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Peter Abramenko

Twin Buildings and Applications to S-Arithmetic Groups

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Author

Peter Abramenko
Fachbereich Mathematik
Universität Frankfurt
Robert-Mayer-Str. 6–10
D-60054 Frankfurt, Germany
e-mail: abramenk@math.uni-frankfurt.de

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To my mother

and

to my brother

Preface

The present Lecture Notes volume combines aspects of two mathematical domains which are closely connected to each other: group theory and the theory of buildings. On the basis of investigations concerning twin buildings and subcomplexes of spherical buildings, finiteness properties of some S-arithmetic groups are derived (cf. the introduction for more details).

Large parts of this book are devoted to the theory of (twin) buildings and not only written with group theoretic applications in mind. The first two sections of Chapter I can serve very well as an introduction to twin buildings. §1 describes the group theoretic background of this new theory. The basic definitions and facts (see in particular Lemma 2) are introduced in §2. Though these results are mainly due to Tits, the complete proofs are given here since they are hard to find in the original papers or not yet published. The following two sections present some of my own investigations concerning twin buildings. These are applied at the end of Chapter I in order to yield Theorem A. This theorem is one major step on the way towards the results about S-arithmetic groups presented in Chapter III. The second main theorem needed in this context is proved in the course of Chapter II. It generalizes the well known Solomon–Tits theorem and states that certain subcomplexes of spherical buildings are homotopy equivalent to bouquets of spheres "in general" (cf. Section 4 of the introduction). The techniques of the proof combine Tits' classification of spherical buildings with some combinatorial ideas. This part of the book is accessible to every reader who has studied Tits' Lecture Notes volume on spherical buildings.

I gladly take the opportunity to thank at least a few of those who helped in one way or the other that this book could be written. First of all, I am greatly indebted to Prof. H. Behr for his personal interest in me, for the opportunities he offered to me and for many stimulating mathematical discussions. I am also very obliged to Prof. H. Abels who invited me several times to the SFB 343 in Bielefeld where I spent more than 17 months altogether and where parts of these notes first took shape. In this context, I would also like to thank the DFG for the financial support I received during that time. Last but not least I express my warmest thanks to Mrs. Ch. Belz for preparing the TEX-version of the present book and for doing this exceptionally well.

Frankfurt am Main, September 1996

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Introduction

In the following pages, I will try to describe briefly the background of this book, the key questions, the progress that has been achieved and some of the problems which are left for future work.

1. Finiteness properties of S–arithmetic groups

Large parts of the present notes are devoted to the theory of buildings and are of interest in their own right. However, the origin of these investigations was a group theoretic question which I am going to describe now.

Since the last century, groups of invertible matrices have been studied extensively, partly because of their geometric significance (one may think of $O_m(\mathbb{R})$ and other "groups of motion"). Since the books of Weyl (1946) and Dieudonné (1948/55), important classes of these linear groups, namely the general and special linear, the orthogonal (often also the spin-), the symplectic and the unitary groups over skew fields, have been subsumed under the notion of "**classical groups**". A common feature of all classical groups is the fact that they can be defined by algebraic equations over a commutative subfield k of the skew field K in question if the latter is finite-dimensional over its center. In order to mention at least one example, I recall that any orthogonal group is of the form $O_m(k,Q) = \{g \in GL_m(k) \,|\, g^tQ\,g = Q\}$, showing that the entries g_{ij} of $g \in O_m(k,Q)$ satisfy a system of quadratic equations.

Starting with the papers of Borel and Chevalley in the mid 1950's, a systematic abstract theory of **linear algebraic groups** has been developed. Classical groups belong to the central subjects of this theory which was also strongly influenced by Lie theory on the other side. Chevalley's classification of semisimple algebraic groups over arbitrary algebraically closed fields, motivated by and at the same time vastly generalizing the corresponding result concerning semisimple complex Lie groups, represents one of the highlights in the theory of linear algebraic groups.

From the beginning, not only the Lie groups but also their arithmetic subgroups like e.g. $SL_m(\mathbb{Z})$, $Sp_{2m}(\mathbb{Z})$ have been of interest. By the way, "most" discrete subgroups of finite covolume in semisimple Lie groups are arithmetic by a celebrated theorem due to Margulis (for a precise statement and much more information about S-arithmetic groups, I refer to [M]). However, the notion of an arithmetic group in its original meaning (involving only $\mathbb{Q}$-groups and the Ring $\mathbb{Z}$) is too restrictive in many respects. As S-arithmetic subrings of global fields are natural generalizations of $\mathbb{Z}$, arithmetic groups are generalized by **S-arithmetic groups**. The prototype of an S-arithmetic group is represented by $\mathcal{G}(\mathcal{O}_S)$, where $\mathcal{G}$ is an algebraic group defined over a global field k and $\mathcal{O}_S$ is the ring of S-integers in k. In general, all subgroups of $\mathcal{G}(k)$ commensurable with $\mathcal{G}(\mathcal{O}_S)$ are called S-arithmetic. The applications mentioned in the title of this book refer to S-arithmetic groups of the form $\mathcal{G}(\mathbb{F}_q[t])$ or $\mathcal{G}(\mathbb{F}_q[t,t^{-1}])$, where $\mathcal{G}$ is a semisimple algebraic $\mathbb{F}_q$-group (cf. Chapter III, § 2, Theorem C, Corollary 20 and Remark 17 iv)).

Regarding the structure of an S-arithmetic group Γ, some questions are suggesting themselves. Can one find a finite set of generators for Γ ? Is Γ finitely presented? What about higher (homological) finiteness properties? In this context, I recall the following: Γ is said to be of **type $\mathbf{FP_n}$** ($n \in \mathbb{N}_0 \cup \{\infty\}$) if there exists a projective resolution of the trivial Γ-module $\mathbb{Z}$ such that the first $n+1$ projective Γ-modules are finitely generated. This implies for example that all homology and cohomology groups $H_i(\Gamma)$, $H^i(\Gamma)$ are finitely generated abelian groups for $0 \le i \le n$. I mention in passing that commensurable groups are of the same FP-type. The properties FP_1 and finitely generated are equivalent; finite presentability implies FP_2 and "often" coincides with FP_2. (But there exist groups of type FP_2 which are not finitely presented as was shown recently, cf. [BB].) I refer to [Bi] for further interesting consequences of the FP_n-property. Modifying this notion slightly, one says that Γ is of **type $\mathbf{F_n}$** if there exists an Eilenberg–MacLane complex of type $K(\Gamma,1)$ with finite n-skeleton (respectively, with finite m-skeleton for all $m \in \mathbb{N}$ if $n = \infty$). This is equivalent to requiring FP_n plus finite presentability in case $n \ge 2$ (cf. [Br1], Ch. VIII, §7).

As for answers to the questions stated above, one has to distinguish between the number field and the function field case. Suppose that Γ is an S-arithmetic subgroup of the linear algebraic group $\mathcal{G}$ defined over the global field k. We first assume that k is a number field. If Γ is arithmetic in the narrow sence (i.e. $k = \mathbb{Q}$ and $S = \{\infty\}$), it is always finitely presentable and even of type F_∞ according to results of Raghunathan (cf. [Ra]), respectively of Borel and Serre (cf. [BoS1]). If Γ is just S-arithmetic, a similar statement is not true any longer. For example, the additive group of $\mathbb{Z}[\frac{1}{p}]$ is not finitely generated. However, if $\mathcal{G}$ is **reductive**, Γ is again of type F_∞ as was shown by Borel and Serre in [BoS2]. In fact much more is proved there, for example that Γ is virtually of type FL and a duality group. Finally, the finitely presented S-arithmetic groups were completely characterized by Abels in the number field case (cf. [A1]).

Next we assume that k is a global function field, i.e. a finite extension of a rational function field $\mathbb{F}_q(t)$. We additionally suppose that $\mathcal{G}$ is reductive and isotropic over k (if $\mathcal{G}$ is k-anisotropic, then Γ is cocompact and hence of type F_∞ according to Theorem 4 of [Se1]). Contrary to the number field case, Γ need not even be of type F_1 here. It was first observed by Nagao in 1959 that $SL_2(\mathbb{F}_q[t])$ is not finitely generated (cf. [N]). Using group actions on trees, this fact was explained very nicely by Serre some years later (cf. [Se2], Ch. II, §1.6). Since 1959 several mathematicians contributed to the solution of the problem regarding finite generation and finite presentability of Γ (cf. the references in [Be1] and [Be2]). Eventually in 1992, Behr was able to give a full proof for the complete characterization — conjectured by him already years before — of all finitely presented S-arithmetic subgroups of reductive groups defined over global function fields (cf. [Be2]). The complete solution of the corresponding problem concerning higher finiteness properties of Γ will perhaps require another couple of decades. At the moment, the result is only known for some classes of S-arithmetic groups. It is shown in Stuhler's paper [Stu] that $SL_2(\mathcal{O}_S)$ is of type F_{s-1} but not of type FP_s for any S-arithmetic function ring $\mathcal{O}_S$ with $\# S = s$. (By the way, a similar result concerning the subgroup of all upper triangular matrices in $SL_2(\mathcal{O}_S)$ is derived in [Bu].) On the other side, $SL_{n+1}(\mathbb{F}_q[t])$ is of type F_{n-1} but not of type FP_n provided that q is "sufficiently big" (cf. [Ab1] and [A2]). Analogous results are derived — presupposing Theorem B (cf. Chapter II, § 8), the proof of which is published here for the first time — in [Ab3] for all classical Chevalley groups over $\mathbb{F}_q[t]$. They will reappear as special cases of Theorem C below. Apart from Stuhler's paper and from the quantitatively slightly better result for $SL_{n+1}(\mathbb{F}_q[t])$ derived in [Ab1] (cf. Remark 17 ii)), this Theorem C contains all what is known at the moment concerning higher finiteness properties of S-arithmetic subgroups of reductive groups in the function field case.

2. Filtrations of Bruhat–Tits buildings

Almost all the results mentioned in the last paragraphs were proved by using topological methods. The definition of the property F_n already indicates that finiteness properties of groups are closely connected with topology. Even problems regarding finite generation and finite presentability, though in principle accessible to the methods of algebraic K-theory, are sometimes more successfully attacked by studying the action of the group in question on an appropriate topological space. This is well demonstrated by the proof of Behr's theorem given in [Be2].

Now for a given S-arithmetic subgroup Γ of a reductive group $\mathcal{G}$ defined over a global field k , a suitable space X with natural Γ-action can be obtained as follows: Denote by k_v the completion of k relative to v . Let X_v be the quotient space of $\mathcal{G}(k_v)$ modulo a maximal compact subgroup if v is archimedian, respectively the Bruhat–Tits building associated to $\mathcal{G}(k_v)$ as described in [BrT1,2] if v is non-archimedian. Then consider $X = \prod_{v \in S} X_v$ with diagonal Γ-action.

Though space and action enjoy "nice" properties (X is contractible and the Γ-action is proper), finiteness properties for Γ cannot be deduced directly unless the quotient X/Γ is compact. Essentially two methods have been applied so far in order to treat the non-cocompact case. The first consists in compactifying X/Γ suitably, thus yielding a compact $K(\Gamma, 1)$-complex. This idea was successfully exploited in the number field case (cf. [Ra] and [BoS1,2]), showing in particular that Γ is of type F_∞. A different approach has to be used if k is a function field. Most of the results in this case are based on an idea due to Stuhler. Studying $\Gamma = SL_2(\mathcal{O}_S)$, he filtered $X = \bigcup_{j \in \mathbb{N}_0} X_j$ by an increasing sequence of Γ-invariant subcomplexes $X_0 \subseteq \ldots \subseteq X_j \subseteq X_{j+1} \subseteq \ldots$ with compact quotients X_j/Γ . Since the filtration constructed in [Stu] induces isomorphisms $\pi_i(X_j) \xrightarrow{\sim} \pi_i(X_{j+1})$ for all (sufficiently big) j and all $0 \leq i \leq s-2$, all these homotopy groups are trivial in view of the contractibility of X . This implies that $SL_2(\mathcal{O}_S)$ is of type F_{s-1} . Using additionally a criterion due to Brown (cf. [Br2]), it is also easily deduced from the properties of the filtration that Γ is not of type F_{s-1} (Stuhler gave a different proof for this statement).

Stuhler's method was applied independently by Abels and me in order to determine the "finiteness length" (i.e. the maximal m such that Γ is of type F_m) of $\Gamma = SL_{n+1}(\mathbb{F}_q[t])$. Applying the "reduction theory" for X/Γ , one has many choices to construct filtrations of X which are finite modulo Γ . The problem is to verify the desired homotopy properties. The filtration used in [Ab1] yields a slightly better result (I refer again to Remark 17ii)) but the proof given in [A2] is more elegant and accessible to generalizations. It is Abel's filtration which will be applied in the present book (cf. Chapter I, §5).

3. Twin buildings

The action of $SL_{n+1}(\mathbb{F}_q[t])$ on the corresponding Bruhat–Tits building admits a simplicial fundamental domain in the strictest sense. More generally, given a (simply connected) Chevalley group $\mathcal{G}$, it was shown by Soulé in [So] that $X/\mathcal{G}(\mathbb{F}_q[t])$ can be identified with a "quartier" in the Bruhat–Tits building X associated to $\mathcal{G}(\mathbb{F}_q((t^{-1})))$. However, Soulé's proof ist not very transparent since it depends on calculations and not on geometric arguments.

A better understanding of this result is provided by the theory of **twin buildings**. The group $G = \mathcal{G}(\mathbb{F}_q[t,t^{-1}])$ possesses a **twin BN–pair** such that the two components Δ_+, Δ_- of the corresponding twin building are canonically isomorphic to the Bruhat–Tits buildings associated to $\mathcal{G}(\mathbb{F}_q((t^{-1})))$, $\mathcal{G}(\mathbb{F}_q((t)))$ (cf. Chapter I, §1, Example 3). $\mathcal{G}(\mathbb{F}_q[t])$ and $\mathcal{G}(\mathbb{F}_q[t^{-1}])$ are opposite maximal parabolic subgroups in G and are therefore stabilizers in G of two opposite vertices $0_- \in \Delta_-$ and $0_+ \in \Delta_+$. It follows (cf. Chapter I, §3, Proposition 3 and Corollaries 1,2) that the action of $\Gamma = \mathcal{G}(\mathbb{F}_q[t])$ on $X = \Delta_+$ admits the same simplicial fundamental domain as the action of $\mathcal{G}(\mathbb{F}_q[t^{-1}]) = \mathrm{Stab}_G\, 0_+$, namely a quartier in Δ_+ .

Starting with this observation, it has turned out in many respects that the action of Γ on Δ_+ is better understood if one interprets $\Gamma = \mathcal{G}(\mathbb{F}_q[t])$ as the stabilizer of a vertex in Δ_- , the "twin" of Δ_+ . Many arguments used in [Ab3] which I first thought to be dependent on specific features of Bruhat–Tits buildings can in fact be deduced more transparently in the framework of twin buildings (cf. in particular Ch. I, §5). At the same time, this approach admits more general results, for example concerning classical $\mathbb{F}_q$-groups instead of Chevalley groups over $\mathbb{F}_q[t]$ but also regarding certain Kac–Moody groups over $\mathbb{F}_q$.

Therefore, Chapter I is completely devoted to twin buildings. Motivated by the theory of Kac–Moody groups (cf. in particular [T8]), these objects which are generalizations of spherical buildings were introduced by Ronan and Tits. Roughly speaking, a twin building is a pair of buildings (Δ_+, Δ_-) together with an opposition relation between the chambers of Δ_+ and Δ_- possessing similar properties as the opposition relation in a spherical building. Only parts of what is known concerning twin buildings are published yet (cf. [T9-11] and [MR]; for the special case of twin trees see also [RT]). However, firstly the group theoretic background regarding twin BN-pairs, emphasizing the most important examples, and secondly the basic definitions and lemmata (which are either contained in [T9-11] or in [AR]) are recalled in the first two sections of Chapter I. § 3 treats, as already mentioned, questions concerning fundamental domains for group actions on twin buildings. In order to deduce certain local properties of the filtration described in § 5, one has to introduce "coprojections" in twin buildings. This is done in § 4, the main result being Proposition 4, where coprojections are expressed by means of ordinary projections and the opposition relation. (In case the reader is interested in a suitable notion of "convexity" for twin buildings, I also refer to the appendix of § 4.)

Finally, the goal of Chapter I, namely **Theorem A**, is deduced in § 6. It states the following: Given a group G acting "strongly transitively" (cf. Definition 5 in § 2) on a twin building (Δ_+, Δ_-) , where Δ_+, Δ_- are thick n-dimensional buildings, and a simplex $\emptyset \neq a_- \in \Delta_-$. Then the stabilizer G_{a_-} is of type F_{n-1} but not of type FP_n provided that certain conditions, namely (LF), (F) and (S), are satisfied. (LF) states that the apartments of Δ_+, Δ_- are infinite and locally finite which amounts to saying that they are either of irreducible affine or of compact hyperbolic type. (F) requires the finiteness of the intersections $G_{a_-} \cap G_{b_+}$ for all $\emptyset \neq b_+ \in \Delta_+$ and is equivalent to the finiteness of the ground field in most examples (cf. Corollary 7 in § 6). The crucial condition (S) will be discussed below. As for applications, one should think of the example $G = \mathcal{G}(\mathbb{F}_q[t,t^{-1}])$, $a_- = 0_-$ and $G_{a_-} = \mathcal{G}(\mathbb{F}_q[t])$ described above. Another application is concerned with groups acting on twin trees (cf. Corollary 8 and Proposition 6) and generalizes the Nagao–Serre theorem. Further consequences of Theorem A will be stated below. But before I have to say a few words concerning (S).

4. Spherical subcomplexes of spherical buildings

It is usually difficult to determine the homotopy properties of a filtration $(X_j)_{j\in\mathbb{N}_0}$ directly. However, in [Stu], [A2], [Ab 1,3] and in [Be2], this problem could be reduced to questions concerning the **local** structure of the respective Γ-complex X . In all these cases, the isomorphisms $\pi_i(X_j) \xrightarrow{\sim} \pi_i(X_{j+1})$ were established up to a certain level of i by showing that the occurring "relative links" $\ell k_{X_{j+1}}(\sigma) \cap X_j$ have the "right" connectedness properties for all (poly-) simplices $\sigma \in X_{j+1} \setminus X_j$.

A similar proceeding is also possible with regard to the filtration described in Chapter I, §5, provided that the condition (LF) is satisfied. The latter implies that the full links of non-void simplices in $X = \Delta_+$ are spherical buildings. Then the relative links with respect to the filtration are determined by Corollary 6 in §5. They are of the form $\Theta^0(a)$ with $a \in \Theta = \ell k_X(\sigma)$, where $\Theta^0(a)$ denotes the subcomplex of Θ generated by all chambers of Θ which contain a simplex opposite to a . Now the desired homotopy properties of the filtration of Δ_+ can be deduced from the following condition.

(S) If Θ is the full link of a non-void simplex in Δ_+ , then $\Theta^0(a)$ is $(\dim \Theta)$-spherical for any $a \in \Theta$.

Recall that (the geometric realization of) a d-dimensional simplicial complex is said to be **d-spherical** if it is $(d-1)$-connected. By the well known Solomon–Tits theorem, every spherical building Θ is $(\dim \Theta)$-spherical. Chapter II of the present book is devoted to the question whether the same is true for the subcomplexes $\Theta^0(a)$.

This question does not occur here for the first time. Already in connection with the determination of the finiteness length of $SL_{n+1}(\mathbb{F}_q[t])$, it was essential. It has also been investigated for other purposes than studying finiteness properties of groups. In [T7], Tits considers (among other things) the question whether $\Theta^0(c)$ is simply connected for a chamber c of an irreducible spherical rank 3 building Θ and translates it into a group theoretic problem (cf. also Chapter II, §2, Lemma 19). For finite rank 2 buildings, the connectedness of $\Theta^0(a)$ is investigated in [Brou]. By the way, two new results concerning generalized m-gons are outlined in Chapter II, §2, namely the Propositions 7 and 9. Proposition 7 can be used in order to verify in "almost all" cases the condition "(co)" introduced in [MR]. In that paper, an extension theorem for isometries between twin buildings is proved under the assumption (co) that $\Theta^0(c)$ is connected whenever Θ is a rank 2 link in one of the two components of the twin building and c is a chamber of Θ .

Unfortunately, it is definitely possible that $\Theta^0(a)$ is not spherical. To mention at least one example (others are discussed in §2 of Chapter II), I recall that $\Theta^0(c)$ is a torus if Θ is the A_3 building over $\mathbb{F}_2$ and c a chamber of Θ (cf. [T7], Section 16). Counter-examples of this type show that $\Theta^0(a)$ can only be expected to be spherical if Θ is "thick enough", i.e. if every panel (:= codimension 1 face of a chamber) is contained in sufficiently many chambers. However, as Proposition 9 demonstrates, this does not suffice. Thus we are led to the following

Conjecture 1: Let Θ be a spherical Moufang building of rank $d+1$ which is "thick enough". Then $\Theta^0(a)$ is d-spherical for any $a \in \Theta$.

The proof of this conjecture for "classical buildings", i.e. for spherical buildings corresponding to classical groups (a definition not referring to groups is given in Chapter II, §3), occupies the largest part of Chapter II. The result is the following:

Theorem B: *Let Θ be a building of type A_{d+1}, C_{d+1} or D_{d+1} but not an exceptional C_3 building. Assume that every panel of Θ is contained in at least (2^d+1) chambers in the A_{d+1} case, respectively in at least $(2^{2d+1}+1)$ chambers in the two other cases. Then $\Theta^0(a)$ is d-spherical for any $a \in \Theta$.*

The A_{d+1} case is considerably easier than the other two and was already established in [AA]. The general method underlying the proof of Theorem B is discussed in some detail in §3 of Chapter II. It should be applicable to buildings of exceptional type as well. However, the corresponding proofs will become technically complicated to such an extent that I have dispensed with trying to carry them out.

Some characteristic features of the proof of Theorem B are the following: One has to treat the spherical buildings case by case (this is already necessary for rank 2 Moufang buildings, cf. Proposition 7). In each case, one represents the buildings as flag complexes of certain geomtries and uses induction on the rank. In order to obtain sufficiently strong induction hypotheses, one also has to consider other subcomplexes than those of type $\Theta^0(a)$. It is one of the main difficulties (at least in the D_n case) to choose the "right" class of subcomplexes. Decreasing the rank of the buildings is connected with increasing the number of conditions defining the subcomplexes to be considered. One ends up with bounds as stated in the theorem though the complexes $\Theta^0(a)$ are probably already spherical under much milder assumptions. Apart from obvious quantative questions, there is also an interesting qualitative one: Is there a fixed constant $T \in \mathbb{N}$ such that Theorem B remains true after replacing 2^d+1 , respectively $2^{2d+1}+1$ by T ? The opinions about the answer to be expected diverge; my guess would be "no".

5. Group theoretic consequences

Equipped with Theorem B, it is now easy to draw conclusions from Theorem A. As already mentioned, the main application is concerned with certain S-arithmetic groups. For the first time, one also obtains some results regarding higher finiteness properties in the function field case where the linear algebraic groups are **non–split**. I will just state a variant — appearing as Corollary 20 in Chapter III, §2 — of the more detailed Theorem C of Chapter III here.

Theorem C': *Let $\mathcal{G}$ be an absolutely almost simple $\mathbb{F}_q$-group which is not of exceptional type. Denote by n the $\mathbb{F}_q$-rank of $\mathcal{G}$. Suppose $n \geq 1$ and $q \geq 2^{2n-1}$. Then $\mathcal{G}(\mathbb{F}_q[t])$ and $\mathcal{G}(\mathbb{F}_q[t,t^{-1}])$ are of type F_{n-1} , and $\mathcal{G}(\mathbb{F}_q[t])$ is not of type FP_n .*

For the restrictions occurring in this statement, Theorem B is responsible. But a similar result should also be true for the exceptional types.

Conjecture 2: The statement of Theorem C' holds for any absolutely almost simple $\mathbb{F}_q$-group of $\mathbb{F}_q$-rank $n \geq 1$ provided that q is "big enough".

I am careful about the cases with small q since I know meanwhile that Theorem A becomes definitely wrong if one cancels assumption (S) (cf. the last remark concerning (S) in Chapter I, § 6, directly before Theorem A).

As far as $G = \mathcal{G}(\mathbb{F}_q[t,t^{-1}])$ is concerned, Theorem C' just represents a preliminary result since the action of G on the corresponding twin building is not fully exploited yet (cf. Chapter I, §6, Remark 7). I expect that the following is true:

Conjecture 3: Let $\mathcal{G}$ be an absolutely almost simple $\mathbb{F}_q$-group of $\mathbb{F}_q$-rank $n \geq 1$ and assume that q is "sufficiently big". Then $\mathcal{G}(\mathbb{F}_q[t,t^{-1}])$ is of type F_{2n-1} but not of type FP_{2n} .

Of course, much more general statements than the two conjectures mentioned above may be suspected in the function field case. However, since there are so few results, it does not seem to be appropriate at the moment to formulate these speculations explicitly.

Instead, I conclude this introduction by noting a further consequence of Theorem A which is obtained as a by-product. As already mentioned, the methods of Chapter I can also be applied to certain **Kac–Moody groups over $\mathbb{F}_q$**. From Theorem A (to be more precise: from the Corollaries 10 and 11) and from Proposition 11 in Chapter II, §2, one can deduce the following

Example: Let $\mathcal{G}_D$ be a Kac–Moody group functor as described in [T8] (cf. also Example 5 below). Assume that the Coxeter system (W, S) associated to D is of rank 4 and of compact hyperbolic type. Let q be a prime power ≥ 16 . Then $G = \mathcal{G}_D(\mathbb{F}_q)$ is finitely presented. All proper parabolic (with respect to one of the two natural BN-pairs in G) subgroups of G are finitely presented but not of type FP_3 .

I Groups acting on twin buildings

§ 1 Twin BN–pairs and RGD–systems

"BN–pairs", later on also called "Tits systems", were introduced by Tits in the context of linear algebraic groups. Tits extracted the BN–axioms from Chevalley's work (cf. in particular [C1] and [C2]) on semisimple groups and showed together with Borel that this axiomatization applies as well to arbitrary, not necessarily split reductive groups (cf. [BoT], §5, or [Bo], §21). BN–pairs have proved to be a powerful tool in group theory since, mainly for two reasons. Firstly, much is known of the structure of a group if it possesses a BN–pair (key-words: Bruhat decomposition, parabolic subgroups, criterions for simplicity). Secondly, a group with a BN–pair naturally acts on a simplicial complex, namely the building associated to it. This renders certain group theoretic problems accessible to geometric interpretations and solutions.

If a BN–pair belongs to the group $G = \mathcal{G}(k)$ of k-rational points of a reductive k-group $\mathcal{G}$ as described by Borel and Tits, it possesses some additional features due to the properties of the family $(\mathcal{U}_\alpha(k))_{\alpha\in\Phi}$ of unipotent subgroups associated to the (relative) root system $\Phi = {}_k\Phi$ of $\mathcal{G}$. These properties were axiomatized by Bruhat and Tits in [BrT1], §6.1, where they defined "root data" ("données radicielles"). The Tits system corresponding to a root datum always possesses a finite Weyl group because the latter coincides with the Weyl group of the root system indexing the root datum. Therefore, the associated building is of spherical type. In particular, it is possible to define when two chambers or two (minimal) parabolic subgroups are opposite. These opposition relations are among the important additional features of root data which have no analogues in the general theory of buildings and BN–pairs. Nevertheless, the notion of "oppositeness" can also be applied to certain situations where the Weyl groups are infinite. For example, every "minimal" Kac–Moody group G over a field gives rise to two BN–pairs (G, B_+, N) and (G, B_-, N) with the same Weyl group W (cf. [T8]). Though B_+ and B_- are not conjugate if W is infinite, they are related to each other in the same way as the opposite minimal parabolic subgroups B and $w_0 B w_0^{-1}$ of a BN–pair with finite Weyl group, where w_0 denotes the element of maximal length of the latter. A precise formulation of the relationship between B_+ and B_- leads to the axioms of a "twin BN–pair" which will be recalled below. The geometric structures corresponding to twin BN–pairs are "twin buildings". They were introduced by Ronan and Tits (cf. [T9], [T11] and [RT]) and will be treated in