FAMOUS PROBLEMS

AND OTHER MONOGRAPHS

FAMOUS PROBLEMS OF
ELEMENTARY GEOMETRY

BY F. KLEIN

FROM DETERMINANT
TO TENSOR

BY W. F. SHEPPARD

INTRODUCTION TO
COMBINATORY ANALYSIS

BY P. A. MACMAHON

THREE LECTURES ON
FERMAT'S LAST THEOREM

BY L. J. MORDELL

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EDITOR'S PREFACE

This work, like its companion volume, *Squaring the Circle, and other Monographs* by Hobson et al., consists of a reprint in one volume of several books on mathematics that were originally published as separate volumes.

The reason for the selection of the four books that comprise this volume is that each is a valuable and important work and that each is of interest to a fairly wide circle of mathematicians and students.

The reason for their inclusion in a single volume is neither learned nor recondite. The reason is purely economic: Reprinted separately, the books would have to be priced at not much less than the price of the whole present volume (if they could be so reprinted at all). Anyone who buys the book for the sake of one of the four volumes that it contains will surely find the other three of interest and will consider them to be a worthwhile and welcome addition to his library.

FAMOUS PROBLEMS

OF

ELEMENTARY GEOMETRY

THE DUPLICATION OF THE CUBE
THE TRISECTION OF AN ANGLE
THE QUADRATURE OF THE CIRCLE

AN AUTHORIZED TRANSLATION OF F. KLEIN'S
VORTRÄGE ÜBER AUSGEWÄHLTE FRAGEN DER ELEMENTARGEOMETRIE
AUSGEARBEITET VON F. TÄGERT

BY

WOOSTER WOODRUFF BEMAN
1850—1922

AND

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SECOND EDITION REVISED, AND ENLARGED WITH NOTES

BY

RAYMOND CLARE ARCHIBALD
PROFESSOR OF MATHEMATICS IN BROWN UNIVERSITY

PREFACE.

THE more precise definitions and more rigorous methods of demonstration developed by modern mathematics are looked upon by the mass of gymnasium professors as abstruse and excessively abstract, and accordingly as of importance only for the small circle of specialists. With a view to counteracting this tendency it gave me pleasure to set forth last summer in a brief course of lectures before a larger audience than usual what modern science has to say regarding the possibility of elementary geometric constructions. Some time before, I had had occasion to present a sketch of these lectures in an Easter vacation course at Göttingen. The audience seemed to take great interest in them, and this impression has been confirmed by the experience of the summer semester. I venture therefore to present a short exposition of my lectures to the Association for the Advancement of the Teaching of Mathematics and the Natural Sciences, for the meeting to be held at Göttingen. This exposition has been prepared by Oberlehrer Tägert, of Ems, who attended the vacation course just mentioned. He also had at his disposal the lecture notes written out under my supervision by several of my summer semester students. I hope that this unpretending little book may contribute to promote the useful work of the association.

F. KLEIN.

GÖTTINGEN, Easter, 1895.

TRANSLATORS' PREFACE.

At the Göttingen meeting of the German Association for the Advancement of the Teaching of Mathematics and the Natural Sciences, Professor Felix Klein presented a discussion of the three famous geometric problems of antiquity, — the duplication of the cube, the trisection of an angle, and the quadrature of the circle, as viewed in the light of modern research.

This was done with the avowed purpose of bringing the study of mathematics in the university into closer touch with the work of the gymnasium. That Professor Klein is likely to succeed in this effort is shown by the favorable reception accorded his lectures by the association, the uniform commendation of the educational journals, and the fact that translations into French and Italian have already appeared.

The treatment of the subject is elementary, not even a knowledge of the differential and integral calculus being required. Among the questions answered are such as these: Under what circumstances is a geometric construction possible? By what means can it be effected? What are transcendental numbers? How can we prove that e and π are transcendental?

With the belief that an English presentation of so important a work would appeal to many unable to read the original, Professor Klein's consent to a translation was sought and readily secured.

In its preparation the authors have also made free use of the French translation by Professor J. Griess, of Algiers, following its modifications where it seemed advisable.

They desire further to thank Professor Ziwet for assistance in improving the translation and in reading the proof-sheets.

W. W. BEMAN.
D. E. SMITH.

August, 1897.

EDITOR'S PREFACE.

Within three years of its publication thirty-five years ago Klein's little work was translated into English, French, Italian, and Russian[1]. In the United States it filled a decided need for many years, and not a few teachers regretted that the work was allowed to go out of print. No other work supplied in such compact form just the information here found. Hence it seemed desirable to have a new edition with at least some of the slips of the first edition rectified, and with added notes illuminating the text.

The corrections and notes of the present edition are little more than revised extracts from my article in *The American Mathematical Monthly*[2], 1914. I am indebted to the Editors for courteously allowing the reproduction of this material.

R. C. A.

February, 1930.

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[1] French translation by Griess, Paris, Nony, 1896; Italian by Giudice, Turin, Rosenberg e Sallier, 1896; Russian by Parfentiev and Sintsov, Kazan, 1898. This last translation seems to have been unknown to the editors of Klein's *Abhandlungen* (see v. 3, 1923, p. *28*).

[2] Remarks on Klein's "Famous Problems of Elementary Geometry", v. 21, p. 247—259.

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INTRODUCTION.

THIS course of lectures is due to the desire on my part to bring the study of mathematics in the university into closer touch with the needs of the secondary schools. Still it is not intended for beginners, since the matters under discussion are treated from a higher standpoint than that of the schools. On the other hand, it presupposes but little preliminary work, only the elements of analysis being required, as, for example, in the development of the exponential function into a series.

We propose to treat of geometrical constructions, and our object will not be so much to find the solution suited to each case as to determine the *possibility* or *impossibility* of a solution.

Three problems, the object of much research in ancient times, will prove to be of special interest. They are

1. *The problem of the duplication of the cube* (also called the *Delian problem*).
2. *The trisection of an arbitrary angle.*
3. *The quadrature of the circle, i.e.*, the construction of π.

In all these problems the ancients sought in vain for a solution with straight edge and compasses, and the celebrity of these problems is due chiefly to the fact that their solution seemed to demand the use of appliances of a higher order. In fact, we propose to show that a solution by the use of straight edge and compasses is impossible.

The impossibility of the solution of the third problem was demonstrated only very recently. That of the first and second is implicitly involved in the Galois theory as presented to-day in treatises on higher algebra. On the other hand, we find no explicit demonstration in elementary form unless it be in Petersen's text-books, works which are also noteworthy in other respects.

At the outset we must insist upon the difference between *practical* and *theoretical* constructions. For example, if we need a divided circle as a measuring instrument, we construct it simply on trial. Theoretically, in earlier times, it was possible (*i.e.*, by the use of straight edge and compasses) only to divide the circle into a number of parts represented by 2^n, 3, and 5, and their products. Gauss added other cases by showing the possibility of the division into parts where p is a prime number of the form $p = 2^{2^\mu} + 1$, and the impossibility for all other numbers. No practical advantage is derived from these results; *the significance of Gauss's developments is purely theoretical.* The same is true of all the discussions of the present course.

Our fundamental problem may be stated: *What geometrical constructions are, and what are not, theoretically possible?* To define sharply the meaning of the word "construction," we must designate the instruments which we propose to use in each case. We shall consider

1. Straight edge and compasses,
2. Compasses alone,
3. Straight edge alone,
4. Other instruments used in connection with straight edge and compasses.

The singular thing is that elementary geometry furnishes no answer to the question. We must fall back upon algebra and the higher analysis. The question then arises: How shall we use the language of these sciences to express the employment of straight edge and compasses? This new method of attack is rendered necessary because elementary geometry possesses no general method, no *algorithm*, as do the last two sciences.

In analysis we have first *rational* operations: addition, subtraction, multiplication, and division. These operations can be directly effected geometrically upon two given segments by the aid of proportions, if, in the case of multiplication and division, we introduce an auxiliary unit-segment.

Further, there are *irrational* operations, subdivided into *algebraic* and *transcendental*. The simplest algebraic operations are the extraction of square and higher roots, and the solution of algebraic equations not solvable by radicals, such as those of the fifth and higher degrees. As we know how to construct $\sqrt{ab}$, rational operations in general, and irrational operations involving only square roots, can be constructed. On the other hand, every *individual* geometrical construction which can be reduced to the intersection of two straight lines, a straight line and a circle, or two circles, is equivalent to a rational operation or the extraction of a square root. In the higher irrational operations the construction is therefore impossible, *unless we can find a way of effecting it by the aid of square roots.* In all these constructions it is obvious that the number of operations must be limited.

We may therefore state the following fundamental theorem: *The necessary and sufficient condition that an analytic expression can be constructed with straight edge and compasses is that it can be derived from the known quantities by a finite number of rational operations and square roots.*

Accordingly, if we wish to show that a quantity cannot be constructed with straight edge and compasses, we must prove that the corresponding equation is not solvable by a finite number of square roots.

A fortiori the solution is impossible when the problem has *no* corresponding algebraic equation. An expression which satisfies no algebraic equation is called a transcendental number. This case occurs, as we shall show, with the number π.

PART I.

THE POSSIBILITY OF THE CONSTRUCTION OF ALGEBRAIC EXPRESSIONS.

CHAPTER I.

Algebraic Equations Solvable by Square Roots.

The following propositions taken from the theory of algebraic equations are probably known to the reader, yet to secure greater clearness of view we shall give brief demonstrations.

If x, *the quantity to be constructed, depends only upon rational expressions and square roots, it is a root of an irreducible equation* $\phi(x) = 0$, *whose degree is always a power of* 2.

1. To get a clear idea of the structure of the quantity x, suppose it, *e.g.*, of the form

$$x = \frac{\sqrt{a + \sqrt{c + ef}} + \sqrt{d + \sqrt{b}}}{\sqrt{a} + \sqrt{b}} + \frac{p + \sqrt{q}}{\sqrt{r}},$$

where a, b, c, d, e, f, p, q, r are rational expressions.

2. The number of radicals one over another occurring in any term of x is called the *order of the term;* the preceding expression contains terms of orders 0, 1, 2.

3. Let μ designate the *maximum order,* so that no term can have more than μ radicals one over another.