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Modern Geometry— Methods and Applications

Part I. The Geometry of Surfaces,
Transformation Groups, and Fields

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Translated by Robert G. Burns

With 45 Illustrations



Springer-Verlag
New York Berlin Heidelberg Tokyo

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c/o VAAP—Copyright Agency of the U.S.S.R.
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AMS Subject Classifications: 49-01, 51-01, 53-01

Library of Congress Cataloging in Publication Data
Dubrovin, B. A.

Modern geometry—methods and applications.
(Graduate texts in mathematics; 93+)

“Original Russian edition published by Nauka in 1979.”

Contents: pt. 1. The geometry of surfaces, transformation groups, and fields. —

Bibliography: p.

Vol. 1 includes index.

1. Geometry. I. Fomenko, A. T. II. Novikov, Sergei
Petrovich. III. Title. IV. Series: Graduate texts in
mathematics; 93, etc.

QA445.D82 1984 516 83-16851

This book is part of the Springer Series in Soviet Mathematics.

Original Russian edition: *Sovremennaja Geometrija: Metody i Priloženia*. Moskva:
Nauka, 1979.

© 1984 by Springer-Verlag New York Inc.

All rights reserved. No part of this book may be translated or reproduced in any
form without written permission from Springer-Verlag, 175 Fifth Avenue, New York,
New York 10010, U.S.A.

Typeset by Composition House Ltd., Salisbury, England.

Printed and bound by R. R. Donnelley & Sons, Harrisonburg, Virginia.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-90872-2 Springer-Verlag New York Berlin Heidelberg Tokyo
ISBN 3-540-90872-2 Springer-Verlag Berlin Heidelberg New York Tokyo

Preface*

Up until recently, Riemannian geometry and basic topology were not included, even by departments or faculties of mathematics, as compulsory subjects in a university-level mathematical education. The standard courses in the classical differential geometry of curves and surfaces which were given instead (and still are given in some places) gradually came to be viewed as anachronisms. However, there has been hitherto no unanimous agreement as to exactly how such courses should be brought up to date, that is to say, which parts of modern geometry should be regarded as absolutely essential to a modern mathematical education, and what might be the appropriate level of abstractness of their exposition.

The task of designing a modernized course in geometry was begun in 1971 in the mechanics division of the Faculty of Mechanics and Mathematics of Moscow State University. The subject-matter and level of abstractness of its exposition were dictated by the view that, in addition to the geometry of curves and surfaces, the following topics are certainly useful in the various areas of application of mathematics (especially in elasticity and relativity, to name but two), and are therefore essential: the theory of tensors (including covariant differentiation of them); Riemannian curvature; geodesics and the calculus of variations (including the conservation laws and Hamiltonian formalism); the particular case of skew-symmetric tensors (i.e. "forms") together with the operations on them; and the various formulae akin to Stokes' (including the all-embracing and invariant "general Stokes formula" in n dimensions). Many leading theoretical physicists shared the mathematicians' view that it would also be useful to include some facts about

* Parts II and III are scheduled to appear in the Graduate Texts in Mathematics series at a later date.

manifolds, transformation groups, and Lie algebras, as well as the basic concepts of visual topology. It was also agreed that the course should be given in as simple and concrete a language as possible, and that wherever practicable the terminology should be that used by physicists. Thus it was along these lines that the archetypal course was taught. It was given more permanent form as duplicated lecture notes published under the auspices of Moscow State University as:

Differential Geometry, Parts I and II, by S. P. Novikov, Division of Mechanics, Moscow State University, 1972.

Subsequently various parts of the course were altered, and new topics added. This supplementary material was published (also in duplicated form) as

Differential Geometry, Part III, by S. P. Novikov and A. T. Fomenko, Division of Mechanics, Moscow State University, 1974.

The present book is the outcome of a reworking, re-ordering, and extensive elaboration of the above-mentioned lecture notes. It is the authors' view that it will serve as a basic text from which the essentials for a course in modern geometry may be easily extracted.

To S. P. Novikov are due the original conception and the overall plan of the book. The work of organizing the material contained in the duplicated lecture notes in accordance with this plan was carried out by B. A. Dubrovin. This accounts for more than half of Part I; the remainder of the book is essentially new. The efforts of the editor, D. B. Fuks, in bringing the book to completion, were invaluable.

The content of this book significantly exceeds the material that might be considered as essential to the mathematical education of second- and third-year university students. This was intentional: it was part of our plan that even in Part I there should be included several sections serving to acquaint (through further independent study) both undergraduate and graduate students with the more complex but essentially geometric concepts and methods of the theory of transformation groups and their Lie algebras, field theory, and the calculus of variations, and with, in particular, the basic ingredients of the mathematical formalism of physics. At the same time we strove to minimize the degree of abstraction of the exposition and terminology, often sacrificing thereby some of the so-called "generality" of statements and proofs: frequently an important result may be obtained in the context of crucial examples containing the whole essence of the matter, using only elementary classical analysis and geometry and without invoking any modern "hyperinvariant" concepts and notations, while the result's most general formulation and especially the concomitant proof will necessitate a dramatic increase in the complexity and abstractness of the exposition. Thus in such cases we have first expounded the result in question in the setting of the relevant significant examples, in the simplest possible language

appropriate, and have postponed the proof of the general form of the result, or omitted it altogether. For our treatment of those geometrical questions more closely bound up with modern physics, we analysed the physics literature: books on quantum field theory (see e.g. [35], [37]) devote considerable portions of their beginning sections to describing, in physicists' terms, useful facts about the most important concepts associated with the higher-dimensional calculus of variations and the simplest representations of Lie groups; the books [41], [43] are devoted to field theory in its geometric aspects; thus, for instance, the book [41] contains an extensive treatment of Riemannian geometry from the physical point of view, including much useful concrete material. It is interesting to look at books on the mechanics of continuous media and the theory of rigid bodies ([42], [44], [45]) for further examples of applications of tensors, group theory, etc.

In writing this book it was not our aim to produce a "self-contained" text: in a standard mathematical education, geometry is just one component of the curriculum; the questions of concern in analysis, differential equations, algebra, elementary general topology and measure theory, are examined in other courses. We have refrained from detailed discussion of questions drawn from other disciplines, restricting ourselves to their formulation only, since they receive sufficient attention in the standard programme.

In the treatment of its subject-matter, namely the geometry and topology of manifolds, Part II goes much further beyond the material appropriate to the aforementioned basic geometry course, than does Part I. Many books have been written on the topology and geometry of manifolds: however, most of them are concerned with narrowly defined portions of that subject, are written in a language (as a rule very abstract) specially contrived for the particular circumscribed area of interest, and include all rigorous foundational detail often resulting only in unnecessary complexity. In Part II also we have been faithful, as far as possible, to our guiding principle of minimal abstractness of exposition, giving preference as before to the significant examples over the general theorems, and we have also kept the interdependence of the chapters to a minimum, so that they can each be read in isolation insofar as the nature of the subject-matter allows. One must however bear in mind the fact that although several topological concepts (for instance, knots and links, the fundamental group, homotopy groups, fibre spaces) can be defined easily enough, on the other hand any attempt to make nontrivial use of them in even the simplest examples inevitably requires the development of certain tools having no forbears in classical mathematics. Consequently the reader not hitherto acquainted with elementary topology will find (especially if he is past his first youth) that the level of difficulty of Part II is essentially higher than that of Part I; and for this there is no possible remedy. Starting in the 1950s, the development of this apparatus and its incorporation into various branches of mathematics has proceeded with great rapidity. In recent years there has appeared a rash, as it were, of nontrivial applications of topological methods (sometimes

in combination with complex algebraic geometry) to various problems of modern theoretical physics: to the quantum theory of specific fields of a geometrical nature (for example, Yang-Mills and chiral fields), the theory of fluid crystals and superfluidity, the general theory of relativity, to certain physically important nonlinear wave equations (for instance, the Korteweg-de Vries and sine-Gordon equations); and there have been attempts to apply the theory of knots and links in the statistical mechanics of certain substances possessing "long molecules". Unfortunately we were unable to include these applications in the framework of the present book, since in each case an adequate treatment would have required a lengthy preliminary excursion into physics, and so would have taken us too far afield. However, in our choice of material we have taken into account which topological concepts and methods are exploited in these applications, being aware of the need for a topology text which might be read (given strong enough motivation) by a young theoretical physicist of the modern school, perhaps with a particular object in view.

The development of topological and geometric ideas over the last 20 years has brought in its train an essential increase in the complexity of the algebraic apparatus used in combination with higher-dimensional geometrical intuition, as also in the utilization, at a profound level, of functional analysis, the theory of partial differential equations, and complex analysis; not all of this has gone into the present book, which pretends to being elementary (and in fact most of it is not yet contained in any single textbook, and has therefore to be gleaned from monographs and the professional journals).

Three-dimensional geometry in the large, in particular the theory of convex figures and its applications, is an intuitive and generally useful branch of the classical geometry of surfaces in 3-space; much interest attaches in particular to the global problems of the theory of surfaces of negative curvature. Not being specialists in this field we were unable to extract its essence in sufficiently simple and illustrative form for inclusion in an elementary text. The reader may acquaint himself with this branch of geometry from the books [1], [4] and [16].

Of all the books on the topology and geometry of manifolds, the classical works *A Textbook of Topology* and *The Calculus of Variations in the Large*, of Seifert and Threlfall, and also the excellent more modern books [10], [11] and [12], turned out to be closest to our conception in approach and choice of topics. In the process of creating the present text we actively mulled over and exploited the material covered in these books, and their methodology. In fact our overall aim in writing Part II was to produce something like a modern analogue of Seifert and Threlfall's *Textbook of Topology*, which would however be much wider-ranging, remodelled as far as possible using modern techniques of the theory of smooth manifolds (though with simplicity of language preserved), and enriched with new material as dictated by the contemporary view of the significance of topological methods, and

of the kind of reader who, encountering topology for the first time, desires to learn a reasonable amount in the shortest possible time. It seemed to us sensible to try to benefit (more particularly in Part I, and as far as this is possible in a book on mathematics) from the accumulated methodological experience of the physicists, that is, to strive to make pieces of nontrivial mathematics more comprehensible through the use of the most elementary and generally familiar means available for their exposition (preserving however, the format characteristic of the mathematical literature, wherein the statements of the main conclusions are separated out from the body of the text by designating them "theorems", "lemmas", etc.). We hold the opinion that, in general, understanding should precede formalization and rigorization. There are many facts the details of whose proofs have (aside from their validity) absolutely no role to play in their utilization in applications. On occasion, where it seemed justified (more often in the more difficult sections of Part II) we have omitted the proofs of needed facts. In any case, once thoroughly familiar with their applications, the reader may (if he so wishes), with the help of other sources, easily sort out the proofs of such facts for himself. (For this purpose we recommend the book [21].) We have, moreover, attempted to break down many of these omitted proofs into soluble pieces which we have placed among the exercises at the end of the relevant sections.

In the final two chapters of Part II we have brought together several items from the recent literature on dynamical systems and foliations, the general theory of relativity, and the theory of Yang-Mills and chiral fields. The ideas expounded there are due to various contemporary researchers; however in a book of a purely textbook character it may be accounted permissible not to give a long list of references. The reader who graduates to a deeper study of these questions using the research journals will find the relevant references there.

Homology theory forms the central theme of Part III.

In conclusion we should like to express our deep gratitude to our colleagues in the Faculty of Mechanics and Mathematics of M.S.U., whose valuable support made possible the design and operation of the new geometry courses; among the leading mathematicians in the faculty this applies most of all to the creator of the Soviet school of topology, P. S. Aleksandrov, and to the eminent geometers P. K. Raševskii and N. V. Efimov.

We thank the editor D. B. Fuks for his great efforts in giving the manuscript its final shape, and A. D. Aleksandrov, A. V. Pogorelov, Ju. F. Borisov, V. A. Toponogov and V. I. Kuz'minov who in the course of reviewing the book contributed many useful comments. We also thank Ja. B. Zel'dovič for several observations leading to improvements in the exposition at several points, in connexion with the preparation of the English and French editions of this book.

We give our special thanks also to the scholars who facilitated the task of incorporating the less standard material into the book. For instance the

proof of Liouville's theorem on conformal transformations, which is not to be found in the standard literature, was communicated to us by V. A. Zorič. The editor D. B. Fuks simplified the proofs of several theorems. We are grateful also to O. T. Bogojavlenskii, M. I. Monastyrskii, S. G. Gindikin, D. V. Alekseevskii, I. V. Gribkov, P. G. Grinevič, and E. B. Vinberg.

Translator's acknowledgments. Thanks are due to Abe Shenitzer for much kind advice and encouragement, and to Eadie Henry for her excellent typing and great patience.

Contents

CHAPTER I

Geometry in Regions of a Space. Basic Concepts

\$1. Co-ordinate systems	1
1.1. Cartesian co-ordinates in a space	2
1.2. Co-ordinate changes	3
\$2. Euclidean space	9
2.1. Curves in Euclidean space	9
2.2. Quadratic forms and vectors	14
\$3. Riemannian and pseudo-Riemannian spaces	17
3.1. Riemannian metrics	17
3.2. The Minkowski metric	20
\$4. The simplest groups of transformations of Euclidean space	23
4.1. Groups of transformations of a region	23
4.2. Transformations of the plane	25
4.3. The isometries of 3-dimensional Euclidean space	31
4.4. Further examples of transformation groups	34
4.5. Exercises	37
\$5. The Serret-Frenet formulae	38
5.1. Curvature of curves in the Euclidean plane	38
5.2. Curves in Euclidean 3-space. Curvature and torsion	42
5.3. Orthogonal transformations depending on a parameter	47
5.4. Exercises	48
\$6. Pseudo-Euclidean spaces	50
6.1. The simplest concepts of the special theory of relativity	50
6.2. Lorentz transformations	52
6.3. Exercises	60

CHAPTER 2

The Theory of Surfaces

§7. Geometry on a surface in space	61
7.1. Co-ordinates on a surface	61
7.2. Tangent planes	66
7.3. The metric on a surface in Euclidean space	68
7.4. Surface area	72
7.5. Exercises	76
§8. The second fundamental form	76
8.1. Curvature of curves on a surface in Euclidean space	76
8.2. Invariants of a pair of quadratic forms	79
8.3. Properties of the second fundamental form	80
8.4. Exercises	86
§9. The metric on the sphere	86
§10. Space-like surfaces in pseudo-Euclidean space	90
10.1. The pseudo-sphere	90
10.2. Curvature of space-like curves in \mathbb{R}_1^3	94
§11. The language of complex numbers in geometry	95
11.1. Complex and real co-ordinates	95
11.2. The Hermitian scalar product	97
11.3. Examples of complex transformation groups	99
§12. Analytic functions	100
12.1. Complex notation for the element of length, and for the differential of a function	100
12.2. Complex co-ordinate changes	104
12.3. Surfaces in complex space	106
§13. The conformal form of the metric on a surface	109
13.1. Isothermal co-ordinates. Gaussian curvature in terms of conformal co-ordinates	109
13.2. Conformal form of the metrics on the sphere and the Lobachevskian plane	114
13.3. Surfaces of constant curvature	117
13.4. Exercises	120
§14. Transformation groups as surfaces in N -dimensional space	120
14.1. Co-ordinates in a neighbourhood of the identity	120
14.2. The exponential function with matrix argument	127
14.3. The quaternions	131
14.4. Exercises	136
§15. Conformal transformations of Euclidean and pseudo-Euclidean spaces of several dimensions	136

CHAPTER 3

Tensors: The Algebraic Theory

§16. Examples of tensors	145
§17. The general definition of a tensor	145
17.1. The transformation rule for the components of a tensor of arbitrary rank	151

17.2. Algebraic operations on tensors	157
17.3. Exercises	161
§18. Tensors of type $(0, k)$	161
18.1. Differential notation for tensors with lower indices only	161
18.2. Skew-symmetric tensors of type $(0, k)$	164
18.3. The exterior product of differential forms. The exterior algebra	166
18.4. Exercises	167
§19. Tensors in Riemannian and pseudo-Riemannian spaces	168
19.1. Raising and lowering indices	168
19.2. The eigenvalues of a quadratic form	170
19.3. The operator $*$	171
19.4. Tensors in Euclidean space	172
19.5. Exercises	173
§20. The crystallographic groups and the finite subgroups of the rotation group of Euclidean 3-space. Examples of invariant tensors	173
§21. Rank 2 tensors in pseudo-Euclidean space, and their eigenvalues	194
21.1. Skew-symmetric tensors. The invariants of an electromagnetic field	194
21.2. Symmetric tensors and their eigenvalues. The energy-momentum tensor of an electromagnetic field	199
§22. The behaviour of tensors under mappings	203
22.1. The general operation of restriction of tensors with lower indices	203
22.2. Mappings of tangent spaces	204
§23. Vector fields	205
23.1. One-parameter groups of diffeomorphisms	205
23.2. The Lie derivative	207
23.3. Exercises	211
§24. Lie algebras	212
24.1. Lie algebras and vector fields	212
24.2. The fundamental matrix Lie algebras	214
24.3. Linear vector fields	219
24.4. The Killing metric	224
24.5. The classification of the 3-dimensional Lie algebras	226
24.6. The Lie algebras of the conformal groups	227
24.7. Exercises	232
CHAPTER 4	
The Differential Calculus of Tensors	234
§25. The differential calculus of skew-symmetric tensors	234
25.1. The gradient of a skew-symmetric tensor	234
25.2. The exterior derivative of a form	237
25.3. Exercises	243
§26. Skew-symmetric tensors and the theory of integration	244
26.1. Integration of differential forms	244
26.2. Examples of integrals of differential forms	250
26.3. The general Stokes formula. Examples	255
26.4. Proof of the general Stokes formula for the cube	263
26.5. Exercises	265

§27. Differential forms on complex spaces	266
27.1. The operators d' and d''	266
27.2. Kählerian metrics. The curvature form	269
§28. Covariant differentiation	271
28.1. Euclidean connexions	271
28.2. Covariant differentiation of tensors of arbitrary rank	280
§29. Covariant differentiation and the metric	284
29.1. Parallel transport of vector fields	284
29.2. Geodesics	286
29.3. Connexions compatible with the metric	287
29.4. Connexions compatible with a complex structure (Hermitian metric)	291
29.5. Exercises	293
§30. The curvature tensor	295
30.1. The general curvature tensor	295
30.2. The symmetries of the curvature tensor. The curvature tensor defined by the metric	300
30.3. Examples: the curvature tensor in spaces of dimensions 2 and 3; the curvature tensor defined by a Killing metric	302
30.4. The Peterson-Codazzi equations. Surfaces of constant negative curvature, and the "sine-Gordon" equation	307
30.5. Exercises	310
CHAPTER 5	
The Elements of the Calculus of Variations	313
§31. One-dimensional variational problems	313
31.1. The Euler-Lagrange equations	313
31.2. Basic examples of functionals	317
§32. Conservation laws	320
32.1. Groups of transformations preserving a given variational problem	320
32.2. Examples. Applications of the conservation laws	322
§33. Hamiltonian formalism	333
33.1. Legendre's transformation	333
33.2. Moving co-ordinate frames	336
33.3. The principles of Maupertuis and Fermat	341
33.4. Exercises	344
§34. The geometrical theory of phase space	344
34.1. Gradient systems	344
34.2. The Poisson bracket	348
34.3. Canonical transformations	354
34.4. Exercises	358
§35. Lagrange surfaces	358
35.1. Bundles of trajectories and the Hamilton-Jacobi equation	358
35.2. Hamiltonians which are first-order homogeneous with respect to the momentum	363
§36. The second variation for the equation of the geodesics	367
36.1. The formula for the second variation	367
36.2. Conjugate points and the minimality condition	371

CHAPTER 6

The Calculus of Variations in Several Dimensions. Fields and Their Geometric Invariants

§37. The simplest higher-dimensional variational problems	375
37.1. The Euler–Lagrange equations	375
37.2. The energy-momentum tensor	379
37.3. The equations of an electromagnetic field	384
37.4. The equations of a gravitational field	390
37.5. Soap films	397
37.6. Equilibrium equation for a thin plate	403
37.7. Exercises	408
§38. Examples of Lagrangians	409
§39. The simplest concepts of the general theory of relativity	412
§40. The spinor representations of the groups $SO(3)$ and $O(3, 1)$. Dirac's equation and its properties	427
40.1. Automorphisms of matrix algebras	427
40.2. The spinor representation of the group $SO(3)$	429
40.3. The spinor representation of the Lorentz group	431
40.4. Dirac's equation	435
40.5. Dirac's equation in an electromagnetic field. The operation of charge conjugation	437
§41. Covariant differentiation of fields with arbitrary symmetry	439
41.1. Gauge transformations. Gauge-invariant Lagrangians	439
41.2. The curvature form	443
41.3. Basic examples	444
§42. Examples of gauge-invariant functionals. Maxwell's equations and the Yang–Mills equation. Functionals with identically zero variational derivative (characteristic classes)	449
Bibliography	455
Index	459

CHAPTER 1

Geometry in Regions of a Space.

Basic Concepts

§1. Co-ordinate Systems

We begin by discussing some of the concepts fundamental to geometry. In school geometry—the so-called “elementary Euclidean” geometry of the ancient Greeks—the main objects of study are various metrical properties of the simplest geometrical figures. The basic goal of that geometry is to find relationships between lengths and angles in triangles and other polygons. Knowledge of such relationships then provides a basis for the calculation of the surface areas and volumes of certain solids. The central concepts underlying school geometry are the following: the length of a straight line segment (or of a circular arc); and the angle between two intersecting straight lines (or circular arcs).

The chief aim of analytic (or co-ordinate) geometry is to describe geometrical figures by means of algebraic formulae referred to a Cartesian system of co-ordinates of the plane or 3-dimensional space. The objects studied are the same as in elementary Euclidean geometry: the sole difference lies in the methodology. Again, differential geometry is the same old subject, except that here the subtler techniques of the differential calculus and linear algebra are brought into full play. Being applicable to general “smooth” geometrical objects, these techniques provide access to a wider class of such objects.

1.1. Cartesian Co-ordinates in a Space

Our most basic conception of geometry is set out in the following two paragraphs:

- (i) We do our geometry in a certain space consisting of points P, Q, \dots
- (ii) As in analytic geometry, we introduce a system of co-ordinates for the space. This is done by simply associating with each point of the space an ordered n -tuple (x^1, \dots, x^n) of real numbers—the *co-ordinates* of the point—in such a way as to satisfy the following two conditions:
 - (a) Distinct points are assigned distinct n -tuples. In other words, points P and Q with co-ordinates (x^1, \dots, x^n) and (y^1, \dots, y^n) are one and the same point if and only if $x^i = y^i, i = 1, \dots, n$.
 - (b) Every possible n -tuple (x^1, \dots, x^n) is used, i.e. is assigned to some point of the space.

1.1.1. Definition. A space furnished with a system of Cartesian co-ordinates satisfying conditions (a) and (b) is called an *n -dimensional Cartesian space*,† and is denoted by \mathbb{R}^n . The integer n is called the *dimension* of the space.

We shall often refer somewhat loosely to the n -tuples (x^1, \dots, x^n) themselves as the points of the space. The simplest example of a Cartesian space is the real number line. Here each point has just one co-ordinate x^1 , so that $n = 1$, i.e. it is a 1-dimensional Cartesian space. Other examples, familiar from analytic geometry, are provided by Cartesian co-ordinatizations of the plane (which is then a 2-dimensional Cartesian space), and of ordinary (i.e. 3-dimensional) space (Figure 1). These Cartesian spaces are completely adequate for solving the problems of school geometry.

We shall now consider a less familiar but extremely important example of a Cartesian space. Modern physics teaches us that time and space are not separate, non-overlapping concepts, but are merged in a 4-dimensional “space-time continuum.” The following mathematical formulation of the natural ordering of phenomena turns out to be extraordinarily convenient.

The points of our space-time continuum are taken to be events. We assign to each event an ordered quadruple (t, x^1, x^2, x^3) of real numbers, where t is the “instant in time” when the event occurs, and x^1, x^2, x^3 are the co-ordinates of the “spatial location” of the event. With this co-ordinatization, the space-time continuum becomes a 4-dimensional Cartesian space, and we then set aside our interpretation of the co-ordinates (t, x^1, x^2, x^3) as times and locations of the events. The 3-dimensional space of classical geometry is then simply the hyperspace defined by an equation $t = \text{const.}$ The course, or path, in space-time, of an object which can be regarded abstractly at every instant of time as a point (a so-called “point-particle”),

† This terminology is perhaps unconventional. We hope that the reader will not find it too disconcerting.

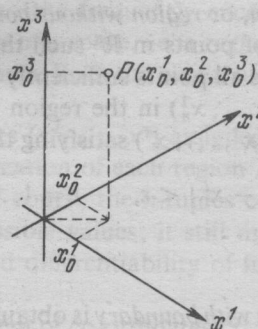


Figure 1

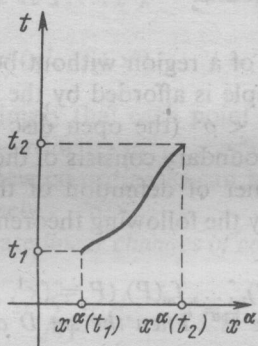


Figure 2. The world-line of an object.

is then identified with a curve segment (or arc) $x^\alpha(t)$, $\alpha = 1, 2, 3$, $t_1 \leq t \leq t_2$, in 4-dimensional space. We call this curve the world-line of the point-particle (Figure 2). We shall be considering also 3-dimensional and even 2-dimensional space-time continua, co-ordinatized by triples (t, x^1, x^2) and pairs (t, x^1) respectively, since for these spaces it is easier to draw intelligible pictures.

1.2. Co-ordinate Changes

Suppose that in an n -dimensional Cartesian space we are given a real-valued function $f(P)$, i.e. a function assigning a real number to each point P of the space. Since each point of the space comes with its n co-ordinates we can think of f as a function of n real variables: if $P = (x^1, \dots, x^n)$, then $f(P) = f(x^1, \dots, x^n)$. We shall be concerned only with continuous (usually even continuously differentiable) functions $f(x^1, \dots, x^n)$. At times the functions we consider will not be defined for every point of the space \mathbb{R}^n , but only on portions, or, more precisely, "regions" of it.