A TABLE OF THE INCOMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

By

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FOREWORD

The tables included in this book were computed to solve a specific problem. It was found that the surface area of certain geometrical bodies could be expressed only in terms of the incomplete elliptic integral of the third and lower kinds. A search of the literature was made, which determined that complete values for the integral of the third kind were not available, and it was therefore decided to compute such a table.

This table will be of greatest use to physicists, engineers, and applied mathematicians who work in the fields of fluid dynamics, heat flow, and related topics. The integral appears in the solution of problems dealing with the motion of the spherical pendulum and related mechanisms, in problems of magnetic potentials due to circular current or of the gravitational potential of a uniform circular disk, and in certain kinds of seismological work.

Formerly, rough approximations or individual values calculated with considerable effort had to serve. It is hoped that the users of this table will find it useful and that the onus of the incomplete elliptic integral of the third kind has been permanently lifted.

The actual computation of this table was performed in 1956 and early in 1957 on an IBM Type 704 Calculator; the greatest difficulty was encountered not in constructing the table but in obtaining satisfactory checking.

The introductory pages have been reviewed for technical accuracy by L. E. Ward, Jr. and D. E. Zilmer; the tables have not been reviewed, as an echo-checking procedure described in the Introduction was used to ensure complete printing, and the computation was verified with a check integral at the completion of each value of α.

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Errata

In the following table the headings on the columns are in error:

For α read α^2,

For θ read ϕ,

For κ read k^2.

INTRODUCTION

ELLIPTIC INTEGRALS OF THE THIRD KIND

The elliptic integral of the third kind is given by the formulas

$$\Pi(\phi, \alpha^2, k) = \int_0^{\phi} \frac{d\theta}{(1 - \alpha^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}$$

$$= \int_0^{y} \frac{dt}{(1 - \alpha^2 t^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

where $y = \sin \phi$, $t = \sin \theta$, $\alpha^2 \neq 1$, $\alpha^2 \neq k^2$.

If $\phi = \pi/2$ $(y = 1)$, the integral is said to be complete and one writes $\Pi(\pi/2, \alpha^2, k) = \Pi(\alpha^2, k)$.

There are two important reference sources for elliptic integrals of the third kind.[1,2] The second of these is a compendium of available tables, and errors for many of them; the first contains some tables and a large supply of useful formulas.

For certain values of α^2 the elliptic integral can be reduced to combinations of Theta and Jacobian Zeta functions of real arguments. Hence, it can be evaluated with available tables (the formulas are listed in the appropriate section). These 'hyperbolic' cases occur if $\alpha^2 > 1$ or $0 < \alpha^2 < k^2$. For the remaining or 'circular' cases, the arguments of the Theta and Zeta functions are complex. This table considers the range $\alpha^2 < 1$, which introduces some excess only for the range $0 < \alpha^2 < k^2$.

In all cases the complete integral can be reduced to com-

[1] Byrd, P. F. and M. D. Friedman. *Handbook of Elliptic Integrals for Engineers and Physicists.* Berlin, Springer, 1954.

[2] Fletcher, Alan. "Guide to Tables of Elliptic Functions," *Math. Tables,* Vol. 3 (1948–1949), pp. 229–281.

binations of integrals of the first and second kind; Fletcher lists the few tables available.[2]

The following sections list some formulas,[3] showing the relations that may be established between the different elliptic integrals.

Complete Integrals

Complete integrals may be given using Heuman's Lambda function [4]

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\,[EF(\beta, k') + KE(\beta, k') - KF(\beta, k')]$$

or the Jacobian Zeta function[5]

$$Z(\beta, k) = E(\beta, k) - EF(\beta, k)/K$$

where

$k'^2 + k^2 = 1, \quad E = E(\pi/2, k), \quad K = F(\pi/2, k).$
$E(\beta, k)$ is the elliptic integral of the second kind.
$F(\beta, k)$ is the elliptic integral of the first kind.

CASE I. $\alpha^2 < 0$

$$\phi = \sin^{-1}\sqrt{\frac{\alpha^2}{\alpha^2 - k^2}} \qquad \beta = \sin^{-1}\frac{1}{\sqrt{1-\alpha^2}}$$

$$\Pi(\alpha^2, k) = \frac{k^2K}{k^2 - \alpha^2} - \frac{\pi}{2}\frac{\alpha^2\Lambda_0(\phi, k)}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}$$

$$\Pi(-k, k) = \frac{1}{4(1+k)}\,[\pi + 2(1+k)K]$$

$$\Pi(\alpha^2, k) = \frac{K}{1-\alpha^2} + \frac{\pi}{2}\frac{\alpha^2[\Lambda_0(\beta, k) - 1]}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}$$

3 Byrd and Friedman, *op. cit.*, p. 600.

4 *Ibid.*, pp. 344–349. See also C. Heuman. "Tables of Complete Elliptic Integrals," *J. Math. Phys.*, Vol. 19–20 (1940–1941), pp. 127–206.

5 *Ibid.*, pp. 336-343.

CASE II. $k^2 < \alpha^2 < 1$

$$\theta = \sin^{-1} \sqrt{\frac{1-\alpha^2}{1-k^2}} \qquad \xi = \sin^{-1} \sqrt{\frac{\alpha^2 - k^2}{\alpha^2(1-k^2)}}$$

$$\Pi(\alpha^2, k) = K + \frac{\pi}{2} \frac{\alpha[1 - \Lambda_0(\theta, k)]}{\sqrt{(\alpha^2 - k^2)(1-\alpha^2)}}$$

$$\Pi(k^2, k) = E/1 - k^2$$

$$\Pi(k, k) = \frac{1}{4(1-k)} [\pi + 2(1-k)k]$$

$$\Pi(\alpha^2, k) = \frac{\pi}{2} \frac{\alpha \Lambda_0(\xi, k)}{\sqrt{(\alpha^2 - k^2)(1-\alpha^2)}}$$

CASE III. $0<\alpha^2<k^2$

$$\beta = \sin^{-1}(\alpha/k)$$

$$\Pi(\alpha^2, k) = K + \frac{\alpha K Z(\beta, k)}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}}$$

$$\Pi(k^2, k) = E/(1-k^2)$$

CASE IV. $\alpha^2>1$

$$\beta = \sin^{-1}(1/\alpha)$$

$$\Pi(\alpha^2, k) = -\frac{\alpha K Z(\beta, k)}{\sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)}}$$

For complex parameter α^2, certain cases may be reduced to combinations of real parameters.[6]

Incomplete Integrals

The formulas for incomplete integrals are far more complicated, and involve Theta functions. For this reason only the formulas for the hyperbolic cases are given.

$$v = \pi F(\phi, k)/2K \qquad \omega(\gamma) = \pi F(\gamma, k)/2K$$

$$p = \pi K'/2K = \pi K(k')/2K \qquad q = e^{-2p}$$

[6] Byrd and Friedman, *op. cit.*, pp. 231-232.

CASE I. $\alpha^2>1$

$$A=\sin^{-1}(1/\alpha)$$

$$\theta_1(v)=2\sum_{m=1}^{\infty}(-1)^{m-1}q^{(m-1/2)^2}\sin(2m-1)v$$

$$\Omega_4=\frac{1}{2}\ln\frac{\theta_1[\omega(A)+v]}{\theta_1[\omega(A)-v]}=\frac{1}{2}\ln\frac{\sin[\omega(A)+v]}{\sin[\omega(A)-v]}$$

$$+\sum_{m=1}^{\infty}q^m\frac{\sin 2mv\sin 2m\omega(A)}{m\sinh 2mp}$$

$$\Pi(\phi,\alpha^2,k)=-\frac{\alpha[F(\phi,k)Z(A,k)-\Omega_4]}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}$$

$$\Pi(\phi,1,k)=\frac{1}{k'^2}[k'^2F(\phi,k)-E(\phi,k)$$

$$+\sqrt{1-k^2\sin^2\phi}\tan\phi]$$

CASE II. $0<\alpha^2<k^2$

$$\beta=\sin^{-1}(\alpha/k)\qquad\theta_0(v)=1+2\sum_{m=1}^{\infty}(-1)^mq^{m^2}\cos 2mv$$

$$\Omega_3=\frac{1}{2}\ln\frac{\theta_0[v+\omega(\beta)]}{\theta_0[v-\omega(\beta)]}=\sum_{m=1}^{\infty}\frac{\sin 2m\omega(\beta)\sin 2mv}{m\sinh 2mp}$$

$$\Pi(\phi,\alpha^2,k)=F(\phi,k)+\frac{\alpha[F(\phi,k)Z(\beta,k)-\Omega_3]}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}}$$

$$\Pi(\phi,k^2,k)=\frac{1}{k'^2}\left[E(\phi,k)-k^2\frac{\sin\phi\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\right]$$

Differential Equations

The third elliptic integral also satisfies the following equation:

$$k^2k'^2(k^2-\alpha^2)\frac{\partial^3\Pi}{\partial k^3}+k(\alpha^2+4k^2+3\alpha^2k^2-8k^4)\frac{\partial^2\Pi}{\partial k^2}$$

$$+(2k^2 - \alpha^2 - 13k^4)\frac{\partial \Pi}{\partial k} - 3k^3\Pi = -\frac{3k^3 \sin\phi \cos\phi}{\sqrt{(1 - k^2 \sin^2\phi)^5}}$$

If $\phi = \pi/2$, the right side of this equation conveniently vanishes. This equation supplied a very useful check on the computations.

Addition Formula

There is one other relation of considerable importance, in that it enables one to consider only the case of $|\alpha^2| \leq 1$.

$$\Pi(\phi, \alpha^2, k) + \Pi(\phi, k^2/\alpha^2, k) = F(\phi, k)$$
$$+ \sqrt{\frac{\alpha^2}{(1-\alpha^2)(\alpha^2 - k^2)}} \tan^{-1} \sqrt{\frac{(1-\alpha^2)(\alpha^2 - k^2)}{\alpha^2(1 - k^2 \sin^2\phi)}} \tan\phi$$

METHOD OF COMPUTATION AND CHECKING

These tables were produced on an IBM Type 704 Calculator, using a simple Simpson's Rule method of integration. The results of integration were stored for each .01 radian, and printed out in groups of ten lines at a time. In order to compensate for accumulated roundoff in the angle, every .1 radian was fed in from a separate list, adjusting the mesh size for one integration, so that at no point is the angle in error by more than 10^{-7}.

The method of printing is naturally of concern, since a large amount of error can occur here. Printing was handled by the computer using a method known as echo-checking. With this process, the type wheels are set as ordered, and then an independent pulse is returned to the computer, indicating what symbol has been printed. A comparison of the return, or echo, with the initial command ensures that what was printed is what was desired. After being printed by the computer, the tabulation was reproduced by the photolithographic process, so that there should be no variation between the initial printing and the final result.

There still remained the problem of checking the computation. This was handled in the following way: Computation proceeded by Simpson's Rule with a mesh size of about .0025 until $\phi = 1.57$. At this point the mesh size was changed so that the next point was computed with $\phi = \pi/2$, yielding all the complete integrals. An entirely different method was then used for obtaining the complete integral, and this has been printed as the last line of each group in the table. Comparison between these lines gives a very excellent indication of the upper limit of the error. There are possible roundoff errors in both the table and the independent check but it seems entirely reasonable that if there is a check for the first n places then the first $n - 1$ places are correct. Furthermore, since the value for $\phi = \pi/2$ was based on all the prior computations for smaller angles, if there is a check at $\pi/2$ the prior computations should have at least the same accuracy.

With the argument as outlined, no attempt has been made to proof or check the printed sheets in any way other than by a comparison of the resultant complete integrals.

Computation of the Check Integral

The complete integral was computed by using the differential equation given earlier, integrating from 0 to K_{15}. For values of $\alpha^2 < 0$ no problems were encountered. For values of $\alpha^2 > 0$ there is the problem of a singularity at $k = \alpha$. This was handled by integrating from 0 to the largest k_i less than α, and then starting again at $k = \alpha$.

Integration was handled by Gill's version of the Runge-Kutta fourth-order method.[7] The necessary starting values are given by the following formulas:

[7] Gill, "A Process for the Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine," *Proc. Camb. Phil. Soc.*, Vol. 47 (1951), pp. 96–108.

$k = 0$

$$\Pi = \pi/2\sqrt{1-\alpha^2}$$

$$\frac{\partial\Pi}{\partial k} = 0$$

$$\partial^2\Pi/\partial k^2 = \frac{\pi}{2\alpha^2}\left(\frac{1}{\sqrt{1-\alpha^2}} - 1\right)$$

$$\frac{\partial^3\Pi}{\partial k^3} = 0$$

$E = E(\pi/2, \alpha)$ and $F = F(\pi/2, \alpha)$ are complete integrals of the second and first kinds, respectively.

$k = \alpha$

$$\Pi = E/1 - \alpha^2$$

$$\partial\Pi/\partial k = \frac{1}{3\alpha(1-\alpha^2)^2}\left[E + \alpha^2 E - (1-\alpha^2)F\right]$$

$$\frac{\partial^2\Pi}{\partial k^2} = \frac{1}{5\alpha(1-\alpha^2)}\left[3\alpha\Pi - (1-13\alpha^2)\frac{\partial\Pi}{\partial k}\right]$$

$$= \frac{E}{15\alpha^2(1-\alpha^2)^3}(4\alpha^4 + 21\alpha^2 - 1)$$

$$+ \frac{F}{15\alpha^2(1-\alpha^2)^2}(1 - 13\alpha^2)$$

$$\partial^3\Pi/\partial k^3 = \frac{E}{105(1-\alpha^2)^4\alpha^3}(36\alpha^6 + 618\alpha^4 + 72\alpha^2 - 6)$$

$$+ \frac{F}{105(1-\alpha^2)^3\alpha^3}(6 - 69\alpha^2 - 297\alpha^4).$$

USE OF THE TABLE

This table is given primarily as a function of α^2. For each value of α^2 fifteen values of k are tabulated for $k = \sin\theta$, $\theta = .1(.1)1.5$, and angular values of $\phi = 0(.01)1.57$ and $\phi = \pi/2$. The last line of each group is a duplicate computation of the complete integral, which serves as a check on the computation. If the two values given for the complete integral agree to n places, this can be considered to give an accurate table to within ± 2 in the nth place. To allow for the slight inaccuracies in angles, and accumulated roundoff, the safest estimate is probably that the values given are correct to $n - 1$ places if the two complete integrals agree to n places.

It will be noticed that certain fractional values print with a sequence of 9's (e.g., .099999 for .1). The particular program used for printing does not make any attempt at rounding off; hence any error that produces a number very slightly less than desired, will produce a sequence of 9's. In all cases, however, the angular values are correct to within 5 in the seventh place.

The table has been printed, for each α, in groups of three. Each group yields a full set of angular values, on four facing pages, for five of the fifteen possible values of k. The entire table is ordered in increasing sequence on α^2, starting at $\alpha^2 = -1.00$.

For values of $\alpha^2 < -1$, one must use the relation given as an addition formula, which yields the value of $\Pi(\phi, \alpha^2, k)$ as a function of $\Pi(\phi, k^2/\alpha^2, k)$. Since large negative values of α^2 will depend on the values for small negative α^2, more values than might be expected are given for $-.1 < \alpha^2 < 0$.

The one missing α^2, namely $\alpha^2 = 0$, is given by $\Pi(\phi, 0, k) = F(\phi, k)$.

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TABLE OF THE
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OF THE THIRD KIND

$\alpha =$ -1.000000

θ	K VALUES				
	.009966711	.039469502	.087332193	.151646642	.229848846
.0000000	.000000	.000000	.000000	.000000	.000000
.0100000	.009999	.009999	.009999	.009999	.009999
.0200000	.019997	.019997	.019997	.019997	.019997
.0300000	.029991	.029991	.029991	.029991	.029992
.0400000	.039978	.039979	.039979	.039980	.039981
.0500000	.049958	.049959	.049960	.049961	.049963
.0600000	.059928	.059929	.059931	.059933	.059936
.0700000	.069886	.069888	.069891	.069894	.069899
.0800000	.079831	.079833	.079837	.079843	.079849
.0900000	.089759	.089763	.089769	.089776	.089786
.1000000	.099670	.099675	.099683	.099694	.099707
.1100000	.109562	.109569	.109579	.109593	.109611
.1200000	.119433	.119441	.119455	.119473	.119496
.1300000	.129281	.129291	.129309	.129332	.129360
.1400000	.139103	.139117	.139138	.139167	.139203
.1500000	.148900	.148916	.148943	.148978	.149022
.1600000	.158668	.158688	.158720	.158763	.158816
.1700000	.168407	.168430	.168469	.168520	.168583
.1800000	.178114	.178142	.178187	.178248	.178323
.1900000	.187788	.187821	.187874	.187946	.188033
.2000000	.197428	.197466	.197528	.197612	.197713
.2100000	.207032	.207076	.207148	.207244	.207361
.2200000	.216599	.216649	.216731	.216842	.216977
.2300000	.226127	.226184	.226278	.226404	.226557
.2400000	.235615	.235680	.235786	.235929	.236103
.2500000	.245062	.245136	.245255	.245415	.245611
.2600000	.254467	.254549	.254683	.254863	.255083
.2700000	.263828	.263920	.264069	.264270	.264515
.2800000	.273145	.273247	.273412	.273636	.273909
.2900000	.282416	.282529	.282712	.282959	.283261
.3000000	.291640	.291764	.291967	.292239	.292573
.3100000	.300817	.300954	.301176	.301475	.301842
.3200000	.309946	.310095	.310338	.310667	.311069
.3300000	.319025	.319188	.319454	.319812	.320252
.3400000	.328054	.328232	.328521	.328912	.329390
.3500000	.337033	.337226	.337540	.337964	.338484
.3600000	.345961	.346169	.346509	.346969	.347533
.3700000	.354836	.355062	.355429	.355926	.356535
.3800000	.363659	.363902	.364298	.364834	.365492
.3900000	.372429	.372690	.373117	.373693	.374401

$\alpha = -1.000000$

θ	K VALUES				
	.009966711	.039469502	.087332193	.151646642	.229848846
.4000000	.381146	.381426	.381884	.382503	.383264
.4100000	.389809	.390109	.390599	.391263	.392078
.4200000	.398417	.398738	.399262	.399973	.400845
.4300000	.406971	.407314	.407873	.408632	.409564
.4400000	.415471	.415836	.416432	.417240	.418235
.4500000	.423915	.424304	.424938	.425798	.426857
.4600000	.432305	.432717	.433391	.434305	.435431
.4700000	.440639	.441076	.441791	.442761	.443956
.4800000	.448918	.449381	.450138	.451165	.452433
.4900000	.457141	.457631	.458431	.459519	.460861
.5000000	.465309	.465826	.466672	.467821	.469240
.5100000	.473422	.473967	.474859	.476072	.477570
.5200000	.481480	.482054	.482993	.484272	.485852
.5300000	.489482	.490086	.491075	.492421	.494086
.5400000	.497429	.498064	.499103	.500519	.502271
.5500000	.505321	.505988	.507079	.508567	.510408
.5600000	.513159	.513858	.515003	.516563	.518497
.5700000	.520942	.521674	.522874	.524510	.526538
.5800000	.528671	.529437	.530693	.532407	.534533
.5900000	.536346	.537147	.538460	.540254	.542479
.6000000	.543967	.544804	.546176	.548051	.550379
.6100000	.551535	.552409	.553841	.555799	.558233
.6200000	.559050	.559961	.561455	.563499	.566040
.6300000	.566513	.567461	.569019	.571150	.573801
.6400000	.573923	.574911	.576533	.578753	.581517
.6500000	.581282	.582309	.583996	.586308	.589188
.6600000	.588589	.589656	.591411	.593816	.596814
.6700000	.595845	.596954	.598777	.601277	.604396
.6800000	.603051	.604201	.606095	.608691	.611933
.6899999	.610206	.611400	.613364	.616060	.619428
.7000000	.617313	.618550	.620587	.623383	.626880
.7100000	.624370	.625652	.627762	.630661	.634289
.7200000	.631379	.632706	.634891	.637895	.641656
.7300000	.638340	.639712	.641974	.645084	.648982
.7400000	.645253	.646672	.649012	.652230	.656267
.7500000	.652120	.653586	.656004	.659333	.663511
.7600000	.658941	.660455	.662953	.666394	.670716
.7700000	.665715	.667278	.669858	.673412	.677881
.7800000	.672445	.674057	.676719	.680390	.685007
.7899999	.679130	.680793	.683538	.687326	.692095