Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Asymptotic Analysis

From Theory to Application

Edited by F. Verhulst

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PREFACE

The composition and writing of these Lecture Notes arose from the realization of two facts. The first one is, that the approach of the fiftieth birthday of W. Eckhaus makes this an appropriate occasion to provide a collection of papers written by a number of mathematicians who were at some stage 'his students'. The papers in this book are dedicated to W. Eckhaus who has been a versatile and influential applied mathematician, working in the United States of America, in France and in the Netherlands. He has contributed fundamentally in the field of non-linear stability theory of partial differential equations, asymptotic analysis and several other branches of applied analysis. The field of asymptotic analysis especially attracted many young mathematicians in the Netherlands with as a result a rapid and varied development of the theory.

This leads us to the second fact. Asymptotic analysis arose from now classical theories in celestial mechanics and fluid mechanics. The flourishing of asymptotics during the last ten years, however, took place within the discipline of applied mathematics and there is an understandable lag in the application of these new theories. One expects the subtleties of matching conditions or the estimation theory by Hilbert-space methods to take some time before being applied in the physical sciences. The realization of this second fact has provided the main idea behind the writing of most of the papers in this book: to show that asymptotic analysis as a branch of mathematics can be applied to develop new theories in such varied fields as biology, plasmaphysics, celestial mechanics, Hamiltonian mechanics, the theory of ocean currents etc. It is in this dynamic sense that the contents of the book should reflect the title.

F. Verhulst

CONTENTS

ON MATCHING PRINCIPLES

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SUMMARY

Matching principles are the key of asymptotic analysis for singular perturbation problems. Starting with some classical definitions in asymptotics we recall the principal results which have been obtained to match asymptotic expansions of a singular function; these classical results are based on Kaplun's extension theorem. After Kaplun and Fraenkel, most of the results are from W. Eckhaus; in fact, he was the first to say clearly that matching is not actually a consequence of overlapping. Following all these ideas, we discuss some theorems and rules which involve matching and try to explore some new ideas with the help of simple examples and counter-examples.

INTRODUCTION

The techniques of matching, which have been proposed to yield a relationship between expansions in a small parameter ε of a singular function $\phi(x,\varepsilon)$, are very important to determine unknown constants or functions occurring in these expansions.

At the beginning of the work on the foundations of matching processes, we find S. Kaplun and P.A. Lagerstrom [4] and W.Eckhaus [3] who try to develop a systematic approach of matching. One way to find some rules is to make the overlap hypothesis; using intermediate variables Kaplun makes the assumption that there exists extended domains of validity for the so-called inner and outer expansion.

Nevertheless, in practice, it is quite useful to get matching rules in a more simple way. As L.E. Fraenkel [4] stated it, the techniques which use the idea of overlapping are often difficult and laborious. When M.D. van Dyke [2] stated his matching rule he thought this to be in the spirit of Kaplun's work; his matching principle is very simple in applications but unfortunately it is not always correct.

Using the ideas of W. Eckhaus [3,6,8] to whom this paper has been dedicated, we try to show heuristically how Van Dyke's matching rule appears to be the best one if we use it in a form written down by W. Eckhaus.

ASYMPTOTIC DEFINITIONS

Let $\phi(x,\varepsilon)$ be a function of the real variable x and the real parameter ε defined in a bounded closed domain $\mathcal{D}$: $0 \leqslant x \leqslant B_0$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$, where B_0 and ε_0 are positive constants. We suppose that this function is regular everywhere except in the neighbourhood of the origin $x = 0$ for $\varepsilon \downarrow 0$.

Thus, there exists a regular expansion of $\phi(x,\varepsilon)$ in $A_0 \leqslant x \leqslant B_0$ where A_0 is a strictly positive constant:

$$(1) \qquad \phi(x,\varepsilon) = \sum_{p=0}^{m} \delta_0^{(p)}(\varepsilon)\phi_0^{(p)}(x) + o(\delta_0^{(m)});$$

the $\delta_0^{(p)}$ are order functions such that

$$\delta_0^{(p+1)} = o(\delta_0^{(p)}) \ , \ \forall p = 0,1,2,\dots \ .$$

In general such a limit process as $\varepsilon \downarrow 0$ is non uniform in the whole domain $\mathcal{D}$; the function ϕ is said to be singular at the origin. To study asymptotic expansions of type (1) near $x = 0$, we introduce local variables,

$$x_\nu = \frac{x}{\delta_\nu(\varepsilon)} \quad \text{with } \delta_\nu(\varepsilon) = o(1) \text{ except } \delta_0(\varepsilon) = 1.$$

(Thus, we have in this notation $x = x_0$).

After this so-called stretching transformation, we assume the existence of local regular expansions in $\mathcal{D}_\nu$: $A_\nu \leqslant x_\nu \leqslant B_\nu$ where A_ν and B_ν are positive constants,

$$(2) \qquad \phi(x,\varepsilon) = \sum_{p=0}^{m} \delta_\nu^{(p)}(\varepsilon)\phi_\nu^{(p)}(x_\nu) + o(\delta_\nu^{(m)}),$$

where of course $\delta_\nu^{(p+1)} = o(\delta_\nu^{(p)}) \ \forall \ p$.

We now use a shorthand notation defining expansion operators as introduced by Eckhaus [6,8].

If $\delta^{(m)}$ is an element of a pre-assigned set of order functions, the expansion operator $E_\nu^{(m)}$ is such that,

$$(3) \qquad \phi - E_\nu^{(m)}\phi = o(\delta^{(m)}) \text{ in } x_\nu \in (A_\nu,B_\nu).$$

Then, the expansion operator has the representation,

$$(4) \qquad E_\nu^{(m)}\phi = \sum_{p=0}^{\nu(m)} \delta_\nu^{(p)}(\varepsilon)\phi_\nu^{(p)}(x_\nu),$$

where $\nu(m)$ is an integer depending on m. This is done to introduce the possibility of cutting the expansion (2) at any pre-assigned order $\delta^{(m)}$. For instance, if for fixed ν, we choose $\delta^{(m)} = \delta_\nu^{(m)}$, we get $\nu(m) = m$.

Nevertheless, we keep also the notation,

$$(5) \qquad E_\nu^{(m)}\phi = \sum_{p=0}^{m} \delta_\nu^{(p)}(\varepsilon)\phi_\nu^{(p)}(x_\nu),$$

such that,

$$(3)' \qquad \phi - E_\nu^{(m)}\phi = o(\delta_\nu^{(m)}).$$

There is no possibility for a mistake since to get (3) and (4) we must start with defining the set $\delta_\nu^{(m)}$.

A CONSEQUENCE OF THE EXTENSION THEOREM

The process which relates expansion operators $E_\nu^{(m)}$ to each other is called matching. This process can take various forms; one of them is the so-called extension theorem of S. Kaplun [1]. This theorem asserts that the domain of uniform convergence of ϕ can, in a sense, be extended to include the origin.

A consequence of this is the following theorem, the proof of which can be found in Eckhaus [3].

Theorem 1. Let $E_{\nu_1}^{(0)}\phi$ and $E_{\nu_2}^{(0)}\phi$ be two local asymptotic approximations, there exists and order function $\delta^* = o(1)$ such that, if $\delta^* \ll \frac{\delta_{\nu_2}}{\delta_{\nu_1}} \ll 1$, then, for δ_μ such that $\delta_{\nu_2} \ll \delta_\mu \ll \delta_{\nu_1}$, we have,

$$E_\mu^{(0)}E_{\nu_1}^{(0)}\phi = E_\mu^{(0)}\phi = E_\mu^{(0)}E_{\nu_2}^{(0)}\phi.$$

This theorem has some importance; it means that we have matching if the two local approximations are sufficiently near to each other. It is the basis of overlapping using intermediate approximations. But one must be very cautious when using such a theorem. In practice, the set of stretching order functions δ_ν is not large enough to apply the result. For instance, if we consider the function

$$\phi(x,\varepsilon) = \frac{1}{\log x} + \frac{e^{-\frac{x}{\varepsilon}}}{\log\varepsilon}$$

with the classical set ε^ν $(\nu \geqslant 0)$, putting

$$x_\nu = \frac{x}{\varepsilon_\nu}, \text{ we get}$$

$$E_0^{(0)}\phi = \frac{1}{\log x}, \quad E_\nu^{(0)}\phi = \frac{1}{\nu\log\varepsilon}, \quad E_1^{(0)}\phi = \frac{1+e^{-x_1}}{\log\varepsilon}$$

Now, if the theorem would apply, there would exist a $\mu < 1$ such that

$$\frac{1}{\log\varepsilon} = E_{\mu}^{(0)}E_{1}^{(0)}\phi = E_{\mu}^{(0)}\phi = \frac{1}{\mu\log\varepsilon}$$

so, we get $\mu = 1$ which contradicts the hypothesis $\mu < 1$. This illustrates one of the reasons to get a rule as Van Dyke's one [2] which is independent of the subset chosen for the stretching order functions. In fact, Van Dyke's rule works perfectly in this case:

$$E_{1}^{(0)}E_{0}^{(0)}\phi = E_{0}^{(0)}E_{1}^{(0)}\phi = \frac{1}{\log\varepsilon}.$$

ASYMPTOTIC MATCHING RULE WITH OVERLAPPING

We shall now discuss a very important theorem of W. Eckhaus [8] which shows that the overlap hypothesis implies the validity of a generalized asymptotic matching principle.
We choose the special subset ε^{ν} for the set of stretching order functions; this is a very usual choice if one tries to find a practical matching principle. $\nu = 0$ now defines the classical outer expansion $E_{0}^{(m)}\phi$, and $\nu = 1$, the inner expansion $E_{1}^{(m)}\phi$.

Theorem 2. If for $0 \leqslant \nu \leqslant 1$, *we have*

$$\phi - E_{\nu}^{(m)}\phi = o(\varepsilon^{m-\gamma})$$

for all $m = 1,2,\ldots$ *where* γ *is an arbitrarily small positive number and if we have overlapping, that is, for any integer* s, *there exist integers* m *and* n *for which,*

(6) $$E_{\nu}^{(s)}E_{0}^{(m)}\phi = E_{\nu}^{(s)}E_{1}^{(n)}\phi = E_{\nu}^{(s)}\phi,$$

then, for any integer s,

(7) $$E_{0}^{(s)}E_{1}^{(s)}\phi = E_{1}^{(s)}E_{0}^{(s)}\phi.$$

In this theorem, $E_{0}^{(m)}\phi$ and $E_{1}^{(m)}\phi$ belong to what Eckhaus calls "a normal class of expansions". This means that expansion operators are supposed to be regularizing operators in such a way that functions as $\phi_{0}^{(p)}(x)$ can be written as a sum of terms of the type $x^{\tau}(\log x)^{\sigma}$; in applications this is not a severe restriction.
Also, the special choice of $\varepsilon^{m-\gamma}$ shows that truncating expansions between logarithms is impossible and thus, we have to pay attention when using this theorem. In [8], one can find an example from Fraenkel [4] which clearly shows that if Van Dyke's rule is wrong, theorem 2 indicates the correct way to match. One should also note that cutting all expansions, outer, inner and intermediate, at the same order, is a very good idea as we shall see later; moreover, in practice, this is easier in constructing a composite expansion.

ASYMPTOTIC MATCHING RULE WITHOUT OVERLAPPING.

As suggested by W. Eckhaus [8] and also by Fraenkel [4], overlap is not a necessary prerequisite for the validity of an asymptotic matching principle. Most of the people working with singular perturbations appear to think exactly the opposite [2,5]. Nevertheless a few things have been done in this way [7], but only for first order matching.
In the following, we want to give some simple examples which will help us to understand why overlapping is not necessary to get a matching rule.

Outer Overlapping.

We assume that, for $\nu < 1$, for all s, there exists m such that

$$E_\nu^{(s)}\phi = E_\nu^{(s)}E_0^{(m)}\phi, \tag{8}$$

but, for $\nu = 1$, this is not possible. This means that if for $\nu < 1$, the intermediate expansion $E_\nu^{(s)}\phi$ is contained in the outer expansion, this is not the case for the inner expansion which is, in a sense, a significant expansion. The process described here is a very classical one for users; we shall call it outer overlapping. If all expansions are taken at the same order we propose the *special matching*,

$$E_\nu^{(s)}E_1^{(s)}E_0^{(s)}\phi = E_\nu^{(s)}E_1^{(s)}\phi \qquad \nu < 1, \tag{9}$$

with the rule

$$E_0^{(s)}E_1^{(s)}E_0^{(s)}\phi = E_0^{(s)}E_1^{(s)}\phi \tag{10}$$

In that case, a composite expansion is given by

$$\phi = E_0^{(s)}\phi + E_1^{(s)}\phi - E_1^{(s)}E_0^{(s)}\phi + o(\delta^{(s)}). \tag{11}$$

Example 1. $\phi(x,\varepsilon) = \frac{1}{\log x} + \frac{e^{-\frac{x}{\varepsilon}}}{\log\varepsilon}$.

$$E_0^{(m)}\phi = \frac{1}{\log x}, \qquad \forall m$$

$$E_\nu^{(3)}\phi = \frac{1}{\nu\log\varepsilon} - \frac{\log x_\nu}{\nu^2(\log\varepsilon)^2},$$

$$E_1^{(3)}\phi = \frac{1+e^{-x_1}}{\log\varepsilon} - \frac{\log x_1}{(\log\varepsilon)^2}.$$

Evidently (8) holds good. In fact, we have

$$E_\nu^{(3)}E_1^{(3)}E_0^{(3)}\phi = \frac{2-\nu}{\log\varepsilon} - \frac{\log x_\nu}{(\log\varepsilon)^2} = E_\nu^{(3)}E_1^{(3)}\phi$$

and the rule,

$$E_0^{(3)}E_1^{(3)}E_0^{(3)}\phi = E_0^{(3)}E_1^{(3)}\phi = \frac{2}{\log\varepsilon} - \frac{\log x}{(\log\varepsilon)^2}\ .$$

For this illustrative example we took the order 3 but it is true for any s. This was demonstrated in [7] for the order 1 on the basis of the continuity of order functions. The composite expansion is the function itself.

Inner Overlapping

In the same way, for $\nu > 0$, for all s, we assume that there exists a number n such that,

$$(12) \qquad E_\nu^{(s)}\phi = E_\nu^{(s)}E_1^{(n)}\phi.$$

However, for $\nu = 0$ this is not possible. In that case, the inner expansion is containing the intermediate expansion; we shall call this inner overlapping. As in the preceding case, if all expansion operators are taken at the same order, we propose

$$(13) \qquad E_\nu^{(s)}E_0^{(s)}E_1^{(s)}\phi = E_\nu^{(s)}E_0^{(s)}\phi \qquad \nu > 0$$

with the rule

$$(14) \qquad E_1^{(s)}E_0^{(s)}E_1^{(s)}\phi = E_1^{(s)}E_0^{(s)}\phi.$$

In that case, a composite expansion is given by

$$(15) \qquad \phi = E_0^{(s)}\phi + E_1^{(s)}\phi - E_0^{(s)}E_1^{(s)}\phi + o(\delta^{(s)}).$$

Example 2. $\phi(x,\varepsilon) = \dfrac{1}{\log x - \log\varepsilon + 1}$

$$E_0^{(3)}\phi = -\frac{1}{\log\varepsilon} - \frac{1+\log x}{(\log\varepsilon)^2},$$

$$E_\nu^{(3)}\phi = -\frac{1}{(1-\nu)\log\varepsilon} - \frac{1+\log x_\nu}{(1-\nu)^2(\log\varepsilon)^2}\ ,$$

$$E_1^{(m)} = \frac{1}{1+\log x_1}\ , \qquad \forall m$$

Now (12) holds good and we can apply (13,14,15).

$$E_\nu^{(3)}E_0^{(3)}E_1^{(3)}\phi = -\frac{1+\nu}{\log\varepsilon} - \frac{1+\log x_\nu}{(\log\varepsilon)^2} = E_\nu^{(3)}E_0^{(3)}\phi$$

and, for $\nu = 1$, we get the rule

$$E_1^{(3)}E_0^{(3)}E_1^{(3)}\phi = E_1^{(3)}E_0^{(3)}\phi = -\frac{2}{\log\varepsilon} - \frac{1+\log x_1}{(\log\varepsilon)^2}.$$

As in the preceding case, (15), the composite expansion is the function ϕ itself.

No overlapping.

In the two cases of semi overlapping, one should note that the composite expansions (11) and (15) are essentially different. However, as in the examples 1 and 2, it was not possible to find a counter example where, in one of the preceding cases, $E_0^{(s)}E_1^{(s)}$ was different of $E_1^{(s)}E_0^{(s)}$. Thus, it is easy, by composition of these two cases, to construct a function $\phi(x,\varepsilon)$ where there is no overlapping at all and it is easy to understand why a matching rule is going to work. This situation has been studied in [7] and the example was treated in [8]:

$$\phi(x,\varepsilon) = \frac{1}{\log x} + \frac{e^{-\frac{x}{\varepsilon}}}{\log \varepsilon} + \frac{1}{\log x - \log \varepsilon + 1}.$$

In this last case, it is very clear that there is no overlapping but the intermediate expansion is contained partly in the outer expansion, partly in the inner approximation. Finally, it is easy to construct an example where this is not the case so that we have no overlapping and no matching rule:

Example 3. $\phi(x,\varepsilon) = \dfrac{1}{\log x - \log \varepsilon + 1} + \dfrac{\log(x+\varepsilon)}{(\log \varepsilon)^2}$;

it is not difficult to compute,

$$E_0^{(2)}E_1^{(2)} = 0 \text{ and } E_1^{(2)}E_0^{(2)} = -\frac{1}{\log \varepsilon}.$$

Of course, we are cutting expansions between logarithms but there is still some work to do in this field. Moreover, since Theorem 2 of W. Eckhaus [8] is the best one we know, we have shown now that the conditions stated there for certain applications are too restrictive. A rather elaborate analysis of all these problems can be found in [9] and more can be expected in the future.

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SINGULAR PERTURBATIONS OF SPECTRA

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ABSTRACT

A mathematical description of free vibrations of a membrane leads to eigenvalue problems for elliptic differential operators containing a small positive parameter ε in the highest order part. The asymptotic behaviour (for $\varepsilon \to +0$) of the eigenvalues is studied in second order problems that reduce to zero-th and first order for $\varepsilon = 0$ and in a fourth order problem that reduces to an elliptic problem of second order. In the case of reduction to zero-th order the density of the eigenvalues on a half-axis grows beyond bound and is proportional to $\varepsilon^{-n/2}$ (in n dimensions). In the case of reduction to first order the relation between the asymptotic behaviour of the spectrum and the critical points of the reduced operator is shown. In the case of reduction to second order an asymptotic series expansion is constructed for every eigenvalue.

1. INTRODUCTION

An important aspect in the mechanical theory of plates and shells is the study of vibrations. In a mathematical model for those shells, the relations between deflections, stresses and loads are described by differential equations, the constraints lead to boundary conditions to be imposed, and the free vibrations are represented by eigenvalue problems for those differential equations. A typical equation which describes small deflections W of a clamped membrane of shape Ω, which is stressed uniformly, is

$$(1.1) \qquad \rho \frac{\partial^2 W}{\partial t^2} = N\Delta W, \qquad W_{|\Gamma} = 0, \quad (\Gamma = \text{boundary of } \Omega) ,$$

where ρ is the density per unit area and N the stress. The determination of the free modes $W(x,y,t) = u(x,y)e^{i\omega t}$ naturally leads to the eigenvalue problem

$$(1.2) \qquad \Delta u + \lambda u = 0, \qquad u_{|\Gamma} = 0, \qquad \lambda = \rho\omega^2/N .$$

A more sophisticated model of the same membrane takes into account that the membrane is a shell with finite (small) thickness h and has a flexural rigidity D,

$$D := Eh^3/12(1 - \nu^2) ,$$

where E is the elasticity and ν is Poisson's ratio. This leads to the improved model equation, cf. Timoshenko [16, ch. 8],

$$(1.3) \qquad \rho \frac{\partial^2 W}{\partial t^2} = -D\Delta^2 W + N\Delta W, \qquad W_{|\Gamma} = \frac{\partial W}{\partial n}_{|\Gamma} = 0 ,$$

in which D is a small parameter. It looks quite natural that the free modes of (1.3) converge to those of (1.1) if D decreases to zero; we shall prove this in section 5.

We get another type of problem if we consider a membrane on which body forces are exerted and whose tension is weak with respect to those body forces, e.g. a thin metallized membrane in an electromagnetic field. This is described by the model equation, cf. [16, ch. 8],

$$(1.4) \qquad \rho \frac{\partial^2 W}{\partial t^2} = N\Delta W + X \frac{\partial W}{\partial x} + Y \frac{\partial W}{\partial y} , \qquad W_{|\Gamma} = 0 ,$$

where (X,Y) is the body force and may depend on (x,y). In this case the behaviour of the free modes (if present) depends heavily on the field (X,Y). The eigenvalues may disappear at infinity, they may remain discrete or tend to a dense set for $N \to 0$. We shall deal with these problems in sections 3-4.

These mechanical models motivate the study of the following eigenvalue problems on a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ,

$$(1.5) \qquad -\varepsilon\Delta u + p(x,y)u = \lambda u, \qquad u_{|\Gamma} = 0 ,$$

$$(1.6) \qquad -\varepsilon\Delta u + p(x,y)\partial_x u + q(x,y)\partial_y u = \lambda u, \qquad u_{|\Gamma} = 0 ,$$

$$(1.7) \qquad \varepsilon\Delta^2 u - \Delta u = \lambda u, \qquad u_{|\Gamma} = \frac{\partial u}{\partial n}_{|\Gamma} = 0 ,$$

where ε is a small positive parameter, where p and q are smooth real functions on Ω and where λ is the (complex) spectral parameter. We shall study how the eigenvalues of these problems behave as ε decreases to zero.

We shall show that the eigenvalues of problem (1.5) decrease with ε, and that their density (above the minimum of p) increases beyond bound for $\varepsilon \to +0$ and is proportional to $1/\varepsilon$. The eigenvalues of the third problem (1.7) decrease also, but they remain well separated and (as we expect) they converge for $\varepsilon \to +0$ to the eigenvalues of Dirichlet's problem $-\Delta u = \lambda u$, $u_{|\Gamma} = 0$; if Γ is smooth enough we can construct asymptotic series in powers of $\varepsilon^{\frac{1}{2}}$ for the eigenvalues and eigenfunctions. The spectral properties of the second problem (1.6) depend heavily on the characteristics of the first order operator $p\partial_x + q\partial_y$: all eigenvalues may recede to infinity (if Ω does not contain critical points of $dy/dx = p/q$), they may tend to a discrete set or their density may grow beyond bound.

The problems (1.5-6-7) are prototypes of much more general elliptic singularly perturbed boundary value problems in n-dimensional space, for which we can obtain analogous results. We have avoided this greater generality, lest the essential techniques should be obscured by the amount of calculations.

Another motivation for the study of the eigenvalue problem $L_\varepsilon u = \lambda u$, where L_ε stands for an operator defined in (1.5-6 or 7), is the study of the steady state equation $L_\varepsilon u = f$ (+ boundary conditions). It may be dangerous to construct inadvertently a formal approximate solution of $L_\varepsilon u = f$, if zero is the (unknown) limit of an eigenvalue. As an example, we refer to [1], [14] and related papers on the singularly perturbed turning point problem (the one-dimensional analogue of (1.6)), where fallacious and contradictory results were obtained by use of merely formal methods. See also [3].

NOTATIONS

Let Ω be a bounded open set in the plane ($\mathbb{R}^2$) with boundary Γ. It satisfies the *cone condition* if for any point $(x,y) \subset \Omega$ we can place a cone of fixed height h and aperture ω with its top at (x,y) in such a way that the cone is contained inside Ω completely. $H^k(\Omega)$, with $k = 0,1,2,\ldots$, is the set of functions on Ω, whose derivatives up to the order k are square integrable; in particular $H^0(\Omega) = L^2(\Omega)$. $H_0^k(\Omega)$ is the subset of $H^k(\Omega)$ of functions whose derivatives up to the order $k-1$ are zero at Γ (provided Γ smooth enough). Functions in $H_0^k(\Omega)$ may be considered as functions on the whole plane if we continue them by zero outside Ω; these continuations are in $H^k(\mathbb{R}^2)$. In $L^2(\Omega)$ the forms $(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual inner product and norm

$$(u,v) := \iint_\Omega u(x,y)\bar{v}(x,y)\,dxdy, \qquad \|u\| := (u,u)^{\frac{1}{2}} ,$$

and in $H^1(\Omega)$ the vectorized forms $(\nabla u,\nabla v)$ and $\|\nabla u\|$ are defined by

$$(\nabla u,\nabla v) := (\partial_x u,\partial_x v) + (\partial_y u,\partial_y v), \qquad \|\nabla u\| := (\nabla u,\nabla u)^{\frac{1}{2}} .$$

The Laplace operator Δ is a formal differential operator, which may act on all functions in $H^2(\Omega)$; it is made to an (invertible) differential operator by restricting it to a suitable domain, e.g. $\Delta_{|D}$ is the restriction to the domain $D \subset H^2(\Omega)$. In general we shall denote the domain of an differential operator T by $\mathcal{D}(T)$ and its range by $\mathcal{R}(T)$.

The symbols ∂_x and ∂_y denote partial derivatives in the x and y-direction and ∂_n denotes the normal derivative in the direction of the outword drawn normal at the boundary.

2. THE EIGENVALUES AND RAYLEIGH'S QUOTIENT

Let T be a selfadjoint operator on a Hilbert space $\mathcal{H}$, let T be semibounded from below (i.e. $(Tu,u) \geq \gamma(u,u)$, $\gamma \in \mathbb{R}$) and let it have a compact inverse. As is well-known, cf. [12, ch. 3, § 6.8], the spectrum of T, $\sigma(T)$, consists of real isolated eigenvalues of finite multiplicity and the set of eigenfunctions corresponding to these eigenvalues is a complete orthonormal set in $\mathcal{H}$. Since T is semibounded with lower bound γ, no eigenvalue can be smaller than γ; hence we can arrange the eigenvalues in a non-decreasing sequence such that

$$(2.1) \qquad \sigma(T) = \{\lambda_k \mid k \in \mathbb{N}\} \quad \text{with} \quad \lambda_{k+1} \geq \lambda_k, \ \forall_k$$

and such that each eigenvalue appears in the sequence as many times as its multiplicity is (the eigenvalue is counted according its multiplicity). To each eigenvalue λ_k corresponds an eigenfunction e_k such that $\{e_k \mid k \in \mathbb{N}\}$ is a complete orthonormal set in $\mathcal{H}$.

Since T is selfadjoint the inner product (Tu,u) is real for all $u \in \mathcal{D}(T)$. Expanding u in the eigenfunctions we find (if $u \neq 0$)

$$(2.2) \qquad \frac{(Tu,u)}{(u,u)} = \sum_{k=1}^{\infty} \frac{\lambda_k (u,e_k)^2}{(u,u)} .$$

Clearly this quotient is minimal if $u = e_1$; it then yields the first eigenvalue. More general, if V is the span of k eigenfunctions, the maximum of the quotient (2.2) is just the largest eigenvalue connected to the eigenfunctions in V; clearly this maximum is minimized and equal to λ_k, if V is the span of the first k eigenfunctions. So it is plausible that λ_k satisfies the minimax characterization

$$(2.3) \qquad \lambda_k = \min_{V \subset \mathcal{D}(T), \dim V = k} \ \max_{u \in V, u \neq 0} \frac{(Tu,u)}{(u,u)} .$$

The quotient (2.2) is called Rayleigh's quotient; the minimax characterization (2.3) is easily proved in the way suggested above, cf. [5, ch. 11].