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Principles of Statistical Radiophysics 1

Elements of Random Process Theory

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With 28 Figures

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Foreword

Principles of Statistical Radiophysics is concerned with the theory of random functions (processes and fields) treated in close association with a number of applications in physics. Primarily, the book deals with radiophysics in its broadest sense, i.e., viewed as a general theory of oscillations and waves of any physical nature[1]. This translation is based on the second (two-volume) Russian edition. It appears in four volumes:

1. Elements of Random Process Theory
2. Correlation Theory of Random Processes
3. Elements of Random Fields
4. Wave Propagation Through Random Media.

The four volumes are, naturally, to a large extent conceptually interconnected (being linked, for instance, by cross-references); yet for the advanced reader each of them might be of interest on its own. This motivated the division of the *Principles* into four separate volumes.

The text is designed for graduate and postgraduate students majoring in radiophysics, radio engineering, or other branches of physics and technology dealing with oscillations and waves (e.g., acoustics and optics). As a rule, early in their career these students face problems involving the use of random functions. The book provides a sound basis from which to understand and solve problems at this level. In addition, it paves the way for a more profound study of the mathematical theory, should it be necessary[2]. The reader is assumed to be familiar with probability theory.

In progressing from one volume to the next, the reader will see that the physical problems under consideration become more complex and the mathematical machinery more involved. This results quite naturally from the course-oriented origin of the book, based as it is on summarized lecture notes. Their extensive teaching experience has convinced the authors that such an approach is preferable to a more uniform presentation. Each chapter is followed by a num-

[1] It should be noted that certain questions in statistical radiophysics are not covered in this book. For example, it is not concerned with quantum radiophysics and quantum electronics, nor with statistical phenomena related to the propagation of waves in nonlinear media, see [1].

[2] Treatments of the mathematical theory are given, for instance, in [2–9], and more specialised applications of the theory to radiophysics and engineering are to be found in [10–19].

ber of problems worked out in detail. The problems not only serve as exercises, but in many cases contain additional theoretical material and further literature references.

Moscow, March 1986

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Preface

This volume concentrates almost exclusively on univariate random functions, i.e., random processes. The main physical questions considered are random pulse Poisson processes, Brownian motion, fluctuations in Thomson self-oscillatory systems and the action of random forces on dissipative dynamic systems, both linear and non-linear. These applications help to introduce the reader to the central concept of random processes, and to several classes of these processes: Gaussian processes, continuous and discontinuous Markov processes, stationary processes, processes with stationary increments, and also to the elements of stochastic convergence and ergodicity, and stochastic differential equations.

Volume 1 was written by S.M. Rytov. The author is indebted to Professors A.M. Yaglom, M.L. Levin, M.A. Isakovich, V.N. Tutubalin, Ya.I. Khurgin, and V.P. Yakovlev, whose valuable suggestions and comments contributed immensely to the quality and arrangement of material in some chapters.

Moscow, March 1986 *S.M. Rytov*

Table of Contents

1. General Introduction

The higher sensitivity of measuring and receiving devices, and hence the improved accuracy of measurement, have resulted in *fluctuations* playing an ever-increasing role in various branches of physics and technology. Fluctuations are known to be random deviations of macroscopic quantities from their mean (for example, thermodynamic equilibrium) values. Along with the theory of equilibrium states and the kinetics of physical processes, the theory of fluctuations is one of the key subjects in statistical physics. The mathematical tools of statistical physics are not only the classical theory of probability concerned with random events and variables, but also the much more general *theory of random functions*. It includes both random *processes* occuring in time (these are considered in Volumes I and II of this work, i.e., this volume and [1.1]), and random *fields* depending on time and space (Volumes III and IV [1.2,3]). The aim of *Principles of Statistical Radiophysics* is to introduce the reader to the theory of random functions at a physical level of rigor, drawing on specific physical problems taken mostly from the field of radiophysics.

A word of clarification on what is meant by "physical level of rigor". We are familiar with the difficulties encountered in seeking a compromise between the requirements of a rigorous presentation of a mathematical theory and the clarity desirable for a first encounter with the subject. The faultless reasoning of a mathematician is on occasion seen by a physicist or engineer as nothing more – just less convincing – than simple and tangible arguments, while these may lack the rigor demanded by the mathematician. The present book furnishes a sufficiently sound mathematical basis for independent work, but is in no way intended as a substitute for a more profound study of the mathematical theory.

As mentioned above, radiophysical problems are used to illustrate applications of the theory, and sometimes to substantiate the need for some of its techniques. In this context, radiophysics as such is understood in a wider sense implying the following.

As early as the 1930s L.I. Mandelshtam substantially extended the heuristically very powerful principle referred to as the "isomorphism of laws". Crudely speaking, it implies that laws discovered in various branches of physics are (under the appropriate conditions) also applicable to other fields. This principle, known in linear optics and acoustics since the time of Rayleigh, has led to the creation and development of the general theory of oscillations and waves (including nonlinear ones) that has encompassed not only the radio-frequency range and optics and electromagnetic processes in general, but also oscillatory and wave phenomena of *any* physical nature, including macroscopic mechanics (in particular, acoustics), quantum mechanics, chemistry, and biology. This general theory of oscillations and waves is naturally concerned with both deterministic processes and fields, and random (stochastic) oscillations and waves, irrespective of their fundamental physical nature, or rather with a wide variety of their physical origin. This is what is meant by "radiophysics understood in the wider sense".

In fact, the very use of the term "radiophysics" is to a certain extent a tribute to the generalization of the principle of isomorphism due to Mandelshtam.

Of course, fluctuations do not exhaust the enormous variety of stochastic phenomena. Yet random variations that occur in completely *deterministic* systems have long been known. By way of example we may cite turbulence in a liquid or gas, developing under the appropriate conditions, but still within a purely deterministic description of the medium. Similar examples, along with the kinetic theory of gases and liquids, strengthened for a long time the conviction that such an "intrinsic" stochasticity (one of the major problems of the so-called ergodic theory) presupposes the presence of an immense number of degrees of freedom in the system considered. About fifteen years ago it was found, however, that this is not required at all: autostochastic behavior appeared to be possible in dynamic systems as well, if the phase space had a dimensionality of three or more. We mention this in order to stress that this book is not concerned with the issues of autostochasticity in dynamic systems. It only considers those random phenomena (including fluctuations) in dynamic systems that occur as a result of external random actions. Mathematically, this implies that the phenomena to be considered are from the very beginning described by random functions explicitly introduced into the equations of motion for the system. This implies the transition from dynamic equations to *stochastic* ones.

Stochastic equations are introduced in this volume, but a more profound development of the theory and applications of these equations are to be found in Volumes III and IV [1,2,3].

Let us take a closer look at physical objects and phenomena covered by stochastic radiophysics. Two types of problem can be distinguished. Probabilistic properties of random actions on a system are either derived from a micromodel, i.e., from stochastic assumptions at a molecular level, or they are from the outset selected at the macroscopic level, i.e, within a phenomenological approach tailored to account properly for the experimentally observed behavior.

Problems of the first kind are, for instance, concerned with the *thermal motion* of microparticles, such as microcharges – electrons, ions, and so on. At the macrolevel this results in *thermal fluctuations* of a wide variety of quantities, e.g., density, pressure, temperature, current, voltage, the strength of electric and magnetic macroscopic fields, etc., etc. Thermal fluctuations give rise to

Brownian motion, molecular scattering of light (both within and on the surface of a medium), so-called thermal noise in radio-engineering, optics and acoustics, thermal radiation of bodies, and other phenomena.

Another example is the random variation of the number of particles in the electron fluxes in thermal- and photo-emission – the so-called *shot effect* ensuing from the discrete nature of microcharge carriers.

Worthy of special mention is a fairly common type of fluctuation – *the flicker effect* – which occurs in vacuum tubes (where it is associated with local variations in the emission of electrons from the cathode), contacts, semiconductors, electrolytes, and nonmetal resistors. Evidently, the prevalence of the flicker effect cannot be associated with a general cause such as, for example, the finite dimensions of a macrosystem (the importance of its dimensions being the greater the slower the fluctuations).

One more example is *magnetic noise* (including the Barkhausen effect) due to chaotic remagnetization of domains in ferromagnets subjected to time-dependent magnetic fields.

Problems of the second kind, where the stochastic behavior of random actions is simply specified at the macrolevel, also involve either random processes or random fields. In the case of *discrete systems*, random functions may be introduced to describe fluctuations of the parameters of the system and/or of the external forces. We deal here with stochastic *ordinary* differential equations. Under this heading is included the important special case of self-oscillatory systems (the present book and Volume II [1.1]). If the oscillations are periodic but not harmonic then the presence of fluctuations results in non-monochromaticity of the self-oscillations. This nonmonochromaticity is to a large extent associated with the stability of the frequency and the accuracy with which it can be measured (and hence the accuracy of time measurements).

Such a "macrostochastic" approach can also be applied to *continuous* systems described by stochastic *partial* differential equations which include a large variety of problems concerned with wave propagation in media with random inhomogeneities which cause wave scattering, random pulsation of refraction, fluctuations of the intensity and phase of a wave at the point of observation, and so forth. In these cases, the inhomogeneities of the medium are described by random fields with given stochastic characteristics.

Similar problems occur not only in wave propagation in free space, but also along guided lines (wire feeders, wave-guides, optical fibers), whose inhomogeneities can be thought of as random. However, though Volumes III and IV [1.3,4] are mainly devoted to wave propagation in random media, they do not refer to these particular questions. The same holds for the important problems arising from random inhomogeneities in emitters (in particular, in composite antenna systems; see [1.4]).

It should be emphasized that autostochasticity does not exclude such a "macrostochastic" approach. For instance, media may be random owing to turbulence. But the emergence and development of turbulence as such are the subject of the ergodic theory. However, when turbulence is considered from the point of view of its influence on the propagation of mechanical or electromagnetic waves, then the inhomogeneities arising from it may be described by introducing aptly chosen random fields into wave equations.

For transverse waves, questions related to the interference of oscillations and waves, their temporal and spatial coherence, and also polarization, constitute a rich and important field of interest. In this context the so-called *correlation theory* of random functions is widely used. Though it represents a fairly restricted part of the general theory of random functions, it is so important that the whole of Volume II [1.1] is devoted to it. The second volume deals with the correlation theory of random processes, but some of its generalizations to random fields are contained in Volume III [1.2].

When receiving radio signals, the quality of the signal is influenced by fluctuations (noise) in the receiver and the measuring device, the transformation of external and internal noise (in a wide variety of signal transducers), the noise immunity of the receiving systems, and so on. Some of these problems, though of primary practical importance, are of no physical interest. Nevertheless, a number of them are discussed to illustrate the techniques of the theory of random functions (this volume and [1.1]). A further reason for not going into these questions at length is that they are amply dealt with in a multitude of existing monographs and textbooks. The reader is referred to this literature, see also the references given in the Preface.

The preceding comments allow us to put into the proper perspective the motives underlying the selection of the subject matter and structure of the book [1.1–3].

On the one hand the above, rather incomplete list of pertinent problems demonstrates that the field of statistical radiophysics covers a tremendous variety of phenomena – whether the term is understood in its wider or narrower sense. It is almost impossible to reflect all the diversity of the statistical problems the subject covers. Moreover, such an attempt could hardly be justified since an extensive overview would necessarily be superficial – this alone rendering it practically useless. On the other hand, the mathematical techniques of statistical radiophysics are more involved and powerful than those of the classical theory of probability. A thorough treatment of the theory of random functions would constitute a special mathematical course.

In view of all this, we believe it appropriate to single out a few radiophysical issues interesting and important in their own right, which also introduce the reader to the elements of the theory of random functions, its techniques and applications. The treatment presupposes an acquaintance with the classical theory of probability, although in the first two chapters of Volume I we recall some of its fundamental concepts and theorems for the benefit of the reader.

2. The Bernoulli Problem

This and the following chapter provides an introduction to the theory of random processes. Therefore, in these two chapters we confine ourselves to the classical theory of probability and deal only with random events and quantities. The purpose of this chapter is two-fold:

First, to clarify the connection between the mathematical concept of probability and the ways in which this concept is applied. One frequently encounters a lack of clear understanding of the fact that the measurement of the probability of an event through the relative frequency of its occurrence is based on an independent postulate not included among the axioms of the mathematical theory. At the same time, this postulate is necessary for any applications of the mathematical theory, to physics in particular (Sects. 2.1,2).

Second, referring to the binomial law (Sect. 2.3), a simple example of a discrete probability distribution, it is demonstrated that a common approach can be used for some seemingly different problems such as: the fluctuations of gas density, the fluctuations of the intensity of the sum of harmonic oscillations with random amplitudes, the one-dimensional Brownian motion, and the shot effect (Sects. 2.4,5). Some of these problems involve processes evolving in time.

The binomial law has the additional feature of having extremely important asymptotics: the discrete Laplace and Poisson distributions (Sects. 2.5,6) and the continuous Gauss distribution (Sect. 2.7).

2.1 The Physical Concept of Probability

Experience has shown that a purely mathematical presentation of probability theory does not always lead to a clear understanding of the principal concepts important for the physicist. We shall, therefore, dwell briefly on what probability is and how the physicist uses this concept.

It is well-known that probabilities had been calculated long before the true meaning of probability was perceived. The theory originated from attempts to calculate the chance of success, of fair stakes, etc., in so-called games of chance such as the throwing of dice, the tossing of coins, and some card games, i.e., those games in which the outcome depends on chance alone and in no way on the player's ability and dexterity. In this sense these games might be called frivolous. But, as mathematicians like to remark, the intellectual game of chess has made practically no contribution to science, whereas the frivolous game of dice has contributed so much. Why?

The truth is that chess obeys its own special rules, and all the situations encountered here lie within the domain of these rules, i.e., they cannot be generalized. In contrast, the throwing of dice is elementary and exhibits a pure form of an extremely general statistical law – the stability of relative frequencies with increasing number of tests. If n throws of a die yield n_i times i points, then the relative frequency n_i/n shows a surprising stability as n increases. This empirical fact is independent of whether the die is good or bad. An a priori false die, e.g., one containing a piece of lead in one of its faces, exhibits stability of the quotient n_i/n, though different faces have different relative frequencies, not equal to 1/6.

Thus, for n_i/n to be stable does not require a "good" die, and specific "asymptotic" values of n_i/n are not predetermined by the very fact of stability. This follows either from a *statistical experiment* especially performed or from data accumulated previously. By saying that for a good (i.e., suitable for the game) die the values of n_i/n are about 1/6, we give in essence a definition of the die that is to be regarded as good.

It is frequently believed that the value $n_i/n \approx 1/6$ follows from the classical definition of probability due to Laplace

$$P = \frac{\text{favorable outcomes}}{\text{equally likely outcomes}} .$$

This definition is too restrictive since it does not cover those cases where the possible outcomes constitute an infinite set, countable or continuous. But even if we do not wish to expand the scope of this definition, we easily see that this is circular reasoning, because equally likely here means nothing but equally probable, or *equiprobable*. Hence, the probability P is here "defined" through probability. This is simply a rule for calculating the probabilities of favorable outcomes using a preassigned *uniform* distribution of probabilities of all possible outcomes. Although we think that the Laplace definition yields the probability $P = 1/6$ of the appearance of each face of the die, in fact, we get from this definition only what we have incorporated into it, namely, the equally likely appearance of each face. In exactly the same way we assume that (in a single draw from a deck of cards) one is equally likely to obtain any one of its 36 or 52 cards.

What is behind these self-evident assumptions? They are sometimes supported by the use of the so-called *principle of insufficient grounds*, which states that if the die is geometrically accurate and made of uniform material, etc., then there are no grounds to expect $P \neq 1/6$.

Intuitively, this may be clear. A person who has not the slightest idea of what a statistical experiment involves can deduce the equiprobability of the faces of the fair dice or of either of the coin's sides appearing by simply exercising common sense. Here the equiprobability is natural, if not self-evident. This is suggested by the finite number of possible outcomes and the *symmetry*.

But without the symmetry or, say, with continuous possible outcomes, common sense jibs or simply fails. With the false die, only a statistical experiment with the *given* die enables us to estimate the probability of various faces. Difficulties encountered in the selection of equally likely outcomes, constituting a continous set, were demonstrated marvellously by *J.L.F. Bertrand* in a number of geometric problems in his book *Calcule des Probabilitiés* (1888). These so-called *Bertrand's* paradoxes with geometric probabilities are another illustration of the inadequacies of common sense or intuition in the more complex situation under consideration [2.1]. Moreover, when we say that in simple cases an intuitive prediction is *correct*, this very "correctness" only implies that a test by statistical experiment will bear out the prediction. Thus, *in the final analysis* even the self-evident assumption of equiprobability implies an enormous number of tests to be actually performed with certain conditions observed, i.e., accumulated statistical experience.

A detailed history of the evolution of probability theory is beyond the scope of the book. This long history has witnessed the emergence of many varied and conflicting approaches to the concept of probability. The development of probability theory into an axiomatic branch of mathematics took about three centuries.

A mathematical theory originating from some (usually fairly simple) ideas taken from everyday life tends to free itself from its empirical roots, thus striving to attain the status of an axiomatic theory. In the case of the theory of probability the process only terminated in the 1930s, when *A.N. Kolmogorov* formulated his axioms that made probability theory a branch of the measure theory. It should be noted that as early as 1917 S.N. Bernstein stressed the need for an axiomatization of probability theory and suggested a concrete set of axioms, which eventually turned out be appreciably less convenient than those proposed by *A.N. Kolgomorov.*

Today, the mathematician treats probability as a non-negative, unit-normalized, completely additive set function, defined on an algebra of subsets. An event A is represented by a set A of points of the space of possible elementary outcomes of the experiment or test under consideration. The probability $P\{A\}$ of event A is a set function, such that if it is defined on sets $A_1, \ldots, A_n$, it may also be defined on a set consisting of all the points constituting at least one of the sets $A_1, \ldots, A_n$, and also on the set of points belonging simultaneously to all the sets $A_1, \ldots, A_n$ [2.2,3]. This function obeys the following three axioms:

Axiom I. The probability $P\{A\}$ of an event A obeys the inequality $P\{A\} \geq 0$.

Axiom II. For a certain event U the equality $P\{U\} = 1$ is valid.

Axiom III. For mutually exclusive events A_k ($k = 1, 2, \ldots, n$; n being arbitrarily large) the relation

$$P\left\{\sum_{i=1}^{n} A_k\right\} = \sum_{k=1}^{n} P\{A_k\}$$

holds. The summation signs on either side of the equality have different meanings. On the left, the sign relates to the probability of at least one of the events A_k, so that $\sum_{k=1}^{n} A_k$ is an event (A_1 or $A_2, \ldots$, or A_n), whereas on the right it is a conventional sum of non-negative numbers $P\{A_k\}$.

The certain event mentioned in axiom II corresponds to the set U consisting of *all* imaginable elementary outcomes, and hence all the other events A are subsets of U.

Axiom III (the additivity axiom) covers also the case of $n = \infty$. That is why P is referred to as a *completely additive* function.

Axioms I–III, supplemented by the definitions of a number of related concepts (e.g., those of random variable or of mathematical expectation), lie at the foundation of the whole theory of probability.

Of course, examination of these axioms reveals the connection between the abstract quantity P and the empirical relative frequency, but this is a *genetic* connection that relates only to the origin of the axioms, and not to their content, from which all traces of the underlying empirical facts have already been removed. Therefore, after being exposed to this abstract definition of probability, a physicist, an engineer, an economist, etc., i.e, one dealing with real problems and events, is bound to ask what he should do with such a completely additive function, and how he should relate the formulas of this mathematical theory to the real world. The situation here is the same as in any physical theory.

Equations and formulas for certain mathematical quantities (symbols) do not (in themselves) constitute a physical theory. The latter requires a knowledge of how to determine from real things and events the *numbers* to be inserted into mathematical relations. That is, the values of the respective mathematical symbols or, in other words, a way of *measuring* these quantities[1]. Of course, a physical theory is a single entity. Its mathematical and measurement ingredients are by no means independent of each other, yet, one cannot directly substitute one for the other. For this reason, in all physical (and in general practical) applications of the mathematical probability theory, the latter should be supplemented by at least one (realizable and concrete) *measuring technique* for the respective quantity P, the probability. It is only natural that we should turn to the relative frequency when looking for this technique or measurement axiom.

We assume that the probability of an event has been *measured* (approximately, as in any measurement) in terms of the *relative frequency* of its occurrence in a sufficiently long series of tests performed under certain unchanged conditions, relying on a sufficiently large ensemble of homogeneous systems, i.e., in a statistical experiment. Since it is an independent postulate that does not belong to the axioms of the mathematical theory of probability, this assumption is by no means the only one and does not a priori ensure success. Of course, a basic requirement is that it be consistent with the mathematical theory. As we shall see below (Sect. 4.6), the "frequency" measurement axiom

[1] This, at any rate, is the approach to measurement in classical (pre-quantum) physics. But the equations of quantum mechanics also contain quantities whose direct measurement is not required. (See Ref. [2.4], *Lectures in Quantum Mechanics*).