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Mathematical Aspects of Finite Element Methods

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FOREWORD

The contents of this book are based on lectures given at the Meeting on Mathematical Aspects of Finite Element Methods, held in Rome, December 10-12 1975, at the "Consiglio Nazionale delle Ricerche" (C.N.R.) organized by the *Istituto per le Applicazioni del Calcolo "Mauro Picone"* and *Laboratorio di Analisi Numerica*.

The subject of this meeting is of particular interest owing to the importance that the Finite Element Method has in many fields of engineering, not only from the point of view of research but also in the industrial routine. It is well known that this method has been developed by engineers as a concept of structural analysis.

When there was discovered the connection between the Finite Element Method and the Ritz-Galerkin-Faedo procedure applied to the spaces of piecewise polynomial functions, the interest of mathematicians to this method increased enormously: now, in this field, the interests and ideas of engineers and mathematicians converge and overlap and the cooperation between them has become more and more essential.

To this aim, the Istituto per le Applicazioni del Calcolo "Mauro Picone" (IAC) in Roma and Laboratorio di Analisi Numerica (LAN) in Pavia have considered the opportunity of organizing this meeting, by emphasizing the mathematical aspects of the Finite Element Method.

Twenty-five papers were presented and discussed at the Meeting; but only twenty-two lectures have been made available for publication on time

We should like to conclude by thanking the members of IAC for their help in the management of the meeting.

I. Galligani - E. Magenes

Roma, July 1976

CONTENTS

Mathematical Problems of Computational Decisions in the Finite Element Method

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Abstract. Present programs for finite element analysis require the user to make numerous, critical, a-priori decisions. They often represent difficult mathematical problems and may influence strongly the accuracy and reliability of the results, the cost of the computation, and other related factors. This paper discusses some of these decisions and their mathematical aspects in the case of several typical examples. More specifically, the questions addressed here concern the effect of different mathematical formulations of the basic problem upon the results, the influence of the desired accuracy on the efficiency of the process, the selection and comparison of different types of elements, and, for nonlinear problems, the choice of efficient methods for solving the resulting finite dimensional equations. In all cases a consistent use of self-adaptive techniques is strongly indicated.

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1. Introduction

The finite element method has advanced rapidly in the past two decades. The most far-reaching progress probably occurred in the practical application of the method in various fields and especially in continuum mechanics. Numerous, often large, general and special-purpose programs for finite element analysis have been built and are widely applied to increasingly complex problems (see, e.g., [1]).

The mathematical analysis of the method began somewhat later but is also progressing at a quick pace. Without question, the method has now been placed on a firm mathematical foundation.

However, in looking over these advances, it is surprising to notice the relatively weak interaction between the mathematical progress and the practical application of the finite element method. Often, in practice, the method is not interpreted as an approximate solution process of a differential equation of, say, continuum mechanics. On the other hand, the theoretical analysis has principally addressed the mathematical basis of the method and of the related approximation problems. There appears to be an urgent need to extend now this theoretical analysis to all phases of the solution process and their interactions. This involves the selection of the

mathematical formulation of the original problem and the characterization of the desired type of solution. It also includes the variety of questions about the numerical procedures and last, but not least, the many computer science problems arising in the overall implementation.

Today's finite element programs require the user to make numerous, very critical, a-priori decisions which, in fact, often represent difficult mathematical questions. This includes decisions about the mathematical model, as, for example, whether a plate or shell may be considered thin, or whether nonlinear behavior may be disregarded. It also includes the questions of the selection of the elements and the meshes, the specification of the time steps and of various other process parameters, as well as the decisions when updates or refinements are to be used, etc. The architecture of all present--and probably many of the future--programs incorporates the need for all these options. Any of the decisions required from the user may influence strongly the accuracy of his results, the cost of the computation, and so on. It appears that only a consistent use of self-adaptive techniques can significantly alter this situation.

The mathematical problems involved in all this are wide-ranging, and in part, novel in nature, especially when it comes to the computer science questions. Some starting points for such studies may well be the many questions raised by the often-startling results reported by experienced practitioners of the finite element method. Our aim here is to delineate some such questions for several typical examples. More specifically, in Section 2 we show the effect of different formulations of the basic mathematical problem upon the results. Then Section 3 addresses the influence of the desired accuracy of the solution upon the efficiency of the solution process and the need for further types of asymptotic analyses. Section 4 considers some aspects related to the theoretical comparison of different types of elements, and finally Section 5 shows that, especially in the nonlinear case, the methods for solving the resulting finite dimensional equations depend once again strongly on the selection of the mathematical formulation of the problem.

2. Formulation of the Mathematical Model

Most physical problems may be formulated mathematically in a variety of more or less simplified forms, and a numerical method applied to any one such mathematical model introduces a further transformation. Clearly, a principal mathematical question must be the analysis and estimation of the errors resulting from the various simplifications and transformations. This, however, requires a decision as to which formulation is to be considered as the reference model. Here, usually, attention is only focused on the approximation errors introduced by the numerical method, although sometimes these errors are much smaller than those caused by earlier simplifications of an original mathematical model.

This situation arises, in particular, in continuum mechanics where theoretical advances now allow for the formulation of very general mathematical models. Because

of their complexity, numerical procedures are in most cases only applied to considerably simplified formulations which then are also used as the reference models in the error analysis. In this section we illustrate how much we may have to adjust our assessment of the numerical results, if some of the earlier simplifications are taken into account.

As an example, we consider a bending analysis of a simply supported plate for which either a two- or three-dimensional formulation may be used. Let $\bar{\Omega} \subset R^2$ denote the (compact and Lipschitzian) domain of the plate and d its thickness. The material is assumed to be homogeneous and isotrop with Young's modulus E and, for simplicity, Poisson's ratio $\sigma = 0$.

The three-dimensional formulation involves the solution of a system of strongly elliptic equations for the unknown vector $\underline{u} = (u_1,u_2,u_3)$ on

$$\Omega_d = \{(x_1,x_2,x_3) \in R^3;\ (x_1,x_2) \in \Omega,\ |x_3| < \frac{d}{2}\}, \tag{2.1}$$

subject to certain boundary conditions on $\partial\Omega_d$. In its weak form this boundary value problem requires the determination of

$$u_1 \in H^1(\Omega_d),\quad u_2 \in H^1(\Omega_d),\quad u_3 \in H^1_{[0]}(\Omega_d)$$

such that

$$E\int_{\Omega_d} \sum_{i,k=1}^{3} \frac{1}{4}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) dx_1dx_2dx_3 = \int_{\Omega} v_3(x_1,x_2,\tfrac{d}{2}) f(x_1,x_2)dx_1dx_2 \tag{2.2}$$

holds for any

$$v_1 \in H^1(\Omega_d),\quad v_2 \in H^1(\Omega_d),\quad v_3 \in H^1_{[0]}(\Omega_d).$$

Here $f \in L_2(\Omega)$ is given, $H^1(\Omega_d)$ denotes the usual Sobolev space and

$$H^1_{[0]}(\Omega_d) = \{u \in H^1(\Omega_d) \mid u(x) = 0,\quad (x_1,x_2) \in \partial\Omega,\ |x_3| < \tfrac{d}{2}\}.$$

The two-dimensional formulation leads to the well-known (see, e.g., [2],[3]) biharmonic problem of finding $w \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\frac{Ed^3}{12}\int_{\Omega}\left[\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + 2\frac{\partial^2 w}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1\partial x_2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2}\right] dx_1dx_2 \tag{2.3}$$

$$= \int_{\Omega} v\, f\, dx_1dx_2,\ \forall\ \ v \in H^2(\Omega) \cap H^1_0(\Omega)\ .$$

Physically, we expect that approximately $w(x_1,x_2) = u_3(x_1,x_2,0)$ for $(x_1,x_2) \in \Omega$. The formulation (2.3) may be derived from (2.2) by requiring that $\underline{u}$ (and correspondingly $\underline{v}$) satisfies

$$u_1 = -x_3\frac{\partial w}{\partial x_1},\ u_2 = -x_3\frac{\partial w}{\partial x_2},\ u_3 = w, \tag{2.4}$$

that is, by restricting the space $H^1(\Omega_d)$. Accordingly, the finite element method for (2.3) may be interpreted as a method for (2.2) with special elements which incor-

porate two "small" parameters, namely, the size h of the elements in the x_1,x_2-directions, and the thickness d with $d \ll h$, in most cases.

A direct three-dimensional finite element solution of (2.2) is most likely inefficient for small d. At the same time, (2.4) is certainly not the only possible restriction. For example, we may use

$$u_1(x_1,x_2,x_3) = -x_3\varphi_1(x_1,x_2)$$
$$u_2(x_1,x_2,x_3) = -x_3\varphi_2(x_1,x_2) \tag{2.5}$$
$$u_3(x_1,x_2,x_3) = \varphi_3(x_1,x_2) ,$$

where now $\varphi_1,\varphi_2 \in H^1(\Omega)$, $\varphi_3 \in H_0^1(\Omega)$. This leads to a system of three equations of second order involving a small parameter, in contrast to the one equation of fourth order in the case of (2.4) without such a small parameter. Hence, the finite element discretization may now involve C^0-elements, instead of the C^1-elements needed before. Of course, the design of the elements has to account for the small parameter (d). A direct use of, say, piecewise linear elements would lead to very inaccurate results when $d \ll h$ is small.

However, there is a significant difference between the restrictions (2.4) and (2.5), namely, their dependence upon the domain. Let $B \subset R^2$ be the open unit ball in R^2 and $B^n \subset B$, $n \geq 3$, the regular n-sided (open) polygon inscribed in B. The corresponding three-dimensional domain B_d is defined as in (2.1). Then we have the following result for (2.2).

Theorem 2.1: Let $\underline{u}$ and $\underline{u}^{(n)}$ denote the solutions of (2.2) (with $f \in L_2(\Omega)$) for $\Omega = B$ and $\Omega = B^n$, respectively. Then $\lim_{n\to\infty} \underline{u}^{(n)} = \underline{u}$ in $H^1(D) \times H^1(D) \times H^1(D)$ for any compact $\bar{D} \subset B_d$.

On the other hand, for (2.3), that is, the restriction (2.4) of (2.2), the following theorem holds:

Theorem 2.2: Let w and $w^{(n)}$ be the solutions of (2.3) (with $f \equiv 1$) for $\Omega = B$ and $\Omega = B^n$, respectively. Then

$$\lim_{n\to\infty} w^{(n)} = W = \frac{3}{16d^3E}(3-4r^2+r^4), \quad (r^2=x_1^2+x_2^2) ,$$

in $H^2(D)$ for any compact $\bar{D} \subset B$, and

$$w = \frac{3}{16d^3E}(5-6r^2+r^4) \neq W .$$

This result was proved in [4] (see also [5], [6]) and its meaning discussed in various papers (see, e.g., [7]) under the name "Babuška's paradox."

In the case of the two-dimensional problem (2.2)/(2.5), obtained from (2.2) by applying the restriction (2.5), we have once again a result of the type of Theorem 2.1 (see [8]).

Theorem 2.3: Let $\underline{u}$ and $\underline{u}^{(n)}$ be the solutions of the two-dimensional problem (2.2)/(2.5) (with $f \in L_2(\Omega)$) for $\Omega = B$ and $\Omega = B^n$, respectively. Then $\lim_{n\to\infty} u^{(n)} = u$ on $H^1(\mathcal{D}) \times H^1(\mathcal{D}) \times H^1(\mathcal{D})$ for any compact $\bar{\mathcal{D}} \subset B_d$.

The questions discussed here belong to the general range of problems of dimension reduction. There exists a large literature in this area especially for problems related to plates and shells. We mention here, for instance, [9], [10], [11], [12], [33], [34], [35], [36], [37], where additional references may be found. It should be noted, however, that all these presentations assume a smooth solution and hence cannot distinguish between the reductions (2.4) and (2.5) that led to the Theorems 2.1 and 2.2.

Some of the results of these three theorems may be summarized as follows:

(1) For small d and $\Omega = B$ the formulations (2.2),(2.3), and (2.2)/(2.5) essentially give the same results. However, if Ω has corners, as does B^n, then, even for large n, the results for (2.3) may be very different from those of (2.2) and (2.2)/(2.5) if h is small.

(2) In order to overcome the effect of Theorem 2.2 we may combine (2.4) and (2.5). More specifically, (2.5) is used in a neighborhood of the boundary and (2.4) elsewhere in the domain. Of course, in doing so we need to take account of the fact that $d \ll h$.

(3) The restriction (2.5) may be generalized to include higher order polynomials in x_3. This may be desirable when d is not sufficiently small and the resulting two-dimensional formulations are still less costly to solve than (2.2).

(4) Dimensional reduction may be treated as a special selection principle for the elements. Asymptotic analysis alone is insufficient to determine the influence of this type of reduction upon the solution. It is an open question how these problems may be approached theoretically. From a computational viewpoint, the only realistic way may be the use of self-adaptive techniques.

Although we discussed here only one particular example, it should be evident that similar situations may arise in connection with various other problems. We certainly encounter them in shell theory, but analogous questions also occur, for instance, when linearizations are introduced. In practice, the decision about the choice of the specific mathematical model is almost entirely left up to "experience". There appears to be a need for new theoretical and computational approaches.

3. Accuracy and Asymptotic Behavior

In practical applications of the finite element method, the accuracy required of the solution is rarely very high; in fact, an error of 10-20% is often fully acceptable. This means that the standard asymptotic error analysis may not provide us with sufficient insight. Since asymptotic approaches are hardly to be avoided, this suggests that we should consider a variety of different asymptotic analyses for characterizing more completely the computational process.

As an example, we consider a simple version of a type of problem occurring in

reactor computations in the presence of interfaces. More specifically, let the following boundary value problem be given:

$$\frac{\partial}{\partial x_1}\left(a \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(a \frac{\partial u}{\partial x_2}\right) = 1, \ \forall \ (x_1,x_2) \in \Omega \tag{3.1a}$$

$$u = 0 \text{ on } \partial\Omega$$

where

$$\Omega = \{(x_1,x_2) \in R^2;\ |x_1| < \tfrac{1}{2}\ ,\ |x_2| < \tfrac{1}{2}\} \tag{3.1b}$$

and

$$a = \begin{cases} \Phi \text{ on } \Omega_1 = \{(x_1,x_2);\ 0 < x_1 < \frac{1}{2}\ ,\ 0 < x_2 < \frac{1}{2}\} \\ 1 \text{ on } \Omega - \Omega_1 \end{cases} \tag{3.1c}$$

The solution has a singularity of the type $r^{\beta}\varphi(\theta)$ at the origin[1] with β and φ depending on Φ. For an analysis of the solution of such interface problems, see [13] and [14].

Singularities of this type influence considerably the convergence of the finite element or finite difference methods. We refer to [15], [16], [17], and [18] for thorough studies of these problems. Suppose that a regular triangular finite element mesh with grid size h is used, as shown in Figure 3.1. Theoretically, the error then satisfies

$$\|e\|_{L_\infty} \geq ch^{\beta+\varepsilon},$$

with arbitrary $\varepsilon > 0$, and this result is independent of the order of the elements.

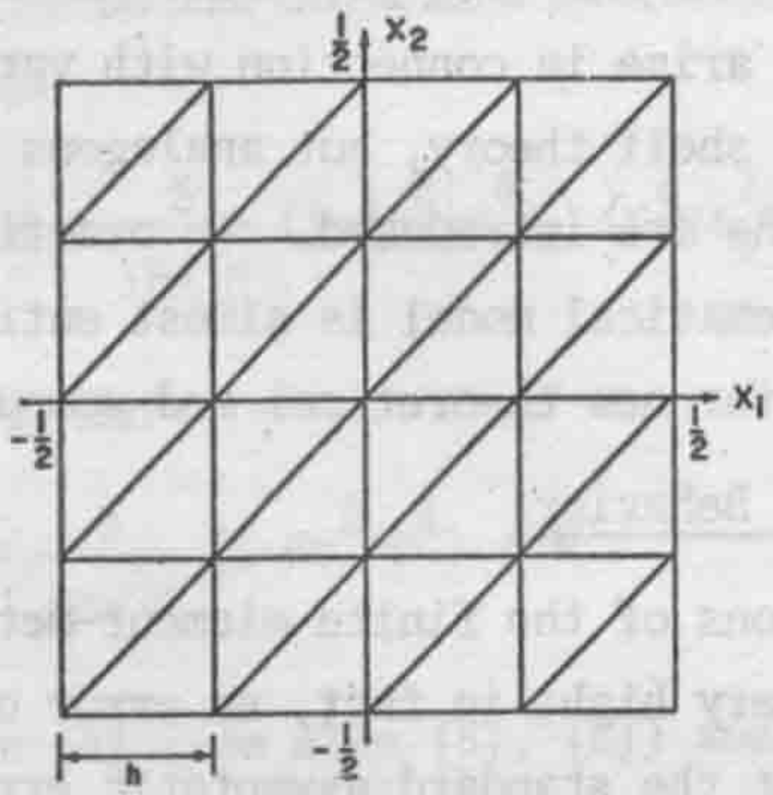


Figure 3.1

[1] Here, and further below, r and θ represent polar coordinates.

However, this error behavior has not been observed in reactor computations. In the case of problem (3.1), the experimentally-observed rate of convergence for 1% accuracy as a function of Φ is indicated in Figure 3.2. For small $\Phi > 1$ the rate of convergence is better than the expected asymptotic value $\alpha(\Phi) < 1$; while for large Φ it essentially equals the predicted rate. The reason is that for small Φ the effect of the singularity is negligible in comparison to the desired 1% accuracy. The higher the required accuracy, the more the effect of the singularity becomes visible.

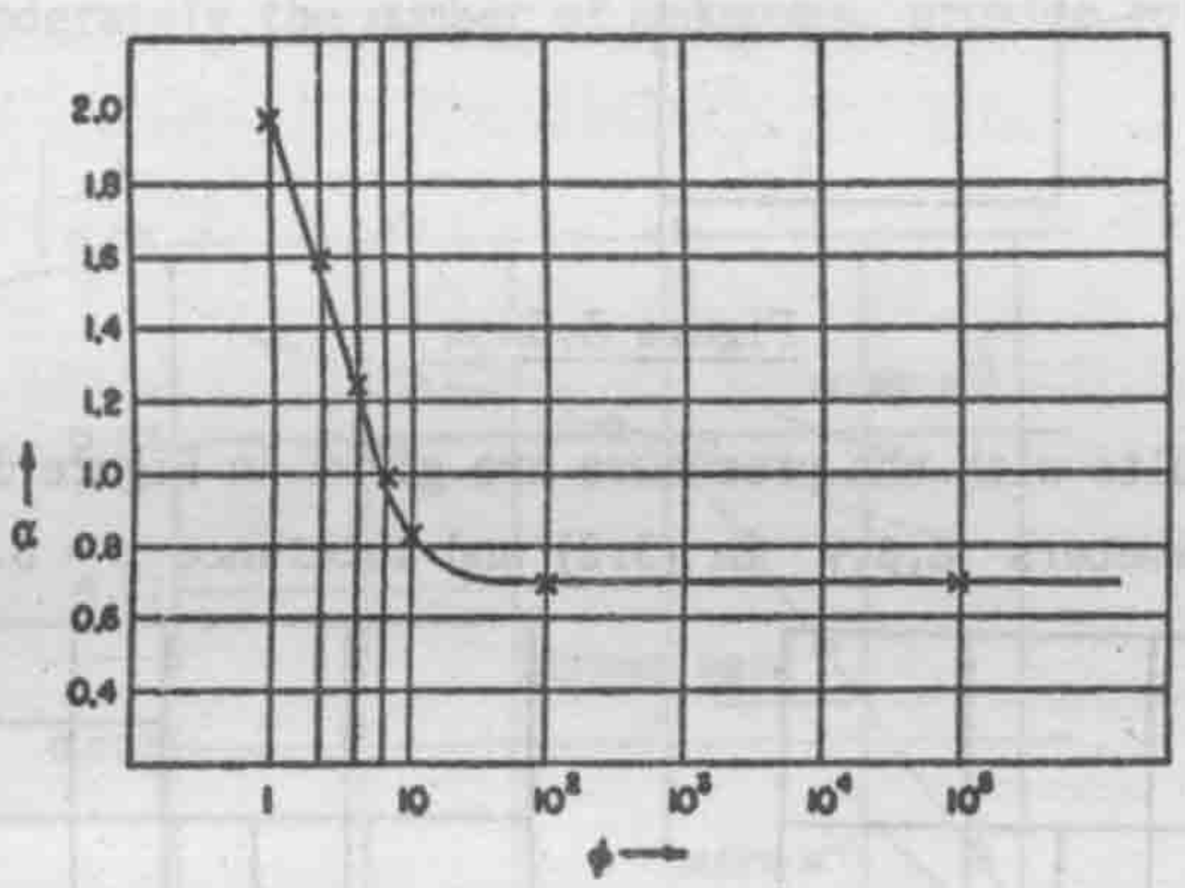


Figure 3.2

Generally, when a singularity of this type is present, it is known that for a regular mesh its effect on the accuracy will be felt throughout the region, and not just in some neighborhood. On the other hand, it has been shown (see, e.g., [19]) that there exist refinements of the mesh such that the resulting rate of convergence is the same as if no singularity were present.

These observations together indicate that for efficient computation a mesh should be constructed which incorporates a proper degree of refinement commensurate with the effect of the singularity at the desired accuracy. Such a mesh can hardly be designed a priori; instead, it must be evolved adaptively during the course of the computation.

In [20] (see also [21]) a procedure has been described for such a self-adaptive mesh refinement. More specifically we considered the numerical solution of the Dirichlet problem for Laplace's equation on an L-shaped domain (see Figure 3.3). The Dirichlet boundary conditions were chosen such that the exact solution has the form

$$u = \alpha r^{2/3} \sin 2/3\,\theta + \gamma e^{\beta x_1} \cos \beta x_2 \,. \tag{3.2}$$

A piecewise regular triangular mesh was used analogous to that shown in Figure 3.1.

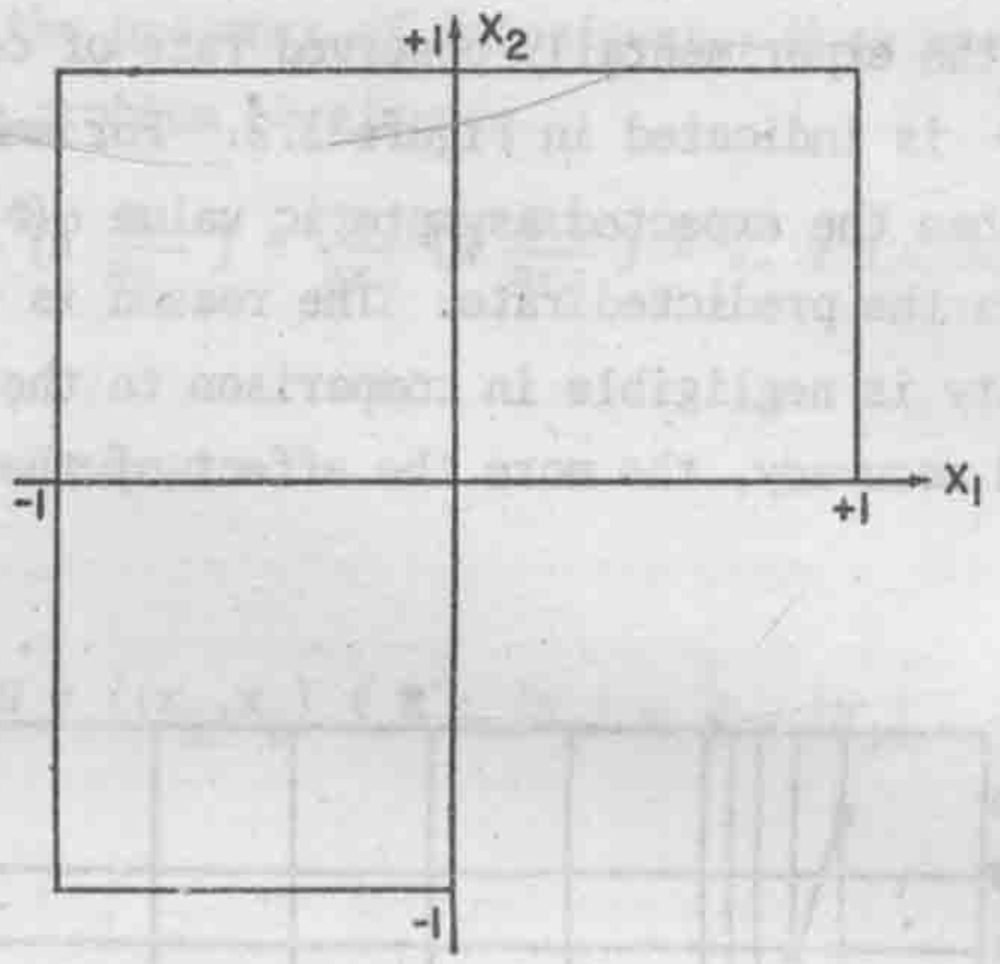
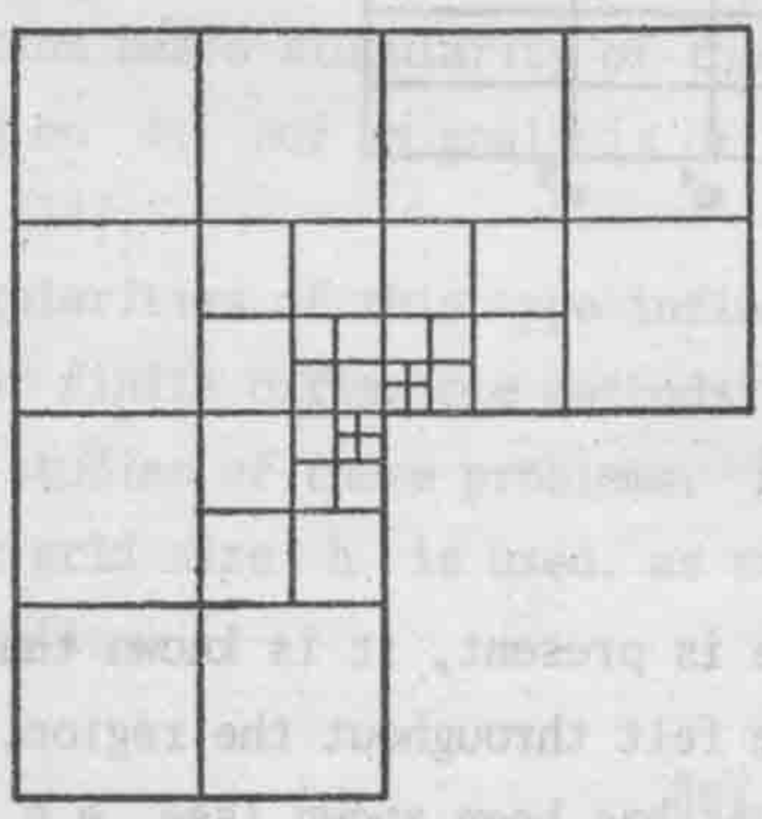


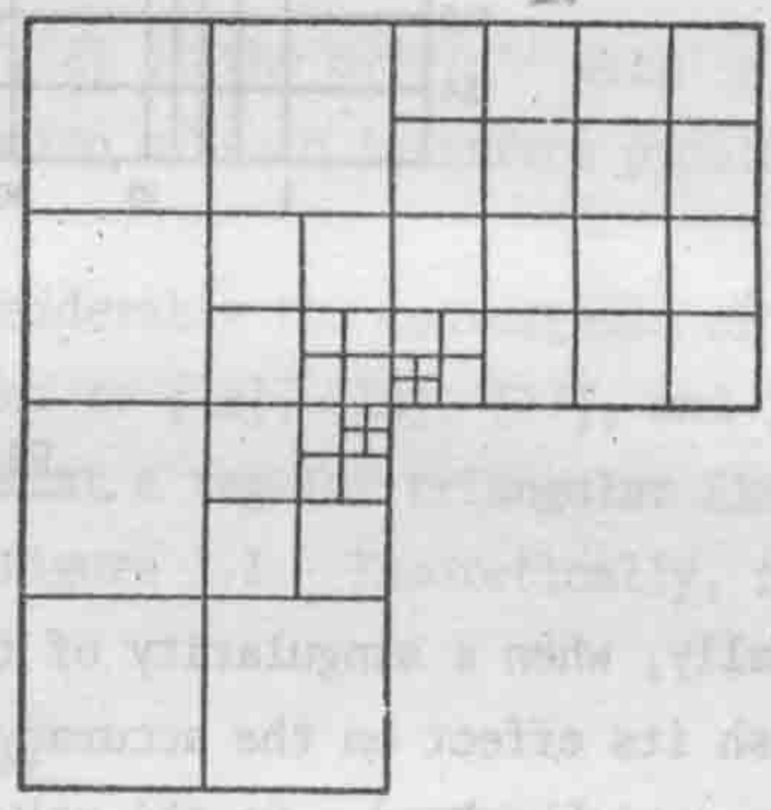
Figure 3.3

Some computational results with the procedure are given in Figure 3.4 for three different sets of the parameters α,β,γ in (3.2) and tolerance $\tau = 0.050$. Every



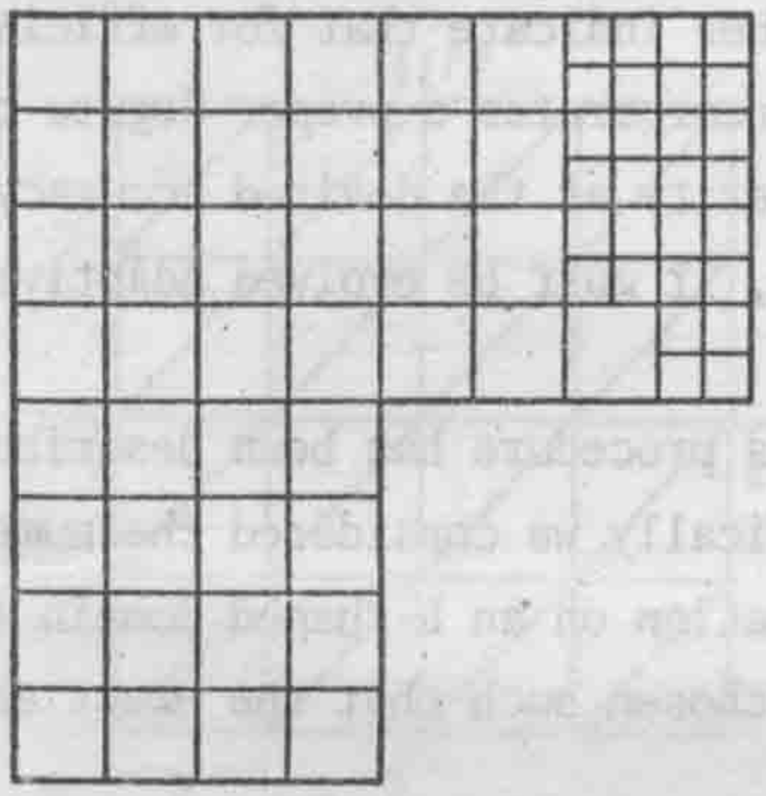
α=1.0
β=0
γ=0

τ=0.05



α=1.0
β=1.0
γ=1.0

τ=0.05



α=0
β=1.0
γ=2.5

τ=0.05

Figure 3.4

square represents here a block of 32 equally sized triangles. Figure 3.5 shows the dependence of the error in the C-norm on the number of unknowns N (that is, the number of nodal points) obtained for different tolerances τ. The dashed lines correspond to the use of a regular mesh while the solid lines give the results obtained with the adaptive mesh generator. The behavior is analogous when the L_2-norm or the H^1-norm is used. It is interesting that before the onset of the asymptotic behavior the rate of convergence is actually better than its theoretical, asymptotic bound. The explanation is that far from the singularity the mesh is already much too fine for the desired accuracy. Therefore local refinements around the corner, which increase only moderately the number of unknowns, provide for a large increase in accuracy.

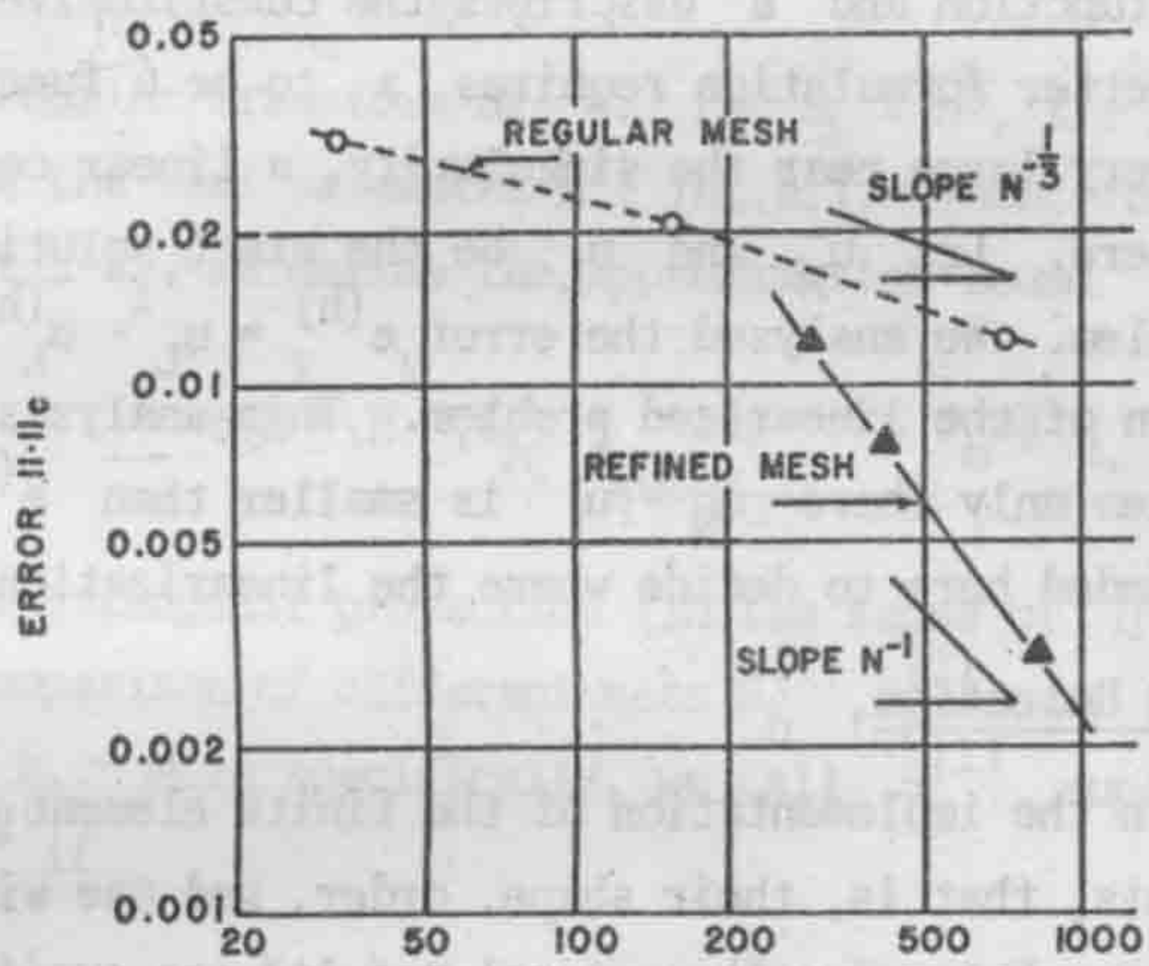


Figure 3.5

Studies of this type suggest the following conclusions:

(1) For efficient computations the finite element mesh should correspond to the desired accuracy. In any a-priori construction of the mesh it is difficult, if not impossible, to avoid over- or under-refinements in some parts of the domain resulting in decreased efficiency or accuracy or both.

(2) A self-adaptive procedure should be based on some asymptotic analysis. In our case, the behavior for $\tau \to 0$ was used, and the numerical experiments indicate that the results of this type of analysis may have a wider range of applicability than those of the standard asymptotic analysis in terms of element size.

(3) The accuracies used in the computations were relatively high and are probably not achieved in practical problems. This indicates that the discrepancy between the predictions of today's asymptotic theories and the results of practical computations should be larger than those shown here.

(4) Any self-adaptive mesh-refinement procedure depends critically on the complexity of the data structures it entails. There appears to be considerable need for studies of the data management problems of such mesh-refinements.

(5) For large problems in which the size and structure of the elements is significantly influenced by the geometry, the question remains how to obtain reasonable estimates of the reliability of computational results. Once again asymptotic results are needed here, since rigorous and realistic a-posteriori estimates are not likely to be obtainable. Some mathematical problems related to this question are addressed in the next section.

(6) In problem (3.1) the coefficient function was assumed to be constant inside the two subdomains Ω_1 and $\Omega - \Omega_1$. If, say, (3.1) represents a torsion problem, then u is the stress function and a describes the constitutive law. In this case, a theoretically better formulation requires a to be a function of grad u, and since grad u is very large near the singularity, a linear constitutive law is clearly unacceptable there. Let u_N and u_L be the exact solutions of the nonlinear and the linearized problem. We analyzed the error $e^{(h)} = u_L - u_L^{(h)}$ between u_L and the approximate solution of the linearized problem. This analysis remains relevant for the nonlinear problem only where $u_N - u_L$ is smaller than $e^{(h)}$. Once again an adaptive approach is needed here to decide where the linearization may be used.

4. Problems of Element Selection

A major question in the implementation of the finite element method is the best selection of the elements, that is, their shape, order, and use within the subdivision of the given domain. Extensive theoretical results are available about various element types, their properties, and influence upon the convergence, etc. But there are many other factors which need to enter into consideration. For instance, we should take account of the complexity of the input problem--a most laborious part of the method.

Practical experience has generated many opinions about the performance of different elements, and many articles have been devoted to experimental results on this topic. For example, it is widely agreed that the square bilinear element in R^2 performs slightly better than the corresponding square constructed of two triangular linear elements.

The performance of an element differs with the context and we should distinguish whether the element is used (i) in the interior of the domain, (ii) at its boundary, or (iii) in the presence of irregularities such as singularities, etc. We shall restrict ourselves here to some results about case (i), although some of the ideas are easily generalized to (ii). The third category (iii) requires special approaches (see, e.g., [18]).

Recent results [22], [23], and [24] about interior estimates for elliptic equations show that the error has two essential parts, a global and a local one. The global error is well understood and is generally of higher order than the local one.

Hence different elements of the same order have to be compared in terms of their local performance.

For computational ease there is good reason for the elements to be distributed locally regularly. Under this assumption, we may concentrate on the performance of meshes in R^n with translation properties. As an example, we present an analysis of a simple case which lends itself easily to considerable generalizations.

We denote by $H^k \equiv H^k(R^2)$ the standard Sobolev space over R^2, and assume that its norm is written in the form

$$\|u\|^2_{H^k} = \int_{R^2} |Fu|^2 (1+(x_1^2+x_2^2))^k \, dx \tag{4.1}$$

where F is the Fourier transform of u. Let $S_h \subset H^t$, $t \geq 0$, be a family of functions depending on the real parameter $h \in [h_0,h_1]$. Then, for any two spaces H^{k_1},H^{k_2} with $k_2 \geq k_1$ and $t \geq k_1$, we define the approximation bound

$$\Phi(H^{k_1},H^{k_2},S_h) = \sup_{\|u\|_{H^{k_2}} \leq 1} \|u-P_h u\|_{H^{k_1}} \tag{4.2}$$

where P_h is the orthogonal projection (in the sense of H^{-k_1}) of H^{k_1} onto S_h. It allows for a comparison of different sets $S_h^{[1]},S_h^{[2]}$ over the same parameter interval $h_0 \leq h \leq h_1$. More specifically, we call $S_h^{[1]}$ superior to $S_h^{[2]}$ with respect to H^{k_1},H^{k_2} if

$$\Phi(H^{k_1},H^{k_2},S_h^{[1]}) < \Phi(H^{k_1},H^{k_2},S_h^{[2]}) \, . \tag{4.3}$$

In the case of equality, the sets are said to be equivalent with respect to the two spaces.

As examples, we consider the sets:

(i) $S_h^{[1]}$, the space of continuous, piecewise bilinear functions on squares of size h as shown in Figure 4.1;

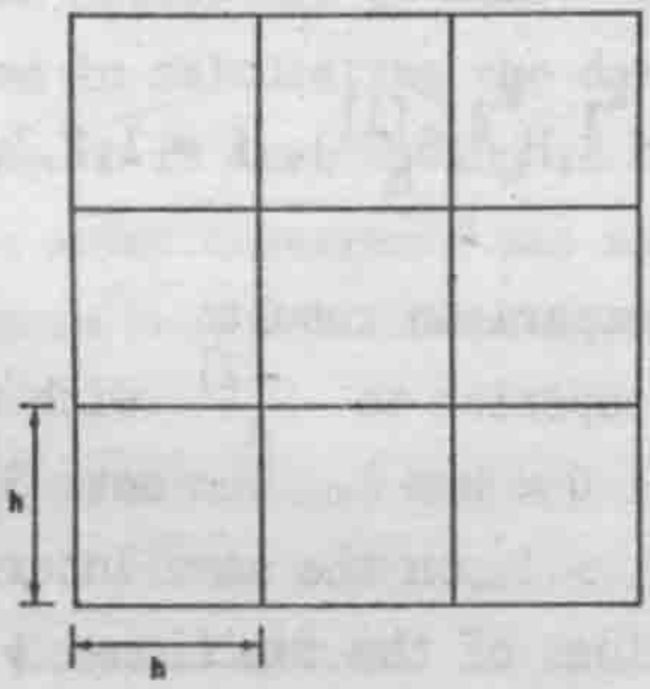


Figure 4.1

(ii) $S_h^{[2]}$, the space of continuous, piecewise linear functions on right triangles as shown in Figure 4.2;

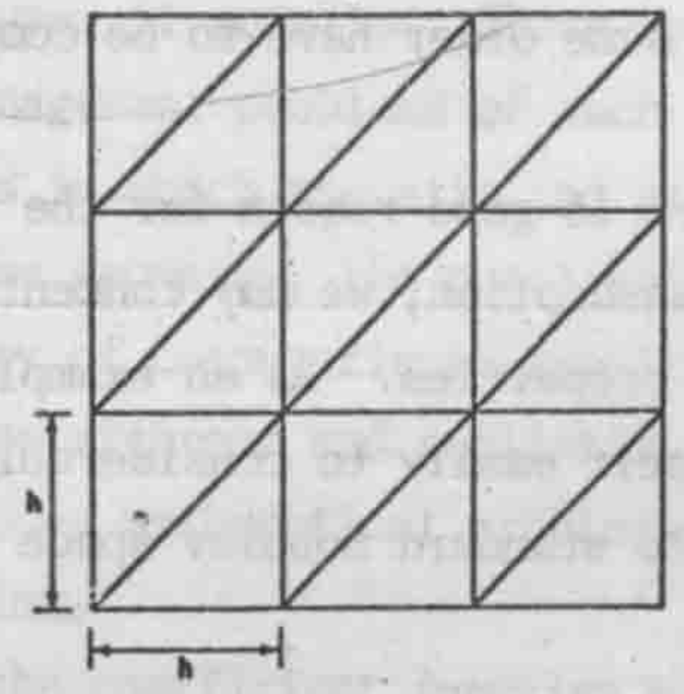


Figure 4.2

(iii) $S_h^{[3]}$, the space of continuous, piecewise linear functions on triangles as shown in Figure 4.3.

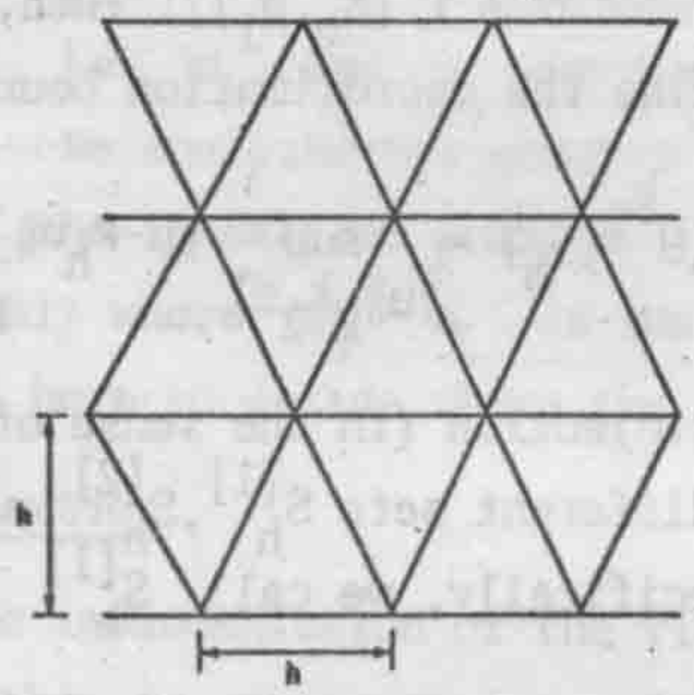


Figure 4.3

All three spaces have the same density of nodal points. They represent so-called $S_h^{t,k}$-spaces (see, e.g., [25]) with $t = 2$, $k = 3/2 - \varepsilon$, and any $\varepsilon > 0$. Then for any pair $k_1 \le k_2$, $k_1 \le k$ we have

$$\Phi(H^{k_1},H^{k_2},S_h^{t,k}) \le C(k_1,k_2,S_h^{t,k})h^{\mu},\ \mu = \min(k_2-k_1,t-k_1)\ . \tag{4.4}$$

This suggests in our case the definition

$$\psi(H^{k_1},H^{k_2},S_h^{[i]}) = h^{-\mu}\Phi(H^{k_1},H^{k_2},S_h^{[i]}),\ i = 1,2,3,\ \mu = \min(k_2-k_1,2-k_1). \tag{4.5}$$

Now we have the following comparison result:

Theorem 4.1: The set $S_h^{[1]}$ is superior to $S_h^{[2]}$ with respect to H^1 and H^{k_2}, $k_2 > 1$, on the parameter interval $0 < h \le 1$. The set $S_h^{[1]}$ is equivalent to $S_h^{[3]}$ with respect to H^1 and H^{k_2}, $k_2 > 1$, on the same interval $0 < h \le 1$.

Table 1 below gives some values of the functions ψ of (4.5) for $i = 1,2$. All numbers are rounded to two digits.