Lecture Notes in Mathematics

1579

Norihiko Kazamaki

Continuous Exponential Martingales and *BMO*

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Preface

This book consists of three chapters and we shall deal entirely with continuous local martingales. Let M be a continuous local martingale and let

$$\mathcal{E}(M) = \exp\left(M - \frac{1}{2}\langle M\rangle\right)$$

where $\langle M\rangle$ denotes the increasing process associated with M. As is well-known, it is a local martingale which plays an essential role in various questions concerning the absolute continuity of probability laws of stochastic processes. Our purpose here is to make a full report on the exciting results about BMO in the theory of exponential local martingales. BMO denotes the class of all uniformly integrable martingales $M = (M_t, \mathcal{F}_t)$ such that

$$\sup_T \left\| E\left[|M_\infty - M_T| \,\middle|\, \mathcal{F}_T\right]\right\|_\infty < \infty$$

where the supremum is taken over all stopping times T. A martingale in BMO is a probabilistic version of a function of bounded mean oscillation introduced in [31] by F. John and L. Nirenberg.

In Chapter 1 we shall explain in detail the beautiful properties of an exponential local martingale. In Chapter 2 we shall collect the main tools to study various properties about continuous BMO-martingales. The fundamentally important result is that the following are equivalent:

(a) $M \in BMO$.

(b) $\mathcal{E}(M)$ is a uniformly integrable martingale which satisfies the reverse Hölder inequality :

(R_p) $\quad E[\mathcal{E}(M)^p_\infty \mid \mathcal{F}_T] \leq C_p \mathcal{E}(M)^p_T$

for some $p > 1$, where T is an arbitrary stopping time.

(c) $\mathcal{E}(M)$ satisfies the condition:

(A_p) $\quad \sup_T \left\| E\left[\{\mathcal{E}(M)_T/\mathcal{E}(M)_\infty\}^{\frac{1}{p-1}} |\mathcal{F}_T\right]\right\|_\infty < \infty$

for some $p > 1$.

These three conditions were originally introduced in the classical analysis. For example, the (A_p) condition is a probabilistic version of the one introduced in [62] by B. Muckenhoupt. In Chapter 3 we shall prove that it is a necessary and sufficient condition for the validity of some weighted norm inequalities for martingales. Furthermore, we shall study two important subclasses of BMO, namely, the class L_∞ of all bounded martingales and the class H_∞ of all martingales M such that $\langle M\rangle_\infty$ is bounded. In general, BMO is neither L_∞ nor H_∞ and it is obvious that there is no inclusion relation between L_∞ and H_∞. In this chapter we shall establish very interesting relationships between the condition (R_p) and the distance to L_∞ in the space BMO. One of them is the result that M belongs to the BMO-closure of L_∞ if and only if $\mathcal{E}(\lambda M)$ satisfies all (R_p) for every real number λ. In addition, we shall prove that the (A_p) condition is remotely related to the distance to H_∞ in the space BMO.

The reader is assumed to be familiar with the martingale theory as expounded in [12] or [60].

I am happy to acknowledge the influence of three of my teachers T. Tsuchikura, C. Watari, and P. A. Meyer. I would also like to thank my colleagues M. Izumisawa, M. Kaneko, M. Kikuchi, M. Okada, T. Okada, T. Sekiguchi, and Y. Shiota for many helpful discussions. Finally, thanks to Mrs.Yoshiko Kitsunezuka for the help in preparing this manuscript.

N. Kazamaki

Contents

Chapter 1

Exponential Martingales

1.1 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions. The usual hypotheses means that

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$,
(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ for all $t \geq 0$.

Definition 1. 1. *A real valued stochastic process* $M = (M_t, \mathcal{F}_t)$ *is called a* martingale (*resp.*supermartingale, submartingale) *if*

(i) *each* M_t *is* $\mathcal{F}_t$*-measurable, i.e,* M *is adapted to the filtration* $(\mathcal{F}_t)$,
(ii) $M_t \in L_1$ *for every* t,
(iii) *if* $s \leq t$, *then* $E[M_t \mid \mathcal{F}_s] = M_s$ *a.s.* (*resp.* $E[M_t \mid \mathcal{F}_s] \leq M_s$, *resp.*$\geq M_s$).

As is well known, a one-dimensional Brownian motion is a typical martingale. It should be noted that the notion of a martingale depends merely on the filtration $(\mathcal{F}_t)$, but also on the probability measure dP. An adapted process $M = (M_t, \mathcal{F}_t)$ is said to be *a local martingale* if there exists a sequence of increasing stopping times T_n with $\lim_{n\to\infty} T_n = \infty$ a.s. such that $(M_{t\wedge T_n} \mathrm{I}_{\{T_n>0\}}, \mathcal{F}_t)$ is a martingale for each n. Such a sequence (T_n) of stopping times is called *a fundamental sequence.* Recall that a stopping time T is a random variable taking values in $[0, \infty]$ such that $\{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.
Throughout this survey, we suppose that any local martingale adapted to this filtration is continuous. It is well-known that the Brownian filtration satisfies this assumption. Note that the following three properties are equivalent :

(a) any local martingale is continuous,
(b) any stopping time is predictable,
(c) for every stopping time T and every $\mathcal{F}_T$-measurable random variable U, there exists a continuous local martingale M with $M_T = U$ a.s.

The equivalence of (a) and (b) is well-known, and the equivalence of (a) and (c) was established by M. Emery, C. Stricker and J. A. Yan ([16]). It seems to me that the essential feature of our problems discussed here appears in this case, which is the reason that we deal entirely with continuous local martingales. We generally assume that $M_0 = 0$. Let us denote by $\langle M\rangle$ the continuous increasing process such that $M^2 - \langle M\rangle$ is also a local martingale. Let $t > 0$ and let $\{T_i^n\}_{i=0,1,\cdots,k_n}$ be a sequence of stopping times such that $0 = T_0^n \le T_1^n \le \cdots \le T_{k_n}^n = t$ and $\lim_{n\to\infty}\max_{0\le i<k_n}(T_{i+1}^n - T_i^n) = 0$. Then from a celebrated result of C. Doléans-Dade([8]) it follows that

$$\sum_{i=0}^{k_n-1}(M_{T_{i+1}^n} - M_{T_i^n})^2 \longrightarrow \langle M\rangle_t$$

in probability as $n \to \infty$. An adapted process $X = (X_t, \mathcal{F}_t)$ is said to be a *semimartingale* if X_t can be written as $M_t + A_t$ where M is a local martingale and A is a stochastic process that is locally of bounded variation. Let $\langle X\rangle = \langle M\rangle$ as usual.

The next formula plays an extremaly important role in stochastic calculus.

Theorem 1. 1. (Itô's formula)
Let $X = M + A$ be a continuous semimartingale, and let f be a real valued function on R which is twice continuously differentiable. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s. \tag{1.1}$$

Note that the second term on the right hand side is the stochastic integral.

Proof. We shall sketch its proof. Let $f \in C^2$. The proof rests essentially on Taylor's theorem :

$$f(y) - f(x) = f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2 + R(x,y) \tag{1.2}$$

where $|R(x,y)| \le r(|y-x|)(y-x)^2$, such that $r : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an increasing function satisfying $\lim_{u\downarrow 0} r(u) = 0$. Let now X be a continuous semimartingale. Without loss of generality we can take $X_0 = 0$, and further by stopping at $T_m = \inf\{t : |X_t| \ge m\}$, we may assume that X is bounded. Let t be a fixed positive number, and let $\{T_i^n\}_{i=0,1,\cdots,k_n}$ be a sequence of stopping times such that $0 = T_0^n \le T_1^n \le \cdots \le T_{k_n}^n = t$ and $\lim_{n\to\infty}\max_{0\le i<k_n}(T_{i+1}^n - T_i^n) = 0$. Then it is not difficult to see that

$$\sum_{i=0}^{k_n-1}(X_{T_{i+1}^n} - X_{T_i^n})^2 \longrightarrow \langle X\rangle_t$$

in probability as $n \to \infty$. From (1.2) it follows that

$$\begin{aligned}
&f(X_t) - f(X_0)\\
&= \sum_{i=0}^{k_n-1}\left\{f\left(X_{T_{i+1}^n}\right) - f\left(X_{T_i^n}\right)\right\}\\
&= \sum_{i=0}^{k_n-1} f'\left(X_{T_i^n}\right)\left(X_{T_{i+1}^n} - X_{T_i^n}\right)\\
&\quad + \frac{1}{2}\sum_{i=0}^{k_n-1} f''\left(X_{T_i^n}\right)\left(X_{T_{i+1}^n} - X_{T_i^n}\right)^2 + \sum_{i=0}^{k_n-1} R\left(X_{T_i^n}, X_{T_{i+1}^n}\right)
\end{aligned}$$

The first sum in the last expression converges in probability to the stochastic integral $\int_0^t f'(X_s)dX_s$, and the second sum converges in probability to $\frac{1}{2}\int_0^t f'(X_s)d\langle X\rangle_s$. So it remains to consider the third sum. To estimate it, observe that

$$\lim_{n\to\infty}\max_i r\left(\left|X_{T_{i+1}^n} - X_{T_i^n}\right|\right) = 0,$$

which follows from the assumption that X is bounded and continuous. Then

$$\left|\sum_{i=0}^{k_n-1} R\left(X_{T_i^n}, X_{T_{i+1}^n}\right)\right| \leq \max_{0\leq i<k_n} r\left(\left|X_{T_{i+1}^n} - X_{T_i^n}\right|\right) \sum_{i=0}^{k_n-1} (X_{T_{i+1}^n} - X_{T_i^n})^2,$$

and the right-hand side converges in probability to 0 as $n \to \infty$. Thus (1.1) holds. □

The Ito formula shows that the class of semimartingales is invariant under composition with C^2- function.

Theorem 1. 2. *If M is a continuous local martingale, then*

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right) \quad (0 \leq t < \infty) \tag{1.3}$$

is also a local martingale such that $\mathcal{E}(M)_0 = 1$.

Proof. Applying Itô's formula with $X = M - \frac{1}{2}\langle M\rangle$ and $f(x) = e^x$ we obtain

$$\mathcal{E}(M)_t = 1 + \int_0^t \mathcal{E}(M)_s dM_s,$$

which completes the proof. □

In [55] B. Maisonneuve gave a nice proof without using Ito's formula, which will be presented at the end of this section.

The generalization to non-continuous local martingales was done in 1970 by C. Doléans-Dade ([9]). She proved that if X is a semimartingale with $X_{0-} = 0$, then the solution Y of the stochastic integral equation

$$Y_t = 1 + \int_0^t Y_{s-} dX_s$$

is given by the formula

$$Y_t = \exp\left(X_t - \frac{1}{2}\langle X^c\rangle_t\right) \prod_{0\leq s\leq t} (1 + \triangle X_s)e^{-\triangle X_s}$$

where $\triangle X_s = X_s - X_{s-}$ and X^c is the continuous part of X.

A noteworthy fact is that, *supposing that the exponential local martingale $\mathcal{E}(M)$ is uniformly integrable, it is not necessarily a true martingale.* We first give such an example.

Example 1.1. Let $B = (B_t, \mathcal{F}_t)$ be a 3-dimensional Brownian motion starting at a $(a \neq o)$,and for $0 < r < |a|$ let $\tau_r = \inf\{t : |B_t| \leq r\}$. Then it is well-known that $P(\tau_r < \infty) = r/|a|$.

Let now h be the function defined by $h(x) = |a|/|x|$ for $x \in \mathbb{R}^3\backslash\{o\}$ which is obviously superharmonic in $\mathbb{R}^3$ and harmonic in the domain $\{x \in \mathbb{R}^3 : |x| > r\}$. Then the process Z defined by $Z_t = h(B_t)$ $(0 \leq t < \infty)$ is a positive supermartingale such that $Z_0 = 1$. By Doob's convergence theorem Z_t converges almost surely and in L_1 as $t \to \infty$. Then, the family $\{Z_t\}_{0\leq t<\infty}$ being compact in L_1, Z is uniformly integrable. Next, let $T_n = \tau_{1/n}$ $(n = 1, 2, \cdots)$. It is clear that $T_n \uparrow \infty$ a.s. Moreover, $Z^{T_n} = h(B^{T_n})$ is a martingale, because h is harmonic in $\{x \in \mathbb{R}^3 : |x| > 1/n\}$. Namely, Z is a local martingale which is uniformly integrable. However, it is impossible that Z is a martingale. Observe that $Z = \mathcal{E}(M)$ where $M_t = \int_0^t Z_s^{-1} dZ_s$ $(0 \leq t < \infty)$.

Generally, we have $E[\mathcal{E}(M)_t] \leq 1$ for every t, because $\mathcal{E}(M)$ is a positive supermartingale with $\mathcal{E}(M)_0 = 1$. Therefore, it is a martingale if and only if $E[\mathcal{E}(M)_t] = 1$ for every t. But the direct verification is usually hard to carry out. We shall deal the problem of finding sufficient conditions for $\mathcal{E}(M)$ to be a martingale in Sections 2 and 4.
An easy calculation shows that

$$\mathcal{E}(M)\mathcal{E}(-M) = \exp(-\langle M \rangle), \quad \mathcal{E}(M) = \mathcal{E}\left(\frac{1}{2}M\right)^2 \exp\left(-\frac{1}{4}\langle M \rangle\right)$$

From these relations one can immediately derive that

$$\{\mathcal{E}(M)_\infty = 0\} = \{\langle M \rangle_\infty = \infty\}. \tag{1.4}$$

Example 1.2. Let $B = (B_t, \mathcal{F}_t)$ be a one dimensional Brownian motion starting at 0. For each $t > 0$ we have

$$\begin{aligned} E[\mathcal{E}(B)_t] &= \int_{-\infty}^{\infty} \exp\left(x - \frac{t}{2}\right) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-t)^2}{2t}\right) dx \\ &= 1 \end{aligned}$$

and hence $\mathcal{E}(B)$ is a *true* martingale. However, since $\mathcal{E}(B)_\infty = 0$ a.s., it is not a uniformly integrable martingale.

The martingale property of $\mathcal{E}(B)$ is used effectively in the book of Mckean ([56]) for computing the distribution of quantities associated with Brownian motion. In the following we give such an examples.

Example 1.3. Let $\tau = \inf\{t : B_t = at + b\}$ where $a \in R$ and $b > 0$. Then we have

$$P(\tau < \infty) = \exp(-2a^+ b). \tag{1.5}$$

To see this, observe that $\mathcal{E}(\alpha B^\tau) \leq \exp(\alpha b)$. This implies that $\mathcal{E}(\alpha B^\tau)$ is a uniformly integrable martingale. Then

$$1 = E[\mathcal{E}(\alpha B^\tau)_\infty] = E\left[\exp\left(a\alpha\tau + b\alpha - \alpha^2 \frac{\tau}{2}\right) : \tau < \infty\right].$$

If $a > 0$, setting $\alpha = a + \sqrt{a^2 + 2\lambda}$ we have

$$E[\exp(ab + b\sqrt{a^2 + 2\lambda}) \exp(-\lambda\tau) : \tau < \infty] = 1,$$

that is, $E[\exp(-\lambda\tau) : \tau < \infty] = \exp(-ab - b\sqrt{a^2 + 2\lambda})$. Letting $\lambda \to 0$ we obtain (1.5).

Theorem 1. 3. (D. W. Stroock and S. R. S. Varadhan [80])
Let $M = (M_t, \mathcal{F}_t)$ be a continuous process and let $A = (A_t, \mathcal{F}_t)$ be a continuous process of finite variation such that $A_0 = 0$. Suppose that for sufficiently small λ the process $Z^{(\lambda)}$ defined by

$$Z_t^{(\lambda)} = \exp\left(\lambda M_t - \frac{1}{2}\lambda^2 A_t\right)$$

is a local martingale. Then M is a local martingale with $A = \langle M \rangle$.

Proof. We sketch the proof. By the assumption there is a $\lambda_0 > 0$ such that for any λ with $|\lambda| \leq \lambda_0$ $Z^{(\lambda)}$ is a local martingale. Let now $0 \leq s < t$. The usual stopping argumment enables us to assume that both $\exp(\lambda_0 M_t^*)$ and A_t are integrable where $M_t^* = \sup_{0 \leq s \leq t} |M_s|$. Then for every λ with $|\lambda| \leq \lambda_0$ $Z^{(\lambda)}$ is a martingale, so that for every $D \in \mathcal{F}_s$

$$\int_D Z_t^{(\lambda)} dP = \int_D Z_s^{(\lambda)} dP.$$

Differentiating the both sides with respect to λ we have

$$(1.6) \qquad \int_D (M_t - \lambda A_t) Z_t^{(\lambda)} dP = \int_D (M_s - \lambda A_s) Z_s^{(\lambda)} dP.$$

Noticing $Z_0^{(\lambda)} = 1$ and setting $\lambda = 0$ gives

$$\int_D M_t dP = \int_D M_s dP \quad (D \in \mathcal{F}_s).$$

This implies that M is a martingale. Further, taking again the derivatives of the both sides in (1.6) with respect to λ at $\lambda = 0$, we find that $M^2 - A$ is a martingale, that is, $A = \langle M \rangle$. □

Remark 1.1. The continuity of the process A and the condition $A_0 = 0$ are essential for the validity of this theorem (see J. Stoyanov [78],p.256).

The extension of Ito's formula to functions of several semimartingales is the following.

Theorem 1. 4. *If $X = (X^1, X^2, \cdots, X^n)$ is an n-tuple of continuous semimartingales and $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives, then*

$$\begin{aligned} &f(X_t) - f(X_0) \\ &= \sum_{i=1}^n \int_0^t D_i f(X_s) dX_s^i + \frac{1}{2} \sum_{1 \leq i,j \leq n} \int_0^t D_{ij} f(X_s) d\langle X^i, X^j \rangle_s. \end{aligned}$$

For the proof, see [12].

A supplementary note. We close this section with another proof of Theorem 1.2 given in [55] by B. Maisonneuve. It is based on the next two lemmas.

Lemma 1. 1. *Let $f = (f_n, \mathcal{G}_n)_{n=0,1,2,\cdots}$ be a martingale, and let*

$$g_n = \frac{e^{f_n}}{\prod_{i=1}^n E[e^{\triangle f_i} | \mathcal{G}_{i-1}]}$$

where $\triangle f_i = f_i - f_{i-1}$ $(i = 1, 2, \cdots)$. Then $g = (g_n, \mathcal{G}_n)$ is a martingale.

This follows immediately by an elementary calculation.

Lemma 1. 2. *Let $M = (M_t, \mathcal{F}_t)_{0\le t<\infty}$ be a continuous local martingale. Then for each $t > 0$ there exist partitions $\Gamma_n : 0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = t$ of $[0,t]$ such that $\lim_{n\to\infty} \max_{1\le i\le m_n}(t_i^n - t_{i-1}^n) = 0$ and*

$$\prod_{i=1}^{m_n} E\left[e^{\triangle M_{t_i^n}} | \mathcal{F}_{t_{i-1}^n}\right] \longrightarrow \exp(\frac{1}{2}\langle M\rangle_t) \quad a.s \quad (n\to\infty)$$

where $\triangle M_{t_i^n} = M_{t_i^n} - M_{t_{i-1}^n} \quad (i = 1, 2, \cdots, m_n)$.

Proof. Let M be a continuous local martingale. The usual stopping time argument enables us to assume that $|M| \le K$ for some constant $K > 0$.
Let now $h(x) = e^x - 1 - x$ and $k(x) = \log(1+x) - x$. Then

$$0 \le h(x) \le \frac{x^2}{2}e^{|x|} \quad (x \in \mathbb{R}) \tag{1.7}$$

$$\left|h(x) - \frac{x^2}{2}\right| \le C|x|^3 \quad (|x| \le 2K) \tag{1.8}$$

$$-\frac{x^2}{2} \le k(x) \le 0 \quad (0 \le x < \infty), \tag{1.9}$$

where the constant C depends only on K.
For a partition $\Gamma : 0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = t$ of $[0,t]$, we set

$$\begin{aligned}
\|\Gamma\| &= \max_{1\le i\le m}(t_i - t_{i-1}) \\
U_i &= E[h(\triangle M_{t_i})|\mathcal{F}_{t_{i-1}}], \\
W_i &= E\left[h(\triangle M_{t_i}) - \frac{1}{2}\triangle M_{t_i}^2 \middle| \mathcal{F}_{t_{i-1}}\right], \\
\mathcal{P}_\Gamma &= \prod_{i=1}^m E[\exp(\triangle M_{t_i})|\mathcal{F}_{t_{i-1}}].
\end{aligned}$$

Then, combining (1.7), (1.8) and (1.9) with the sample continuity of M shows that

$$\begin{aligned}
E\left[\sum_{i=1}^m |W_i|\right] &\le CE\left[\sum_{i=1}^m |\triangle M_{t_i}|^3\right] \\
&\le CE\left[\sup_{1\le i\le m} |\triangle M_{t_i}| \sum_{j=1}^m \triangle M_{t_j}^2\right] \longrightarrow 0 \quad (\|\Gamma\| \to 0),
\end{aligned}$$

$$\begin{aligned}
E[\sum_{i=1}^m |k(U_i)|] &\le \frac{1}{2}\sum_{i=1}^m E[E[h(U_i)|\mathcal{F}_{t_{i-1}}]^2] \\
&\le \frac{1}{2}\sum_{i=1}^m E\left[E\left[\frac{\triangle M_{t_i}^2}{2}\exp(|\triangle M_{t_i}|)\middle|\mathcal{F}_{t_{i-1}}\right]^2\right] \\
&\le \frac{4K}{8}E\left[\sum_{i=1}^m \triangle M_{t_i}^4\right] \\
&\le \frac{4K}{8}E\left[\sup_{1\le j\le m} \triangle M_{t_j}^2 \sum_{i=1}^m \triangle M_{t_i}^2\right] \longrightarrow 0 \quad (n\to\infty).
\end{aligned}$$

On the other hand, since $E[\exp(\triangle M_{t_i})|\mathcal{F}_{t_{i-1}}] = 1 + U_i$, we have

$$\begin{aligned}
\mathcal{P}_\Gamma &= \prod_{i=1}^{m}(1+U_i) \\
&= \exp\left\{\sum_{i=1}^{m} U_i + \sum_{i=1}^{m} k(U_i)\right\} \\
&= \exp\left\{\frac{1}{2}\sum_{i=1}^{m} E\left[\triangle M_{t_i}^2|\mathcal{F}_{t_{i-1}}\right] + \sum_{i=1}^{m} W_i + \sum_{i=1}^{m} k(U_i)\right\}.
\end{aligned}$$

Recall that

$$\sum_{i=1}^{m} E\left[\triangle M_{t_i}^2|\mathcal{F}_{t_{i-1}}\right] \longrightarrow \langle M\rangle_t \quad \text{in } L_1 \text{ as } \|\Gamma\| \to 0$$

by the well-known result of C. Doléans-Dade. Then there exists a sequence of partitions $\Gamma_n : 0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = t$ of $[0,t]$ such that

$$\mathcal{P}_{\Gamma_n} \longrightarrow \exp\left(\frac{1}{2}\langle M\rangle_t\right) \quad a.s \quad (n \to \infty).$$

Thus the lemma is proved. □

Now, let $0 < s < t$. Without loss of generality we may assume that $s \in \Gamma_n$ for every n, namely, $s = t_{k_n}^n$ for some $k_n \le m_n$. Let Γ_n' be the partition :$0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = s$ of $[0,s]$ and let

$$\mathcal{Q}_{\Gamma_n'} = \prod_{i=1}^{k_n} E\left[\exp(\triangle M_{t_i^n})|\mathcal{F}_{t_{i-1}^n}\right].$$

Then $\mathcal{Q}_{\Gamma_n'} \longrightarrow \exp(\frac{1}{2}\langle M\rangle_s)$ $a.s$ $(n \to \infty)$ as is already stated above, and from Lemma 1.1 it follows that

$$E\left[\frac{e^{M_t}}{\mathcal{P}_{\Gamma_n}}\middle|\mathcal{F}_s\right] = \frac{e^{M_s}}{\mathcal{Q}_{\Gamma_n'}}.$$

Thus, letting $n \to \infty$ we obtain

$$E\left[\frac{e^{M_t}}{e^{\frac{1}{2}\langle M\rangle_t}}\middle|\mathcal{F}_s\right] = \frac{e^{M_s}}{e^{\frac{1}{2}\langle M\rangle_s}},$$

which completes the proof.

1.2 The L_p-integrability of $\mathcal{E}(M)$

As is well-known, exponential martingales play an essential role in various questions concerning the absolute continuity of probability laws of stochastic processes. However, $\mathcal{E}(M)$ is not always a uniformly integrable martingales as stated before, and it is often difficult to verify the uniform integrability of $\mathcal{E}(M)$. In 1960, I.V. Girsanov([23]) showed that if $\langle M\rangle_\infty$ is bounded, then $\mathcal{E}(M)$ is a uniformly integrable martingale. In 1972, this assertion was proved by I.I. Gihman and A.V. Skorohod([22]) when $\exp((1+\delta)\langle M\rangle_\infty) \in L_1$ for some $\delta > 0$ and then by R.S. Lipster and A.N. Shiryayev

([54]) when $\exp((\frac{1}{2}+\delta)\langle M\rangle_\infty) \in L_1$ for some $\delta > 0$. After that, A.A. Novikov([64]) gave a nice criterion. In this section we improve his result.

We first prove the following result, from which a simple criterion can be derived. It is remarkable that the constant $\frac{\sqrt{p}}{2(\sqrt{p}-1)}$ is the best possible, and then its proof is extremely simple.

Theorem 1. 5. *Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Suppose that*

$$\sup_T E\left[\exp\left(\frac{\sqrt{p}}{2(\sqrt{p}-1)} M_T\right)\right] < \infty,$$

where the supremum is taken over all bounded stopping times T. Then $\mathcal{E}(M)$ is an L_q-bounded martingale.

Proof. For $1 < p < \infty$, let $r = (\sqrt{p}+1)/(\sqrt{p}-1)$. Then the exponent conjugate to r is $s = (\sqrt{p}+1)/2$, and note that $(q - \sqrt{q/r})s = \sqrt{p}/\{2(\sqrt{p}-1)\}$ by a simple calculation. Since we have

$$\mathcal{E}(M)^q = \exp\left(\sqrt{\frac{q}{r}} M - \frac{q}{2}\langle M\rangle\right) \exp\left\{\left(q - \sqrt{\frac{q}{r}}\right) M\right\},$$

an application of Hölder's inequality shows that for any stopping time S

$$\begin{aligned} E[\mathcal{E}(M)_S^q] &\le E[\mathcal{E}(\sqrt{qr}M)_S]^{\frac{1}{r}} E\left[\exp\left\{\left(q - \sqrt{\frac{q}{r}}\right) sM_S\right\}\right]^{\frac{1}{s}} \\ &\le \sup_T E\left[\exp\left\{\frac{\sqrt{p}}{2(\sqrt{p}-1)} M_T\right\}\right]^{\frac{1}{s}} < \infty. \end{aligned}$$

This completes the proof. □

Remark 1.2. Let M be a right continuous local martingale such that $\triangle M \ge 0$ and suppose that $E[\exp(\frac{K}{2}[M]_\infty)] < \infty$ for some $K > 0$. Then J. Yan proved in ([86]) that $\mathcal{E}(M) \in H_r$ with $r = \frac{K}{2\sqrt{K}-1}$ if $1 < K \le 4$ and $r = \frac{2K}{K+2}$ if $K > 4$. In [52] D. Lépingle and J. Mémin improved his result.

Remark 1.3. H. Sato gave in [71] the following interesting result : Let M be a stochastically continuous additive process with paths which are right continuous and have left-hand limits at every point. If $M_0 = 0, E[M_t] = 0$ and $\triangle M_t > -1$ for every t, then all of the following statements are equivalent.

(i) $M^* \in L_1$
(ii) M is uniformly integrable.
(iii) M_t converges almost surely as $t \to \infty$.
(iv) $\mathcal{E}(M)_\infty > 0$.
(v) $\mathcal{E}(M)$ is a uniformly integrable martingale.
(vi) $\mathcal{E}(M)^* \in L_1$

Now, by using Theorem 1.5 we can give a simple but usefull criterion for the uniform integrability of $\mathcal{E}(M)$.

Theorem 1. 6. *Suppose that*

$$\sup_T E\left[\exp\left(\frac{1}{2}M_T\right)\right] < \infty,$$

where the supremum is taken over all bounded stopping times T. Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Proof. Let $0 < a < 1$ and choose $p > 1$ such that $\sqrt{p}/(\sqrt{p}-1) < 1/a$. Then by the assumption $\mathcal{E}(aM)$ is an L_q-bounded martingale, and so it is obviously a uniformly integrable martingale. Since $\mathcal{E}(aM) = \mathcal{E}(M)^{a^2}\exp\{a(1-a)M\}$, by using Hölder's inequality with exponents a^{-2} and $(1-a^2)^{-1}$ we find

$$\begin{aligned} 1 = E[\mathcal{E}(aM)_\infty] &\leq E[\mathcal{E}(M)_\infty]^{a^2} E\left[\exp\left(\frac{a}{1+a}M_\infty\right)\right]^{1-a^2} \\ &\leq E[\mathcal{E}(M)_\infty]^{a^2} E\left[\exp\left(\frac{1}{2}M_\infty\right)\right]^{2a(1-a)} \end{aligned}$$

The second term on the right hand side converges to 1 as $a \uparrow 1$. Therefore we have $1 \leq E[\mathcal{E}(M)_\infty]$, which completes the proof. □

Example 1.4. For $0 < a < \infty$, let $\tau_a = \inf\{t > 0 : B_t = a\}$. Let now $\lambda > 0$. Then $\sup_T E\left[\exp\left(\frac{1}{2}B_T^{\tau_a}\right)\right] \leq e^a$ and so $E[\mathcal{E}(\sqrt{2\lambda}B^{\tau_a})_\infty] = 1$ by Theorem 1.6. Then we have

$$E[\exp(-\lambda\tau_a)] = \exp(-a\sqrt{2\lambda}),$$

and, applying the inversion formula for the Laplace transform gives

$$P(\tau_a = t) = \sqrt{2\pi t^3}a\exp\left(-\frac{a^2}{2t}\right).$$

Note that the converse of this theorem is not true (see Example 1.13). As a corollary we can obtain the following criterion, because

$$E\left[\exp\left(\frac{1}{2}M_T\right)\right] \leq E\left[\exp\left(\frac{1}{2}\langle M\rangle_T\right)\right]^{\frac{1}{2}}$$

for every stopping time T.

Corollary 1. 1 (A. A. Novikov [64]). *Suppose that*

$$E\left[\exp\left(\frac{1}{2}\langle M\rangle_\infty\right)\right] < \infty.$$

Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

This is known as Novikov's criterion. Note that 1/2 is the best constant in these criteria. Following the idea of Novikov, we exemplify it below.

Example 1.5. For $0 < a < 1$, let us define the stopping time

$$T = \inf\{t : B_t \leq at - 1\},$$