Lecture Notes in Mathematics

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Bernd Herzog

Kodaira-Spencer Maps in Local Algebra

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Author

Bernd Herzog
Matematiska Institutionen
Stockholms Universitet
S-113 85 Stockholm, Sweden
E-mail: herzog@matematik.su.se

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Introduction

One of the most famous results in commutative algebra is Serre's theorem asserting that the localizations of a regular local ring are again regular. A quantitative refinement of this assertion, proved by Nagata [Na55] in 1955, states that for every local ring A and every prime ideal P of A such that A/P is analytically unramified, there is the following inequality between the multiplicities of the ring A and its localization A_P.

$$e_0(A_P) \leq e_0(A) \tag{1}$$

Already Nagata wondered whether the assumption about A/P to be analytically unramified is necessary and hinted at the fact that one could skip it, if one were able to prove an inequality for the multiplicities of certain special flat couples of local rings. In 1959, C. Lech [Le59] suggested that one should try to prove this inequality for arbitrary flat couples, i.e.

$$e_0(A) \leq e_0(B) \tag{2}$$

for arbitrary flat local homomorphisms $f : A \to B$ of local rings (A, m) and (B, n). To support his suggestion, C. Lech proved this inequality for first special examples. As a further refinement he asked in 1964 whether it is possible to prove even the corresponding inequality between sum transforms of the associated Hilbert functions, i.e., to prove the existence of a non-negative integer i such that

$$\mathrm{H}^i_A(n) \leq \mathrm{H}^i_B(n) \tag{3}$$

for every n. Lech found out that the inequality holds always for $n = 1$ (see also [Va67]) and announced it for arbitrary n, if the special fibre B/mB of $f : A \to B$ is a complete intersection ([Le?], see also [He90]). In 1970, Hironaka [Hi70] asked whether it is possible to prove the inequality even for $i = 1$. He treated the case that the special fibre is a hypersurface and used his result to show that the local Hilbert function cannot increase under permissible blowing up.

The above inequalities (2) and (3) are in fact assertions of local deformation theory. In a more geometric setting they mean that given a flat morphism $f : (X, x) \to (Y, y)$ of locally Noetherian schemes (X, x), (Y, y), i.e., a *deformation germ*, there is the inequality

$$e_0(Y, y) \leq e_0(X, x) \tag{4}$$

for the multiplicities at $y \in Y$ and $x \in X$ and, respectively, the inequality

$$\mathrm{H}^i_{(Y,y)} \leq \mathrm{H}^i_{(X,x)} \tag{5}$$

for the associated Hilbert series. The latter inequality means that all pairs of corresponding coefficients should satisfy the inequality. From the point of view of deformation theory it is quite natural to start with a fixed *local singularity* (i.e., a local scheme (X_0, x_0)) and to ask whether the *deformations* of this singularity (X_0, x_0), i.e., the flat morphisms $f : (X, x) \to (Y, y)$ with *special fibre* $(X_y, x) := f^{-1}(y)$ equal to (X_0, x_0), satisfy Lech's inequality. In this terminology Lech's main result ([Le?],[He90]) states that every deformation of a complete intersection satisfies (5) with $i = 1$. The weaker inequality (4) was established by Lech already in 1959 for the deformations of thick points with local ring

$$K[X_1, \ldots, X_N]/(X_1, \ldots, X_N)^d$$

(where $N \geq 2$) and for the deformations of arbitrary complete intersections. In his paper [He91] the author could show that inequality (5) holds even with $i = 0$ for many other special fibres, but in general he could derive only the inequality

$$\mathrm{H}^1_{Y,y} \cdot \mathrm{H}^0_{X_y,x} \geq \mathrm{H}^1_{X,x} \tag{6}$$

(without the flatness assumption, see [He82]) which is, in some sense, converse to Lech's one. It is natural to ask for conditions ensuring that equality holds in (6) (in which case (5) is trivially satisfied). One such even equivalent condition is the flatness of the morphism $df : C(X, x) \to C(Y, y)$ induced by f on the tangent cones at x and y, respectively. The morphism f is called *tangentially flat* in this case (*graduationally flat* in [He82]). Moreover it is possible to give sufficient conditions for singularities having the property that all their deformations are tangentially flat. One such condition is that Schlessiger's module T^1 of the tangent cone shouldn't have (non-zero) homogeneous elements of degree less than -1 (see [He91]). The condition is fairly easy to verify in many situations, is even necessary for homogeneous singularities, and enables one to find many classes of singularities such that all deformations are tangentially flat and therefore satisfy (5) with $i = 0$ (see [He91]).

Another approach to formula (1) is to find a direct refinement in terms of local Hilbert functions. Such a refinement is the following inequality.

$$\mathrm{H}^{i+1}_{A_P}(n) \leq \mathrm{H}^i_A(n) \tag{7}$$

(n an arbitrary non-negative integer). This inequality was proved by Lech [Le64] in the case of excellent local rings with $i = 1$ (see also [Be70]). If one uses results of B. Singh [Si74] on the behavior of local Hilbert functions under permissible blowing up, Bennett's proof gives inequality (7) with $i = 0$. Note that the gap in Singh's paper (for the proof of the main theorem, Lemma (4.5) is insufficient) can be filled (see [He80'], Lemma (2.4)).

In a remarkable joint paper with T. Larfeldt [L-L], C. Lech proved in 1981 that the two inequalities (3) and (7) are equivalent. However, the fact that (7) is known in the geometric (i.e., excellent) case has no implication with respect to (3). It is not difficult to see that inequality (3) is much closer to geometry than

(7). In particular, (3) is easily reduced to the case of Artinian local rings. So the result of Larfeldt and Lech states, roughly speaking, if something is true in the non-geometric (i.e., non-excellent case), then this has consequences for geometry. This means, to the author's opinion, that either Lech's inequality doesn't hold in the general case or the inequality reflects a rather deep property of singularities related with an anyhow mysterious connection between the non-geometric part of commutative algebra and infinitesimal properties of algebraic varieties.

In case the inequality doesn't hold in general, it will be hard to find a counterexample, for, in view of the above mentioned cases, when the inequality is known, one has to find a fairly small locus on the Hilbert scheme. The formal versal deformations of the most simple singularities, which are not known to satisfy Lech's inequality depend on about 60 variables. The equations defining the base ring of the formal versal deformation (considered up to degree 3, - the degree 2 situation is the first interesting one) fill many pages so that it is hopeless to find a reasonable specialization. Therefore, the best what one can currently do is to look for larger and larger classes of singularities satisfying Lech's inequality and hope that one can get this way some idea of the singularities which might fail to do so. The purpose of the material presented here is to contribute to this aim. More precisely, we want to prove the following weakened version of Lech's inequality.

(8) $$\mathrm{H}^1_{Y,y} \cdot \mathrm{H}^0_{X_y,x} \le \mathrm{H}^1_{X,x} \cdot \prod_{d=2}^{\infty} \left(\frac{1-T^d}{1-T}\right)^{n(d)}.$$

for every residually separable flat morphism $f : (X,x) \to (Y,y)$ of germs (X,x), (Y,y) of locally Noetherian schemes with special fibre $(X_y,x) := f^{-1}(y)$. Here

$$n(d) := \dim \mathrm{T}^1_{C(X_y,x)}(-d)$$

is the dimension of the degree $-d$ part of Schlessinger's T^1 associated with the tangent cone $C(X_y,x)$ of the special fibre.

Formula (8) may be interpreted as a numerical refinement of the criterion [He91], Th. 2.5, characterizing the singularities having exclusively tangentially flat deformations. Contrary to (6), the estimation (8) is in the same direction as Lech's inequality and allows often to prove inequalities of type (4) and (5), particularly for singularities with few elements of negative degree in Schlessinger's T^1.

To prove (8), we have to study the morphism $\phi : (Y,y) \to (M,*)$ that defines f as the result of a base change from the formal versal family of (X_y,x), and the mapping

$$d\phi : C(Y,y) \to C(M,*)$$

induced by ϕ on the tangent cones at the base points. The latter morphism may be considered as a local analogue of the Kodaira-Spencer mapping which plays an important role in the deformation theory of compact complex manifolds (see [Ko86]). In proving (8), the morphism $f : (X,x) \to (Y,y)$ is modified via base change of a very special type that allows one to control the behavior of the Hilbert series and that decreases the negative part of the image of $d\phi$. In case the image of this negative part is sufficiently small, f is proved to be tangentially flat, i.e., (8) is trivially true, and the general case is treated inductively depending upon the size of this image. Thus the proof heavily depends upon a good description of $\mathrm{Im}(d\phi)$, and this is the place where the theorems on tangential flatness are needed in a generalized version. The natural filtration of the local ring (A, m) given by the powers of the maximal ideal must be replaced by a more general type of filtration $F_A := (F_A^d)_{d\in\mathbb{N}}$ where the ideals F_A^d are such that A/F_A^d is Artinian and such that $F_A^d \cdot F_A^{d'} \subseteq F_A^{d+d'}$.

More accurately, one should study the Kodaira-Spencer mapping associated not with the morphism $f : (X, x) \to (Y, y)$ but the one coming from the induced map $df : C(X, x) \to C(Y, y)$ on the tangent cones. This corresponds to the phenomenon that it is Schlessinger's T^1 of the tangent cone $C(X_y, x)$ and not of the fibre X_y itself, which decides whether X_y has tangentially non-flat deformations. But the term "Kodaira-Spencer mapping of df" doesn't make sense at first glance, since df is almost never flat, so there is no base change morphism giving df as a result of base change from a universal family. However, there are many possibilities to construct induced morphisms analogous to df which correspond to certain pairs of filtrations on the local rings at x and y, and many of these filtrations define flat morphisms of cones, hence are suited to construct a Kodaira-Spencer map. So the problem is to find a canonical pair of such filtrations. It turns out that among the filtrations giving flat morphisms of cones there is a minimal pair, and this is our candidate. Almost nothing is known about these minimal filtrations currently, and possibly they are rather exotic in certain cases. So we have to reprove the usual theorems on tangential flatness (see [He82] and [He91]) for the case of filtered local rings.

Most of these theorems have analogues in the yet more general context of filtered modules as one can see from an early version of this monograph [He92]. Since the module situation is extremely technical, we spent much care to get rid of it. Up to a few relicts, the present treatment states the results for filtered local rings where the filtrations satisfy an Artin-Rees type condition. The formulations of the theorems and their proofs are much simpler this way. We hope this will make the ideas behind them more transparent to the reader. In the classical (non-filtered) case, the proofs used, as a basic tool, a relative variant [He91, Proposition (1.6)] of Cohen's theorem that each complete local ring is isomorphic to a factor of a formal power series ring. Unfortunately we couldn't prove a general filtered version of it. This is finally the reason that we originally formulated the theorems on filtered tangential flatness in the language of modules where Cohen factorization translates into the trivial fact that each filtered module is the homomorphic image of a "tangentially free" one. Since Cohen factorization is quite useful when dealing with examples, we include a variant of the theorem for quasi-homogeneous filtrations with positive real weights (see 5.9).

In Chapters 1 to 6 we reprove the elementary part of [He91] (covering [He82] and [He83]) in the new context. Using these results, it is possible to prove also a generalized version of the main theorem in [He91] characterizing singularities with tangentially flat deformations in terms of Schlessinger's T^1 (or, equivalently, in terms of the normal module of the fibre with respect to some formal embedding into a regular scheme). The reader is referred to [Ja90], where this generalized criterion is proved and used to construct new fibres such that Lech's inequality holds.

Chapter 1 contains the most important definitions and a few elementary results on the filtrations we are dealing with. To motivate the later restriction to Artin-Rees filtrations of local rings, we give a characterization of these filtrations in terms of the associated complete rings (see 1.17).

In Chapter 2 we prove variants of well-known lemmas which will be frequently used later. The material is present mainly for reference purposes. The reader might skip it and return to it when appropriate.

Chapter 3 is devoted to the fact that tangential flatness is preserved under surjective base change (see 3.8) and its corollaries. The proofs are along the lines of M. Brundu's paper [Br85], where the case of I-adic filtrations is treated. Further we introduce the basic exact sequence (see 3.14) which is used later to establish the connection of tangential flatness to Hilbert series, and we spend some time to prove a kind of inverse (see 3.15) to the base change theorem 3.8. The analogue 4.4 of Brundu's Main theorem [Br85, see (3.3) and (3.4)] is proved in the next chapter using arguments essentially different from Brundu's. The reason is that we don't understand Brundu's argument in the proof of her Theorem (3.3) claiming that her isomorphism $\beta : S \otimes_A B \to T$ is in fact an isomorphism of filtered rings. So we decided to chose a different (more difficult) approach.

In Chapter 5 we introduce the notion of distinguished basis, which is, in some sense, an analogue in the situation of flat local extensions of the notion of a free generating set and which gives us the possibility to introduce "structure constants". The main result 5.8 in this chapter characterizes tangential flatness in terms of these structure constants. When applied to the versal deformation of a (local) singularity it says that the set of quasi-homogeneous filtrations (with positive real weights) having the property that all deformations in an appropriate filtered category are tangentially flat is a finitely generated convex polyhedron. Unfortunately this polyhedron seems to depend heavily upon the local embedding of the singularity into a non-singular scheme. Nevertheless we expect that it will play an important role in the context of Lech's conjecture.

Chapter 6 relates tangential flatness with properties of Hilbert series. The usual theorems known from [He82] are generalized to the filtered case. The proofs differ somewhat from and are more complicated than the proofs known for the natural filtrations. One has to avoid certain enumerative arguments comparing dimensions of vector spaces, which may be infinite in the general case, and has to work instead with exact sequences. But possibly the deductions are even more natural now; the results are the same. We have added two easy but useful theorems on the composition of tangentially flat morphisms, which we learned about from late Christer Lech in a private communication.

In Chapter 7 we introduce the notion of flatifying filtration for a homomorphism of filtered local rings, which is defined as a filtration on the base ring containing the given filtration and making the given homomorphism tangentially flat. We prove the existence of a unique minimal flatifying filtration. The hard part of the proof is to show that this filtration has the Artin-Rees property (see 7.9). As an interesting side result we get that each naturally filtered flat local homomorphism becomes tangentially flat by a base change of a very special type (see 7.8).

Schlessinger's T^1 and the Kodaira-Spencer mapping are constructed in Chapter 8. The central result here is that the minimality property of a flatifying filtration implies that an associated Kodaira-Spencer mapping is injective in certain negative degrees (see 8.13). Contrary to earlier announcements we cannot establish this result for general local homomorphisms, but must restrict to residually rational ones. This is finally the reason that our main result 9.2 is valid for residually separable homomorphisms only. We spend some care to clarify the reason why we can't prove the general statement, showing that the Kodaira-Spencer map is always injective in appropriate degrees when restricted to a certain subspace (see 8.13). One interpretation of our difficulties in the general case is that the minimal flatifying filtration might contain too much information about the extension of the residue class fields induced by the given morphism f whereas the Kodaira-Spencer map itself forgets all such information (see 8.9(iv)).

In Chapter 9 we use the results obtained so far to prove inequality (8). In a first step we establish an inequality of the indicated type with Schlessinger's T^1 replaced by data depending upon the minimal flatifying filtration of the given homomorphism (see 9.1). The final proof of (8) is essentially a comparison of these data with Schlessinger's T^1 based on the injectivity result of Chapter 8. Since the latter is available only for residually rational homomorphisms, we must first reduce the proof of (8) to this special case, which is done using a standard argument of Cohen structure theory. We conclude Chapter 9 with examples illustrating our result and a list of problems related with Lech's inequality and tangential flatness.

The last chapter has the character of an extended example. We study the germ of the universal family at a fixed point of the Hilbert scheme and describe the minimal flatifying filtration on the local ring at the given point. We use this description to give a simplified proof of the main result from [He91] (in the residually rational case) and to show that the question whether a tangentially flat homomorphism can be lifted preserving tangential flatness depends upon finitely many obstructions. Implicitly the considerations of this chapter show that there is a relation between the minimal flatifying filtration on the local rings of the Hilbert scheme and the graded structure of these rings introduced by Pinkham [Pi74] in the homogeneous case. Since this has implications for Lech's inequality, we will come back to this point in a later publication.

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Notation

Throughout we use the conventions of commutative algebra as in [Ma86] or [Bo61]. Local rings will be always considered to be Noetherian.. A local homomorphism of local rings is called *residually rational*, if the induced homomorphism of the residue classes is an isomorphism. Similarly, it is called *residually separable*, if it induces a separable field extension. Given two submodules N_1 and N_2 of the module M, we will write

$$(N_1 \bmod N_2)$$

to denote the canonical image $(N_1 + N_2)/N_2 \subseteq M/N_2$ of N_1 in M/N_2. The terminology below will be used frequently, often without any further explanation.

$\mathbb{N}$	non-negative integers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$< a, b >$	the scalar product $\sum_{\lambda\in\Lambda} a_\lambda b_\lambda$ of the families $a := (a_\lambda)_{\lambda\in\Lambda}$ and $b := (b_\lambda)_{\lambda\in\Lambda}$ indexed by one and the same set, where one of the families should have only finitely many non-zero elements and the elements of both families are from appropriate rings or modules such that multiplication and summation is possible.
$\Omega(A)$	set of maximal ideals of the ring A
$m(A)$	maximal ideal of the local ring A
$L_A(M)$	length of the module M over the ring A
$M^{(\Lambda)}$	direct sum of copies of the module M where the number of copies is equal to the cardinality of the set Λ
$\mu_A(M)$	minimum number of generators of the module M over the ring A
F_M^d	d-th filtration submodule of the filtered module M
$M(d)$	subgroup of homogeneous elements of degree d, if M is a graded module, or M/F_M^{d+1} in case M is a filtered module
$M(\leq e)$	direct sum of all homogeneous parts $M(d)$ with $d \leq e$ of the graded module M

$M(\geq e)$	direct sum of all homogeneous parts $M(d)$ with $d \geq e$ of the graded module M
$\mathrm{G}(M)$	graded module associated with the filtered module M, see 1.13
$\mathrm{G}_F(M)$	same as $\mathrm{G}(M)$ where the A-module M is considered to be filtered with respect to the filtration F (in case F is a filtration of M) and FM(in case F is a filtration of A), respectively. In the second case FM denotes the filtration of M generated by F, see 1.13
$\cap F_M$	intersection of all filtration submodules of the filtered module M
$\mathrm{ord}_M(x)$	$= \sup\{d \in \mathbb{N} \mid x \in F_M^d\}$, the order of the element x in the filtered module M. Sometimes, M will be a factor module of the module containing x. In this case the symbol denotes the order of the residue class in M of the element x.
$M^\wedge$	completion of the filtered module M, see 1.16
$\mathrm{cl}(N)$	closure of the submodule $N \subseteq M$ in the completion of the filtered module M, see 1.16
$\mathrm{R}_B(x)$	module of syzygies of the family x over the ring B, see 3.3
H_M	Hilbert series of the graded module M or the filtered ring $M = A$, see 6.1 and 6.3, respectively.
H_M^i	i-th sum transform of the Hilbert series of the graded module M or the filtered ring $M = A$, see 6.1 and 6.3, respectively.
$\mathbb{P}_K^N$	projective N-space over the field K
$\mathbb{H}$	Hilbert scheme of length n subschemes in $\mathbb{P}_K^N$ where n is a fixed positive integer
$\mathbb{U} \to \mathbb{H}$	the universal family of the Hilbert scheme $\mathbb{H}$

Kodaira-Spencer maps in local algebra

Contents

1. Ring filtrations

Remark

This chapter contains the most important definitions and a few elementary results on the filtrations we are going to deal with. Artin-Rees filtrations are defined in 1.12 and, in the case of local rings, characterized in terms of the associated complete rings in 1.17(iii).

1.1. Graded rings

A *graded ring* is defined to be a ring G which admits a direct sum decomposition

$$G = \bigoplus_{d=0}^{\infty} G(d)$$

into submodules $G(d)$ over the ring $G(0)$ such that

$$G(d) \cdot G(d') \subseteq G(d + d')$$

for arbitrary $d, d' \in \mathbb{N}$. The elements of $G(d)$ will be called *homogeneous of degree* d, and we will write

$$\deg(x) = d$$

to indicate that $x \in G$ is homogeneous of degree d, i.e., $x \in G(d)$. Note that the zero element may have any degree. The ideal

$$G^{+} := \bigoplus_{d=1}^{\infty} G(d)$$

is called the *irrelevant ideal* of G. A *graded module* over the graded ring G is a module M over G that decomposes into a direct sum,

$$M = \bigoplus_{d=d_0}^{\infty} M(d)$$

($d_0 \in \mathbb{Z}$) of submodules $M(d)$ over $G(0)$ satisfying

$$G(d) \cdot M(d') \subseteq M(d + d')$$

for arbitrary $d \in \mathbb{N}, d' \in \mathbb{Z}$. We will use the convention that

$$G(d) := 0 \text{ for } d < 0 \text{ and } M(d) := 0 \text{ for } d < d_0.$$