)\,dx; \tag{1.4.1}$$

(ii) the conclusion is also true if $f_n(x)$ increases with n for all, or almost all, x provided that $\int f_n(x)\,dx \neq -\infty$. In case (i) we shall say that $f_n(x)$ converges *dominatedly* to $f(x)$. In particular the conditions are satisfied if $f_n(x) \to f(x)$ for all x, and $|f_n(x)| \leqq H$, in which case we shall say that $f_n(x)$ converges *boundedly* to $f(x)$. In case (ii) it is to be understood that the limit function $f(x)$, which certainly exists p.p., may be infinite for some x, and that $\int f(x)\,dx$ may also be infinite, in which case it is to be replaced by ∞ in (1.4.1). Finally, the integrations in (1.4.1) may be taken over the whole interval (a, b) or any measurable set contained in it. A useful addition to (i) is 'Fatou's lemma': if $f_n(x) \geqq 0$ and $f_n(x) \to f(x)$ p.p., then

$$\int f(x)\,dx \leqq \varliminf \int f_n(x)\,dx.$$

We require one theorem concerning the inversion of the order of integration, Fubini's theorem that

$$\int dx \int f\,dy = \int dy \int f\,dx = \iint f\,dx\,dy$$

whenever $f(x, y)$ is integrable. The integrals may be infinite when $f \geqq 0$.

We shall occasionally use the notion of the Stieltjes integral of a continuous function with respect to a function of V, the rule for the integration of such an integral by parts, and the theorem that $\left|\int f\,d\phi\right| \leq \mathrm{MV}$, where M is the maximum of $|f|$ and V the total variation of Φ. In Ch. VI we use Egoroff's theorem in two forms: (i) if $f_n(x) \to f(x)$ p.p. in E, then $f_n(x) \to f(x)$ uniformly in a set E^*

* A complex function is V when its real and imaginary parts are V.

included in E and of measure greater than $mE-\epsilon$; (ii) a similar conclusion holds if $f_h(x)\to f(x)$ as $h\to 0$ p.p. in E provided that each $f_h(x)$ is continuous in E.

1.5. The classes L^p. We shall say that f is L^p if f is measurable and $|f|^p$ is L: we shall always suppose that $p\geqq 1$. When p is 1, L^p is L. If f is L^q, and $1\leqq p<q$, then f is L^p.

We shall write

(1.5.1)
$$\mathbf{N}_p(f)=\|f\|_p=\left(\int_a^b |f|^p\,dx\right)^{1/p},\quad \mathbf{M}_p(f)=\left(\frac{1}{b-a}\int_a^b |f|^p\,dx\right)^{1/p}.$$

If f is not L^p, $\mathbf{N}_p(f)$ and $\mathbf{M}_p(f)$ are infinite. We call $\mathbf{N}_p(f)$ the *norm*, $\mathbf{M}_p(f)$ the *mean*, of f, for the interval (a,b) and index p. They differ only by a factor $(b-a)^{1/p}$; but this difference is important.

If $p>1$, we define p' by

(1.5.2)
$$p'=\frac{p}{p-1},\quad \frac{1}{p}+\frac{1}{p'}=1.$$

Then $p'>1$: if $p<2$, $p'>2$. We call p and p' *conjugate* indices, L^p and $L^{p'}$ conjugate classes. The class L^2 is self-conjugate. If $p=1$, we interpret p' as ∞, and conversely. We shall define the class L^∞, conjugate to L, in a moment.

The means $\mathbf{M}_p(f)$ have three fundamental properties. The first, viz.

(1.5.3)
$$\mathbf{M}_1(fg)\leqq \mathbf{M}_p(f)\,\mathbf{M}_{p'}(g),$$

is *Hölder's inequality* (Schwarz's inequality when $p=2$). The second, viz.

(1.5.4)
$$\mathbf{M}_p(f+g)\leqq \mathbf{M}_p(f)+\mathbf{M}_p(g),$$

is *Minkowski's inequality*. The third, viz.

(1.5.5)
$$\mathbf{M}_q(f)\leqq \mathbf{M}_p(f)\quad (q<p),$$

states that $\mathbf{M}_p(f)$ is, for given f, an increasing function of p. The norms $\mathbf{N}_p(f)$ have the first two properties but not the third.

When $p\to\infty$,

(1.5.6)
$$\mathbf{N}_p(f)\to \mathrm{Max}\,|f|,\quad \mathbf{M}_p(f)\to \mathrm{Max}\,|f|,$$

where $\mathrm{Max}\,|f|$ is the 'effective upper bound' of $|f|$, i.e. the smallest η such that $|f|\leqq\eta$ p.p. It is natural to define L^∞ as the class of f for which $\text{Max}\,|f|$ is finite. This is the class of 'effectively bounded' functions, or functions equivalent to bounded functions. We may now write

$$\mathrm{N}_\infty(f) = \mathrm{M}_\infty(f) = \text{Max}\,|f|; \tag{1.5.7}$$

and it may be verified at once that (1.5.3)–(1.5.5) remain true when p or p' is infinite.

1.6. Space L^p and its metric. The theory of the classes L^p, and of the inequalities associated with them, is much illuminated by the use of geometrical language. The class L^p defines a *functional space*, each function defining a *point* of the space. We do not distinguish two functions equivalent to one another, so that each point is a class of equivalent functions. In particular the *origin* 0 is the class of null functions. The space L^2, which is particularly important, is called *Hilbert space*. We define the *distance* $\delta_p(f,g)$ between f and g, in L^p, by

$$\delta_p(f,g) = \mathrm{N}_p(f-g), \tag{1.6.1}$$

or $\delta(f,g) = \mathrm{N}(f-g)$, omitting the suffix when it is plain of what space we are speaking. In particular, $\mathrm{N}(f)$ is the distance of f from the origin. If $p = \infty$, then

$$\delta(f,g) = \text{Max}\,|f-g|. \tag{1.6.2}$$

We can also define space C, the space of all *continuous* functions: here also distance is defined by (1.6.2), but 'Max' is the ordinary maximum.

If we take $f = f_1 - f_2$, $g = f_2 - f_3$ in (1.5.4), it becomes

$$\delta(f_1,f_3) \leqq \delta(f_1,f_2) + \delta(f_2,f_3) \tag{1.6.3}$$

and appears as an extension of the theorem that a side of a triangle cannot exceed the sum of the other two.

We can now set up a metric in space L^p (or C), and carry over to it the ideas of the ordinary theory of sets of points. One such idea is particularly important for our purposes, that of a class of functions *dense* in a wider class. Suppose that S_1 is a sub-class of L^p and S_2 a sub-class of S_1. Then we say that S_2 is *dense* (L^p) *in* S_1 (or simply dense in S_1) if, given any ϕ of S_1 and any positive ϵ, there is a ψ of S_2 such that $\delta_p(\phi,\psi) < \epsilon$. It follows from (1.6.3) that the relation of density is transitive: if S_2 is dense in S_1, and S_3 in S_2, then S_3 is dense in S_1. In such statements, of course, a fixed metric is presupposed.

It also follows from (1.5.5) that if S_2 is dense (L^p) in S_1, and $1 \leqq q < p$, then S_2 is dense (L^q) in S_1.*

There is one proposition concerning density which we shall use so often that we state it as a formal theorem.

Theorem 1. *If* $1 \leqq p < \infty$, *then the classes* L^q $(q > p)$, L^∞, B, C, *and* C_k *are dense in* L^p.

The theorem remains true if all the functions are restricted by periodicity. We shall prove later that the class of algebraical polynomials is dense in L^p, and the class of trigonometrical polynomials dense in the class of periodic functions of L^p.

1.7. Convergence in L^p (*strong convergence*). If f_n and f are L^p, and

$$\delta_p(f_n, f) \to 0 \tag{1.7.1}$$

when $n \to \infty$, or (what is the same thing) if $\mathrm{N}_p(f_n - f)$ tends to 0, then we say that f_n *tends* (L^p) *to* f, and write

$$f_n \to f (L^p). \tag{1.7.2}$$

We shall also say that f_n *tends strongly to* f *with index* p (omitting the reference to the index if there is no ambiguity). When $p = \infty$, $\delta(f_n, f) = \mathrm{Max}\,|f_n - f|$, and strong convergence is 'uniform convergence p.p.': $f_n \to f\,(L^\infty)$ if $f_n \equiv f_n^*$ and $f_n^* \to f$ uniformly.

A strong limit is 'effectively unique': if $f_n \to f\,(L^p)$ and $f_n \to g\,(L^p)$, then $f \equiv g$.

If $f_n \to f\,(L^p)$ and $1 \leqq q < p$, then $f_n \to f\,(L^q)$.

If $f_n \to f\,(L^p)$, then $\mathrm{N}_p(f_n) \to \mathrm{N}_p(f)$.

If $f_n \to f\,(L^p)$ and $g_n \to g\,(L^{p'})$, then $f_n g_n \to fg\,(L)$ and

$$\int f_n g_n\, dx \to \int fg\, dx. \tag{1.7.3}$$

In particular this is true if g is $L^{p'}$ and $g_n = g$ for all n.

If $f_n \to f$ p.p., $|f_n| \leqq \phi$, where ϕ is L^p and independent of n, and $1 \leqq p < \infty$, then $f_n \to f\,(L^p)$.

* It is not true that $\mathrm{N}_q(\phi - \psi) \leqq \mathrm{N}_p(\phi - \psi)$, but (owing to the arbitrariness of ϵ) the powers of $b - a$ involved in (1.5.5) do not affect the conclusion.

The fundamental theorem concerning strong convergence is

Theorem 2. *In order that f_n should converge strongly, with index p, to a function f of L^p, it is necessary and sufficient that*

$$\int |f_m - f_n|^p \, dx \to 0$$

when m and n tend to infinity. There is then a sub-sequence (n_k) such that $f_{n_k} \to f$ for almost all x.*

Theorem 2 is the analogue, for space L^p, of Cauchy's theorem concerning ordinary limits. When $p = \infty$ it reduces (apart from the neglect of null sets) to the corresponding theorem about uniform convergence. Strong convergence does not imply convergence p.p. (or for any x), nor is the converse true. But it follows from Theorem 2 that, if $f_n \to f\,(L^p)$ and $f_n \to f^*$ p.p., then $f \equiv f^*$.

Finally there is one theorem which we shall often use.

Theorem 3. *If $1 \leqq p < \infty$ and f is L^p, then*

$$\int_a^b |f(x+h) - f(x)|^p \, dx \to 0$$

when $h \to 0$, i.e. $f(x+h) \to f(x)\,(L^p)$.

The integral involves the values of f for certain values of x outside (a, b). We may either suppose these values to be 0, or regard $f(x)$ as a function with period $b - a$.

1.8. The resultant of two periodic functions. The *resultant* (*Faltung*) of two functions f and g, each with period $b-a$, over (a, b), is

(1.8.1) $$r(x) = \frac{1}{b-a}\int_a^b f(x-y)\,g(y)\,dy.$$

The resultant is also periodic, and symmetrical in f, g. The fundamental properties of $r(x)$ are as follows.

Theorem 4. *If f and g are L, then r is L (and so finite for almost all x). Also*

(1.8.2) $$\frac{1}{b-a}\int_a^b r\,dx = \frac{1}{b-a}\int_a^b f\,dx\,\frac{1}{b-a}\int_a^b g\,dx,$$

and

(1.8.3) $$\mathbf{M}_1(r) \leqq \mathbf{M}_1(f)\,\mathbf{M}_1(g).$$

* I.e. that $\int |f_m - f_n|^p \, dx < \epsilon$ for every positive ϵ and $m \geqq M(\epsilon)$, $n \geqq N(\epsilon)$.

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