B. A. Dubrovin
A. T. Fomenko
S. P. Novikov

Modern Geometry—Methods and Applications

Part I. The Geometry of Surfaces, Transformation Groups, and Fields

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Translated by Robert G. Burns

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B. A. Dubrovin
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B. Bronnaya 6a
Moscow 103104
U.S.S.R.

A. T. Fomenko
3 Ya Karacharavskaya
d.b. Korp. I. Ku. 35
109202 Moscow
U.S.S.R.

S. P. Novikov
L. D. Landau Institute
for Theoretical Physics
Academy of Sciences of the U.S.S.R.
Vorobevskoe Shosse, 2
117334 Moscow
U.S.S.R.

R. G. Burns (*Translator*)
Department of Mathematics
Faculty of Arts
York University
4700 Keele Street
Downsview, ON, M3J 1P3
Canada

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Preface*

Up until recently, Riemannian geometry and basic topology were not included, even by departments or faculties of mathematics, as compulsory subjects in a university-level mathematical education. The standard courses in the classical differential geometry of curves and surfaces which were given instead (and still are given in some places) gradually came to be viewed as anachronisms. However, there has been hitherto no unanimous agreement as to exactly how such courses should be brought up to date, that is to say, which parts of modern geometry should be regarded as absolutely essential to a modern mathematical education, and what might be the appropriate level of abstractness of their exposition.

The task of designing a modernized course in geometry was begun in 1971 in the mechanics division of the Faculty of Mechanics and Mathematics of Moscow State University. The subject-matter and level of abstractness of its exposition were dictated by the view that, in addition to the geometry of curves and surfaces, the following topics are certainly useful in the various areas of application of mathematics (especially in elasticity and relativity, to name but two), and are therefore essential: the theory of tensors (including covariant differentiation of them); Riemannian curvature; geodesics and the calculus of variations (including the conservation laws and Hamiltonian formalism); the particular case of skew-symmetric tensors (i.e. "forms") together with the operations on them; and the various formulae akin to Stokes' (including the all-embracing and invariant "general Stokes formula" in *n* dimensions). Many leading theoretical physicists shared the mathematicians' view that it would also be useful to include some facts about

* Parts II and III are scheduled to appear in the Graduate Texts in Mathematics series at a later date.

manifolds, transformation groups, and Lie algebras, as well as the basic concepts of visual topology. It was also agreed that the course should be given in as simple and concrete a language as possible, and that wherever practicable the terminology should be that used by physicists. Thus it was along these lines that the archetypal course was taught. It was given more permanent form as duplicated lecture notes published under the auspices of Moscow State University as:

Differential Geometry, Parts I and II, by S. P. Novikov, Division of Mechanics, Moscow State University, 1972.

Subsequently various parts of the course were altered, and new topics added. This supplementary material was published (also in duplicated form) as

Differential Geometry, Part III, by S. P. Novikov and A. T. Fomenko, Division of Mechanics, Moscow State University, 1974.

The present book is the outcome of a reworking, re-ordering, and extensive elaboration of the above-mentioned lecture notes. It is the authors' view that it will serve as a basic text from which the essentials for a course in modern geometry may be easily extracted.

To S. P. Novikov are due the original conception and the overall plan of the book. The work of organizing the material contained in the duplicated lecture notes in accordance with this plan was carried out by B. A. Dubrovin. This accounts for more than half of Part I; the remainder of the book is essentially new. The efforts of the editor, D. B. Fuks, in bringing the book to completion, were invaluable.

The content of this book significantly exceeds the material that might be considered as essential to the mathematical education of second- and third-year university students. This was intentional: it was part of our plan that even in Part I there should be included several sections serving to acquaint (through further independent study) both undergraduate and graduate students with the more complex but essentially geometric concepts and methods of the theory of transformation groups and their Lie algebras, field theory, and the calculus of variations, and with, in particular, the basic ingredients of the mathematical formalism of physics. At the same time we strove to minimize the degree of abstraction of the exposition and terminology, often sacrificing thereby some of the so-called "generality" of statements and proofs: frequently an important result may be obtained in the context of crucial examples containing the whole essence of the matter, using only elementary classical analysis and geometry and without invoking any modern "hyperinvariant" concepts and notations, while the result's most general formulation and especially the concomitant proof will necessitate a dramatic increase in the complexity and abstractness of the exposition. Thus in such cases we have first expounded the result in question in the setting of the relevant significant examples, in the simplest possible language appropriate, and have postponed the proof of the general form of the result, or omitted it altogether. For our treatment of those geometrical questions more closely bound up with modern physics, we analysed the physics literature: books on quantum field theory (see e.g. [35], [37]) devote considerable portions of their beginning sections to describing, in physicists' terms, useful facts about the most important concepts associated with the higher-dimensional calculus of variations and the simplest representations of Lie groups; the books [41], [43] are devoted to field theory in its geometric aspects; thus, for instance, the book [41] contains an extensive treatment of Riemannian geometry from the physical point of view, including much useful concrete material. It is interesting to look at books on the mechanics of continuous media and the theory of rigid bodies ([42], [44], [45]) for further examples of applications of tensors, group theory, etc.

In writing this book it was not our aim to produce a "self-contained" text: in a standard mathematical education, geometry is just one component of the curriculum; the questions of concern in analysis, differential equations, algebra, elementary general topology and measure theory, are examined in other courses. We have refrained from detailed discussion of questions drawn from other disciplines, restricting ourselves to their formulation only, since they receive sufficient attention in the standard programme.

In the treatment of its subject-matter, namely the geometry and topology of manifolds, Part II goes much further beyond the material appropriate to the aforementioned basic geometry course, than does Part I. Many books have been written on the topology and geometry of manifolds: however, most of them are concerned with narrowly defined portions of that subject, are written in a language (as a rule very abstract) specially contrived for the particular circumscribed area of interest, and include all rigorous foundational detail often resulting only in unnecessary complexity. In Part II also we have been faithful, as far as possible, to our guiding principle of minimal abstractness of exposition, giving preference as before to the significant examples over the general theorems, and we have also kept the interdependence of the chapters to a minimum, so that they can each be read in isolation insofar as the nature of the subject-matter allows. One must however bear in mind the fact that although several topological concepts (for instance, knots and links, the fundamental group, homotopy groups, fibre spaces) can be defined easily enough, on the other hand any attempt to make nontrivial use of them in even the simplest examples inevitably requires the development of certain tools having no forbears in classical mathematics. Consequently the reader not hitherto acquainted with elementary topology will find (especially if he is past his first youth) that the level of difficulty of Part II is essentially higher than that of Part I; and for this there is no possible remedy. Starting in the 1950s, the development of this apparatus and its incorporation into various branches of mathematics has proceeded with great rapidity. In recent years there has appeared a rash, as it were, of nontrivial applications of topological methods (sometimes in combination with complex algebraic geometry) to various problems of modern theoretical physics: to the quantum theory of specific fields of a geometrical nature (for example, Yang–Mills and chiral fields), the theory of fluid crystals and superfluidity, the general theory of relativity, to certain physically important nonlinear wave equations (for instance, the Korteweg–de Vries and sine–Gordon equations); and there have been attempts to apply the theory of knots and links in the statistical mechanics of certain substances possessing "long molecules". Unfortunately we were unable to include these applications in the framework of the present book, since in each case an adequate treatment would have required a lengthy preliminary excursion into physics, and so would have taken us too far afield. However, in our choice of material we have taken into account which topological concepts and methods are exploited in these applications, being aware of the need for a topology text which might be read (given strong enough motivation) by a young theoretical physicist of the modern school, perhaps with a particular object in view.

The development of topological and geometric ideas over the last 20 years has brought in its train an essential increase in the complexity of the algebraic apparatus used in combination with higher-dimensional geometrical intuition, as also in the utilization, at a profound level, of functional analysis, the theory of partial differential equations, and complex analysis; not all of this has gone into the present book, which pretends to being elementary (and in fact most of it is not yet contained in any single textbook, and has therefore to be gleaned from monographs and the professional journals).

Three-dimensional geometry in the large, in particular the theory of convex figures and its applications, is an intuitive and generally useful branch of the classical geometry of surfaces in 3-space; much interest attaches in particular to the global problems of the theory of surfaces of negative curvature. Not being specialists in this field we were unable to extract its essence in sufficiently simple and illustrative form for inclusion in an elementary text. The reader may acquaint himself with this branch of geometry from the books [1], [4] and [16].

Of all the books on the topology and geometry of manifolds, the classical works *A Textbook of Topology* and *The Calculus of Variations in the Large*, of Seifert and Threlfall, and also the excellent more modern books [10], [11] and [12], turned out to be closest to our conception in approach and choice of topics. In the process of creating the present text we actively mulled over and exploited the material covered in these books, and their methodology. In fact our overall aim in writing Part II was to produce something like a modern analogue of Seifert and Threlfall's *Textbook of Topology*, which would however be much wider-ranging, remodelled as far as possible using modern techniques of the theory of smooth manifolds (though with simplicity of language preserved), and enriched with new material as dictated by the contemporary view of the significance of topological methods, and of the kind of reader who, encountering topology for the first time, desires to learn a reasonable amount in the shortest possible time. It seemed to us sensible to try to benefit (more particularly in Part I, and as far as this is possible in a book on mathematics) from the accumulated methodological experience of the physicists, that is, to strive to make pieces of nontrivial mathematics more comprehensible through the use of the most elementary and generally familiar means available for their exposition (preserving however, the format characteristic of the mathematical literature, wherein the statements of the main conclusions are separated out from the body of the text by designating them "theorems", "lemmas", etc.). We hold the opinion that, in general, understanding should precede formalization and rigorization. There are many facts the details of whose proofs have (aside from their validity) absolutely no role to play in their utilization in applications. On occasion, where it seemed justified (more often in the more difficult sections of Part II) we have omitted the proofs of needed facts. In any case, once thoroughly familiar with their applications, the reader may (if he so wishes), with the help of other sources, easily sort out the proofs of such facts for himself. (For this purpose we recommend the book [21].) We have, moreover, attempted to break down many of these omitted proofs into soluble pieces which we have placed among the exercises at the end of the relevant sections.

In the final two chapters of Part II we have brought together several items from the recent literature on dynamical systems and foliations, the general theory of relativity, and the theory of Yang–Mills and chiral fields. The ideas expounded there are due to various contemporary researchers; however in a book of a purely textbook character it may be accounted permissible not to give a long list of references. The reader who graduates to a deeper study of these questions using the research journals will find the relevant references there.

Homology theory forms the central theme of Part III.

In conclusion we should like to express our deep gratitude to our colleagues in the Faculty of Mechanics and Mathematics of M.S.U., whose valuable support made possible the design and operation of the new geometry courses; among the leading mathematicians in the faculty this applies most of all to the creator of the Soviet school of topology, P. S. Aleksandrov, and to the eminent geometers P. K. Raševskiĭ and N. V. Efimov.

We thank the editor D. B. Fuks for his great efforts in giving the manuscript its final shape, and A. D. Aleksandrov, A. V. Pogorelov, Ju. F. Borisov, V. A. Toponogov and V. I. Kuz'minov who in the course of reviewing the book contributed many useful comments. We also thank Ja. B. Zel'dovič for several observations leading to improvements in the exposition at several points, in connexion with the preparation of the English and French editions of this book.

We give our special thanks also to the scholars who facilitated the task of incorporating the less standard material into the book. For instance the proof of Liouville's theorem on conformal transformations, which is not to be found in the standard literature, was communicated to us by V. A. Zorič. The editor D. B. Fuks simplified the proofs of several theorems. We are grateful also to O. T. Bogojavlenskiĭ, M. I. Monastyrskiĭ, S. G. Gindikin, D. V. Alekseevskiĭ, I. V. Gribkov, P. G. Grinevič, and E. B. Vinberg.

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Contents

CHAPTER 1

Geometry in Regions of a Space. Basic Concepts

§1. Co-ordinate Systems

We begin by discussing some of the concepts fundamental to geometry. In school geometry—the so-called "elementary Euclidean" geometry of the ancient Greeks—the main objects of study are various metrical properties of the simplest geometrical figures. The basic goal of that geometry is to find relationships between lengths and angles in triangles and other polygons. Knowledge of such relationships then provides a basis for the calculation of the surface areas and volumes of certain solids. The central concepts underlying school geometry are the following: the length of a straight line segment (or of a circular arc); and the angle between two intersecting straight lines (or circular arcs).

The chief aim of analytic (or co-ordinate) geometry is to describe geometrical figures by means of algebraic formulae referred to a Cartesian system of co-ordinates of the plane or 3-dimensional space. The objects studied are the same as in elementary Euclidean geometry: the sole difference lies in the methodology. Again, differential geometry is the same old subject, except that here the subtler techniques of the differential calculus and linear algebra are brought into full play. Being applicable to general "smooth" geometrical objects, these techniques provide access to a wider class of such objects.

1.1. Cartesian Co-ordinates in a Space

Our most basic conception of geometry is set out in the following two paragraphs:

(i) We do our geometry in a certain space consisting of points $P, Q, \ldots$.
(ii) As in analytic geometry, we introduce a system of co-ordinates for the space. This is done by simply associating with each point of the space an ordered n-tuple $(x^1, \ldots, x^n)$ of real numbers—the *co-ordinates* of the point—in such a way as to satisfy the following two conditions:
   (a) Distinct points are assigned distinct n-tuples. In other words, points P and Q with co-ordinates $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are one and the same point if and only if $x^i = y^i$, $i = 1, \ldots, n$.
   (b) Every possible n-tuple $(x^1, \ldots, x^n)$ is used, i.e. is assigned to some point of the space.

1.1.1. Definition. A space furnished with a system of Cartesian co-ordinates satisfying conditions (a) and (b) is called an *n-dimensional Cartesian space*,† and is denoted by $\mathbb{R}^n$. The integer n is called the *dimension* of the space.

We shall often refer somewhat loosely to the n-tuples $(x^1, \ldots, x^n)$ themselves as the points of the space. The simplest example of a Cartesian space is the real number line. Here each point has just one co-ordinate x^1, so that $n = 1$, i.e. it is a 1-dimensional Cartesian space. Other examples, familiar from analytic geometry, are provided by Cartesian co-ordinatizations of the plane (which is then a 2-dimensional Cartesian space), and of ordinary (i.e. 3-dimensional) space (Figure 1). These Cartesian spaces are completely adequate for solving the problems of school geometry.

We shall now consider a less familiar but extremely important example of a Cartesian space. Modern physics teaches us that time and space are not separate, non-overlapping concepts, but are merged in a 4-dimensional "space-time continuum." The following mathematical formulation of the natural ordering of phenomena turns out to be extraordinarily convenient.

The points of our space–time continuum are taken to be events. We assign to each event an ordered quadruple (t, x^1, x^2, x^3) of real numbers, where t is the "instant in time" when the event occurs, and x^1, x^2, x^3 are the co-ordinates of the "spatial location" of the event. With this co-ordinatization, the space-time continuum becomes a 4-dimensional Cartesian space, and we then set aside our interpretation of the co-ordinates (t, x^1, x^2, x^3) as times and locations of the events. The 3-dimensional space of classical geometry is then simply the hyperspace defined by an equation $t = \text{const.}$ The course, or path, in space-time, of an object which can be regarded abstractly at every instant of time as a point (a so-called "point-particle"),

† This terminology is perhaps unconventional. We hope that the reader will not find it too disconcerting.

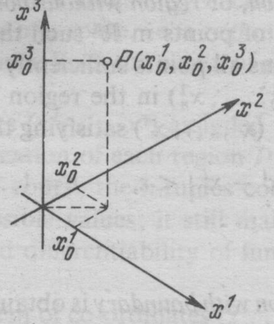

Figure 1

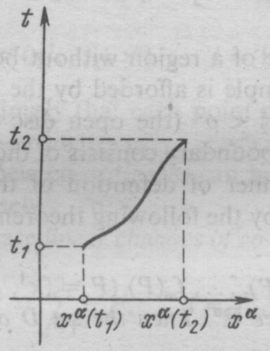

Figure 2. The world-line of an object.

is then identified with a curve segment (or arc) $x^\alpha(t)$, $\alpha = 1, 2, 3$, $t_1 \leq t \leq t_2$, in 4-dimensional space. We call this curve the world-line of the point-particle (Figure 2). We shall be considering also 3-dimensional and even 2-dimensional space–time continua, co-ordinatized by triples (t, x^1, x^2) and pairs (t, x^1) respectively, since for these spaces it is easier to draw intelligible pictures.

1.2. Co-ordinate Changes

Suppose that in an n-dimensional Cartesian space we are given a real-valued function $f(P)$, i.e. a function assigning a real number to each point P of the space. Since each point of the space comes with its n co-ordinates we can think of f as a function of n real variables: if $P = (x^1, \ldots, x^n)$, then $f(P) = f(x^1, \ldots, x^n)$. We shall be concerned only with continuous (usually even continuously differentiable) functions $f(x^1, \ldots, x^n)$. At times the functions we consider will not be defined for every point of the space $\mathbb{R}^n$, but only on portions, or, more precisely, "regions" of it.