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Edited by A. Dold and B. Eckmann

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PREFACE

This volume of LECTURE NOTES IN MATHEMATICS contains the proceedings of a research Seminar-Workshop on recent progress in the Analytic Theory of Continued Fractions held at Loen, Norway from June 5 to June 30, 1981. In recent years there has been a renewed interest in the subject of continued fractions. This is due in part to the advent of computers and the resulting importance of the algorithmic character of continued fractions. It is also due to the close connection between continued fractions and Padé approximants and their application to theoretical physics. Primary emphasis at the Workshop was on the analytic aspects of the subject; however, considerable attention was also given to applied and computational problems. These interests are reflected in the Workshop proceedings.

The sessions at Loen were devoted not only to reports on recent work but also to the development of new results and the formulation of further problems. The authors whose papers appear in these proceedings either attended the Workshop or, if unable to attend, had their work presented and discussed at Loen.

The Seminar-Workshop was organized by Haakon Waadeland of the University of Trondheim and was made possible by grants from the Norwegian Research Council for Science and the Humanities (NAVF) and from the University of Trondheim. Support for travel expenses of the American participants came from the United States National Science Foundation, the University of Colorado at Boulder, Colorado State University, NAVF, and Fridtjof Nansen's and Affiliated Funds for the Advancement of Science and the Humanities. The latter also supported a visit to the University of Colorado for follow-up discussions of research topics. The University of Colorado provided a small grant for expenses related to publication of the proceedings. We gratefully acknowledge these contributions.

We also wish to thank the director and staff of the Alexandra Hotel in Loen for providing excellent working facilities and a cordial atmosphere for the Workshop. The professional assistance of the technical typists, Burt Rashbaum and Alexandra Hunt, at the Mathematics Department of the University of Colorado is greatly appreciated. Finally, we would like to thank Professor B. Eckmann, ETH Zürich, for accepting this volume for the Springer series of LECTURE NOTES IN MATHEMATICS.

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CONTENTS

BASIC DEFINITIONS AND NOTATION

To help unify the contributions to this volume we have asked all authors to use the same notation for certain basic concepts. These concepts and their definitions are listed here so as to avoid unnecessary duplication in the introduction to various articles. The complex plane shall be denoted by $\mathbb{C}$. For the extended complex plane, that is $\mathbb{C} \cup [\infty]$, we use the notation $\hat{\mathbb{C}}$.

The continued fraction algorithm is a function K that associates with ordered pairs of sequences $\langle\{a_n\},\{b_n\}\rangle$, with $a_n \in \mathbb{C}$, $a_n \neq 0$ for $n \geq 1$ and $b_n \in \mathbb{C}$ for $n \geq 0$, a third sequence $\{f_n\}$ with $f_n \in \hat{\mathbb{C}}$. A continued fraction is an ordered pair

$$\langle\langle\{a_n\}, \{b_n\}\rangle, \{f_n\}\rangle,$$

where the sequence $\{f_n\}$ is defined as follows. Let $\{s_n\}$ and $\{S_n\}$ be sequences of linear fractional transformations (l.f.t.) defined by

(DN1a) $$s_0(w) = b_0 + w, \quad s_n(w) = \frac{a_n}{b_n + w}, \quad n = 1, 2, 3, \ldots$$

and

(DN1b) $$S_0(w) = s_0(w), \quad S_n(w) = S_{n-1}(s_n(w)), \quad n = 1, 2, 3, \ldots .$$

Then

(DN1c) $$f_n = S_n(0), \quad n = 0, 1, 2, \ldots .$$

The numbers a_n and b_n are called the n th partial numerator and n th partial denominator, respectively, of the continued fraction $\langle\langle\{a_n\},\{b_n\}\rangle, \{f_n\}\rangle$; they are also called the elements. f_n is called the n th approximant of the continued fraction. If $f = \lim_{n\to\infty} f_n$ exists in $\hat{\mathbb{C}}$, we say that the continued fraction is convergent and that its value is f. If the limit does not exist we speak of a divergent continued fraction.

More generally we shall also be interested in sequences of l.f.t.'s $\{S_n^{(m)}\}$ defined as follows:

$$S_0^{(0)}(w) = w,$$

(DN2)

$$S_n^{(m)}(w) = s_{m+1} \circ s_{m+2} \circ \cdots \circ s_{m+n}(w), \quad n = 1, 2, 3, \ldots, \quad m = 0, 1, 2, \ldots .$$

Here the symbol $\circ$ denotes functional composition; that is, for functions g and h, $g \circ h(w) = g(h(w))$, provided that the domain of g contains the range of h. It can be seen that

$$S_n^{(m)}(w) = \cfrac{a_{m+1}}{b_{m+1} + \cfrac{a_{m+2}}{b_{m+2} + \cdots + \cfrac{a_{m+n}}{b_{m+n} + w}}} .$$

One also obtains the following

(DN3) $$S_n(w) = s_0 \circ S_n^{(0)}(w) = s_0 \circ s_1 \circ \cdots \circ s_n(w) , \quad n = 0, 1, 2, \ldots,$$

so that

(DN4) $$f_n = S_n(0) = s_0 \circ S_n^{(0)}(0) , \quad n = 0, 1, 2, \ldots .$$

Thus when $b_0 = 0$ (so that $s_0(w) = w$), we have

$$S_n(w) = S_n^{(0)}(w) \quad \text{and} \quad f_n = S_n(0) = S_n^{(0)}(0) .$$

Since $S_n(w)$ is the composition of non-singular l.f.t.'s, it is itself a non-singular l.f.t. It is well known that $S_n(w)$ can be written in the form

$$S_n(w) = \frac{A_n + w A_{n-1}}{B_n + w B_{n-1}} , \quad n = 0, 1, 2, \ldots$$

where the A_n and B_n (called the n th <u>numerator</u> and n th <u>denominator</u>, respectively, of the continued fraction) are defined by the second order linear <u>difference equations</u>

(DN6a) $$A_{-1} = 1 , \ A_0 = b_0 , \ B_{-1} = 0 , \ B_0 = 1 ,$$

$$A_n = b_n A_{n-1} + a_n A_{n-2} , \quad n = 1, 2, 3, \ldots,$$

(DN6b)

$$B_n = b_n B_{n-1} + a_n B_{n-2} , \quad n = 1, 2, 3, \ldots .$$

For simplicity a continued fraction $\langle\langle \{a_n\} , \{b_n\}\rangle , \{f_n\}\rangle$ will be denoted by the symbol

(DN7) $$b_0 + \overset{\infty}{\underset{n=1}{K}} \left(\frac{a_n}{b_n} \right)$$

or

(DN8) $$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots .$$

The symbols (DN7) and (DN8) may be used to denote both the continued fraction and its value when it is convergent.

The notation (DN8) shall also be used for finite combinations. Thus we write

$$S_n^{(m)}(w) = \frac{a_{m+1}}{b_{m+1} +} \frac{a_{m+2}}{b_{m+2} +} \cdots \frac{a_{m+n}}{+ b_{m+n} + w} , \quad m \geq 0 .$$

We shall call the continued fraction

$$\frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \frac{a_{m+3}}{b_{m+3}} + \cdots, \quad (m \geq 0)$$

the m th <u>tail</u> of (DN8) and we denote its n th approximant by $f_n^{(m)}$. Thus

$$f_n^{(m)} = S_n^{(m)}(0), \quad n = 1, 2, 3, \ldots, \quad m = 0, 1, 2, \ldots .$$

If $\lim_{n\to\infty} f_n^{(m)}$ exists in $\hat{\mathbb{C}}$, then the m th tail is convergent and its value is denoted by $f^{(m)}$. One thus has, for $m \geq 0$,

(DN9) $$f^{(m)} = \frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \frac{a_{m+3}}{b_{m+3}} + \cdots .$$

An important role is played in continued fraction theory by the expression h_n defined by

$$h_n = -S_n^{-1}(\infty) = B_n/B_{n-1} .$$

It is easily shown, from the difference equations (DN6), that

(DN10) $$h_1 = 1 \quad \text{and} \quad h_n = b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_2}{b_1}, \quad n = 2, 3, 4, \ldots .$$

In some cases the a_n, b_n are functions of a complex variable z. To emphasize this dependence on z we sometimes write

$$a_n(z),\ b_n(z),\ S_n^{(m)}(z,w),\ A_n(z),\ B_n(z),\ f_n^{(m)}(z),\ f^{(m)}(z),\ \text{etc.}$$

If in a continued fraction all $b_n = 1$ or all $a_n = 1$, the notation outlined above shall nevertheless be employed. If two or more continued fractions are considered at the same time

s_n^*, $S_n^{*(m)}$, $f^{*(m)}$, ... might be used for $K(a_n^*/b_n^*)$ and

s_n', $S_n^{(m)'}$, $f^{(m)'}$ ... for $K(a_n'/b_n')$.

SURVEY OF CONTINUED FRACTION METHODS OF SOLVING MOMENT PROBLEMS AND RELATED TOPICS

William B. Jones and W. J. Thron

1. Introduction. There is a constellation of interrelated topics:

Orthogonal polynomials;

Gauss quadrature;

Integral representation of continued fractions;

Determination of functions having given power series as asymptotic expansions;

Expansions of functions in series of orthogonal polynomials;

Solutions of certain three-term recurrence relations.

Today most of these topics are usually studied without reference to continued fraction theory. Nevertheless, all of them either arose or received important impetus from the theory of continued fractions. Thus Szegö [20, p. 54] asserts that: "historically the orthogonal polynomials $\{p_n(x)\}$ originated in the theory of continued fractions. This relationship is of great importance and is one of the possible starting points of the treatment of orthogonal polynomials."

Wynn in two statements [25, p. 190,191] attempts to delineate the role of continued fractions even more sharply. His first statement is: "many theories originating from the study of continued fractions have, upon reflection, been found to have little to do with them." He next asserts: "The theory of continued fractions has been preeminently an avenue to new and unexpected results...." The results to be described here provide a further confirmation of Wynn's thesis.

The interrelations in the constellation are intricate but, unfortunately, confusing. This is in part due to the fact that many mathematicians have made contributions using the tools, the language and the outlook of their trade, be it continued fractions, orthogonal polynomials, functional analysis or others. Not surprisingly they were, with few notable exceptions, somewhat narrow in their knowledge of the literature and quite ignorant of the earlier history. We too must plead guilty of such narrowness, although we are attempting to acquire a wider point of view and to learn more about the beginnings of the various topics in the constellation. However, at this writing our knowledge is still much more fragmentary than we would like it to be.

Arising from an investigation of correspondence and convergence properties of general T-fractions, we have been led to formulate strong Stieltjes and Hamburger moment problems (SSMP and SHMP, respectively). We showed that certain sequences of Laurent polynomials (L-polynomials), closely related to the denominators of the approximants of the positive T-fractions, are orthogonal with respect to a distribution function which can be derived from the positive T-fraction. In addition we were able to obtain new results for other topics in the constellation.

To facilitate the description of our results we first make a series of definitions.

A general T-fraction is a continued fraction of the form

$$\frac{F_1 z}{1+G_1 z}+\frac{F_2 z}{1+G_2 z}+\frac{F_3 z}{1+G_3 z}+\cdots, \tag{1.1a}$$

where $F_n \neq 0$ for all n. It can also be written in the following equivalent forms:

$$\frac{F_1}{z^{-1}+G_1}+\frac{F_2}{1+G_2 z}+\frac{F_3}{z^{-1}+G_3}+\frac{F_4}{1+G_4 z}+\cdots, \tag{1.1b}$$

$$\frac{z}{e_1+d_1 z}+\frac{z}{e_2+d_2 z}+\frac{z}{e_3+d_3 z}+\cdots, \tag{1.1c}$$

where $e_n \neq 0$ for all n. Here

$$e_1 = 1/F_1, \quad e_{2n-1} = \prod_{k=1}^{n-1} F_{2k} \Big/ \prod_{k=1}^{n} F_{2k-1}, \quad n = 2,3,4,\ldots, \tag{1.2a}$$

$$e_{2n} = \prod_{k=1}^{n} F_{2k-1} \Big/ \prod_{k=1}^{n} F_{2k}, \quad n = 1,2,3,\ldots, \tag{1.2b}$$

$$d_n = G_n e_n, \quad n = 1,2,3\ldots . \tag{1.2c}$$

If all $F_n > 0$ and $G_n > 0$, then (1.1a) (and all forms equivalent to it) is called a positive T-fraction. If $d_{2n-1} > 0$, $e_{2n} > 0$ and e_n, d_n are real for all $n \geq 1$, then (1.1c) (and all forms equivalent to it) is called a semi-positive T-fraction.

For a function $f(z)$ holomorphic at $z = 0$, let us denote by $\Lambda_0(f)$ its Taylor series expansion at 0. Let

$$\sum_{k=0}^{\infty} \alpha_k z^k$$

be a formal power series, and let $\{R_n(z)\}$ be a sequence of rational functions holomorphic at $z = 0$. Then we say that the sequence $\{R_n(z)\}$ corresponds to the series $\Sigma \alpha_k z^k$ at $z = 0$ if the formal power series $\Lambda_0(R_n) - \Sigma a_k z^k$ has the form

$$\Lambda_0(R_n) - \sum_{k=0}^{\infty} \alpha_k z^k = g_{m_n} z^{m_n} + g_{m_n+1} z^{m_n+1} + \cdots,$$

where $m_n \to \infty$ as $n \to \infty$. A continued fraction $K(a_n(z)/b_n(z))$ is said to correspond to a series if the sequence of approximants corresponds to the series.

Analogous definitions are used for correspondence at $z = a$ and, in particular, for $a = \infty$.

A general T-fraction (1.1a) (at least if all $G_n \neq 0$) corresponds to formal powers series L_0 at $z = 0$ and L_∞ at $z = \infty$. It is convenient to write these series as

$$(1.3) \qquad L_0 = \sum_{k=1}^{\infty} - c_{-k} z^k \quad \text{and} \quad L_\infty = \sum_{k=0}^{\infty} c_k z^{-k} .$$

By $\Phi^c(a,b)$ we shall mean the family of all real-valued, functions $\psi(t)$ defined on $a < t < b$, which are bounded, monotone non-decreasing with infinitely many points of increase on (a,b), and for which the integrals

$$(1.4) \qquad c_n = \int_a^b (-t)^n \, d\psi(t)$$

exist for all integers $n \geq 0$. (This additional condition is meaningful if $a = -\infty$ and/or $b = +\infty$, possibilities which we do admit.) The family of functions $\psi \in \Phi^c(a,b)$ for which the c_n in (1.4) also exist for all negative integers n we shall denote by $\Phi(a,b)$. The functions $\psi \in \Phi^c(a,b)$ or $\Phi(a,b)$ are called distribution functions and the c_n defined by (1.4) are called moments with respect to the distribution ψ.

The classical Stieltjes moment problem defined by Stieltjes in 1894 consists in finding conditions on the moments $\{c_n\}_0^\infty$ which would insure the existence of a function $\psi \in \Phi^c(0,\infty)$ for which (1.4) holds, for all $n = 0,1,2,\ldots$, with $a = 0$, $b = \infty$. Stieltjes found necessary and sufficient conditions for the existence of such a ψ. He also found necessary and sufficient conditions for the ψ to be unique. In 1920 Hamburger extended the problem to the interval $(-\infty,\infty)$. This is the classical Hamburger moment problem. In the solutions to the two problems Hankel determinants $H_0^{(m)} = 1$,

$$(1.5) \qquad H_k^{(m)} = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-1} \\ c_{m+1} & c_{m+1} & \cdots & c_{m+k} \\ \vdots & \vdots & & \vdots \\ c_{m+k-1} & c_{m+k} & & c_{m+2k-2} \end{vmatrix}, \quad k = 1,2,3,\ldots$$

as well as J-fractions

$$(1.6) \qquad \frac{k_1}{\ell_1 + z -} \, \frac{k_2}{\ell_2 + z -} \, \frac{k_3}{\ell_3 + z -} \cdots$$

are of importance.

We return now to the main theme of this article. General T-fractions (1.1a) with all $G_n \neq 0$ correspond to formal power series (1.3) at $z = 0$ and $z = \infty$. Thus associated with a general T-fraction (with $G_n \neq 0$) is a double sequence $\{c_n\}_{-\infty}^{\infty}$ of numbers and it makes sense to ask whether a function $\psi \in \Phi(0,\infty)$

exists for which the c_n with $n = 0, \pm 1, \pm 2, \ldots$ are the moments with respect to ψ. This is the strong Stieltjes moment problem (SSMP). It was posed and solved by Waadeland and the authors in [13] by means of positive T-fractions. They also showed that a positive T-fraction has an integral representation of the form

$$G(z) = \int_0^\infty \frac{z d\psi(t)}{z + t},$$

where $\psi \in \Phi(0,\infty)$, and that the function $G(z)$ has the power series (1.3) (to which the positive T-fraction corresponds) as asymptotic expansions at 0 and ∞, respectively. In a later paper [12] the present authors identified and studied the orthogonal functions associated with the SSMP. They turn out to be Laurent polynomials (L-polynomials)

(1.7) $$a_0 x^{-n} + \cdots + a_{2n} x^n .$$

Let the n th denominator of the positive T-fraction in the form (1.1b) be denoted by $V_n(z)$; then the orthogonal L-polynomials are given by $Q_0(z) = 1$,

(1.8) $$Q_{2m-1}(z) = (-1)^m V_{2m-1}(-z),\ Q_{2m}(z) = (-1)^m V_{2m}(-z),\quad n = 1,2,3,\ldots .$$

To pose the strong Hamburger moment problem (SHMP) is the natural next step. It is defined and solved, but without the use of continued fractions, by Njåstad and the authors in [9]. Only a partial solution of the SHMP can be obtained using continued fractions. This was also worked on jointly with Njåstad and is in the process of being written. We present here an outline of the solution in terms of semi-positive T-fractions, various ramifications such as integral representations, and the question when does correspondence imply asymptoticity. In addition, generalized approximants, which play a role in resolving the question of uniqueness of the solution to the SHMP, are also discussed. The generalized approximants arise from a "modification" (in the sense of [23] in these Proceedings) of semi-positive T-fractions.

An overview of the contents of this article is as follows. In Section 2 we give a summary of the historical background of the topics to be discussed here. In Section 3, general T-fractions, their correspondence to power series at $z = 0$ and $z = \infty$, as well as the partial fraction decomposition of semi-positive T-fractions are presented. In that section the generalized approximants are introduced and convergence of semi-positive T-fractions are taken up.

In Section 4 the results of the preceding section are used to obtain integral representations for all convergent subsequences of generalized approximants of semi-positive T-fractions.

Section 5 is devoted to solutions of the SSMP and SHMP. In Section 6 L-polynomials, orthogonality, Favard's theorem on recurrence relations, and related topics are considered. In particular, the identification of sequences of orthogonal L-polynomials with the denominators of positive T-fractions will be described.

Section 7 is concerned with Gaussian quadratures and convergence results that can be obtained from them. In Section 8 the discussion shifts back to semi-positive T-fractions. Here sufficient conditions for limit functions of convergent subsequences of generalized approximants to have the series L_0 and L_∞ as asymptotic expansions will be derived.

2. Summary of early history. Even though Legendre discovered the sequence of polynomials named after him in 1782, and was aware of the orthogonality property of the sequence with even subscripts as early as 1785, it was really Gauss who got the subject started.

In an article in 1812, Gauss studied hypergeometric functions and obtained, among other results, a continued fraction expansion for ratios of hypergeometric functions. In a second paper in 1814 he posed and solved a new quadrature problem (earlier work had been done by Cotes and Newton among others), namely, to find an approximation to an integral

$$\int_{-1}^{+1} f(t)dt$$

of the form

$$\sum_{k=1}^{n} \lambda_k^{(n)} f(\tau_k^{(n)}) , \tag{2.1}$$

where $\lambda_k^{(n)}$ and $\tau_k^{(n)}$ are to be determined in such a way that the approximation is exact for all polynomials $f(t)$ of degree not greater than $2n-1$. The proof makes use of the continued fraction expansion

$$\int_{-1}^{+1} \frac{dt}{z+t} = \log\left(\frac{z+1}{z-1}\right) = \frac{2}{z} - \frac{1/3}{z} - \frac{2^2/3\cdot5}{z} - \frac{3^2/5\cdot7}{z} - \cdots , \tag{2.2}$$

which was known to Gauss from his work in 1812. Let $K_n(z)/L_n(z)$ be the $2n$ th approximant of (2.2). Then the roots of $L_n(z)$ are all real and distinct and

$$\frac{K_n(z)}{L_n(z)} = \sum_{k=1}^{n} \frac{\lambda_k^{(n)}}{z+\tau_k^{(n)}} , \tag{2.3}$$

where the $\lambda_k^{(n)}$ and $\tau_k^{(n)}$ are exactly the constants needed in (2.1). Thus the Gaussian quadrature formula can be proved and the constants involved in it can be obtained from continued fraction considerations. Gauss actually computed some of the $\lambda_k^{(n)}$ and $\tau_k^{(n)}$.

Gauss considered this work important and expected that it would be used extensively in practical problems. As it turned out Gaussian quadrature and its generalizations were found to be of considerable theoretical interest throughout the 19th century. After the advent of computers it again attracted the attention of applied mathematicians.

Jacobi in a series of papers, proved the quadrature formula without using continued fractions. In the second of his papers he pointed out that the $L_n(z)$ are indeed the Legendre polynomials.

The people who extended the Gauss quadrature formula during the nineteenth century, among whom Christoffel, Heine, Tchebycheff and Stieltjes are probably the most notable (but Mehler, Radon, Markoff and Posse should also be mentioned), made essential use of continued fraction considerations.

The pattern developed as follows (using the notation introduced by Stieltjes only toward the end of the period). To obtain an approximation to

$$\int_a^b f(t)\, d\psi(t) ,$$

where $\psi(t) \in \Phi^c(a,b)$, one obtains a J-fraction (1.6) which converges to the integral

$$\int_a^b \frac{d\psi(t)}{z+t} ,$$

for $z \notin [-b, -a]$. Let $K_n(z)/L_n(z)$ be the n th approximant of this J-fraction. The constants $\lambda_k^{(n)}$, $\tau_k^{(n)}$ in the quadrature formula

$$\int_a^b f(t)\, d\psi(t) \approx \sum_{k=1}^{n} \lambda_k^{(n)} f(\tau_k^{(n)})$$

are determined by the partial fraction decomposition of $K_n(z)/L_n(z)$ as in (2.3). In the general case, the $\lambda_k^{(n)}$, $\tau_k^{(n)}$ depend on ψ , but are independent of f . The approximation is exact if $f(z)$ is a polynomial of degree at most $2n-1$.

Using the correspondence between the integral and the J-fraction, one then can prove the following:

$$\text{(2.4)} \qquad \int_a^b t^k L_m(-t)\, d\psi(t) = 0 , \quad k < m , \quad m = 1,2,3,\ldots,$$

$$\text{(2.5)} \qquad \int_a^b L_n(-t)L_m(-t)\, d\psi(t) = \prod_{\nu=1}^{n} k_\nu \, \delta_{n,m} ,$$

where $\delta_{n,m}$ is the Kronecker δ . One also has

$$\text{(2.6)} \qquad K_n(z) = \int_a^b \frac{L_n(z)-L_n(-t)}{z+t} d\psi(t) .$$

If one sets $P_n(z) = (-1)^n L_n(-z)$, it follows that $\{P_n(z)\}$ is a sequence of orthogonal polynomials with respect to $\psi(t)$, normalized so that the coefficient of z^n in $P_n(z)$ is 1 . It is for this reason that (to paraphrase Gautschi [5, p. 82]) throughout the 19th century orthogonal polynomials were generally viewed as the denominators $L_n(z)$ of the nth approximant of a J-fraction. It is now easily seen that

$$\text{(2.7)} \qquad P_n(z) = (z-\ell_n)P_{n-1}(z) - k_n P_{n-2}(z) , \quad n = 2,3,\ldots ,$$

where $k_n > 0$ and $\ell_n \in \mathbb{R}$. The result, that a sequence $\{P_n(z)\}$ satisfying (2.7) is the sequence of orthogonal functions with respect to some ψ , is usually attributed to Favard who stated it in 1935. There are other claimants to having been the first to have obtained this result. However, as Chihara [2, p. 209] puts it very well: "This multiple discovery is not surprising since the theorem is really implicitly contained in the theory of continued fractions. It seems quite likely that mathematicians who worked with continued fractions were well aware of the theorem, but never bothered to formulate it explicitly. Nevertheless, the explicit formulation was a real contribution since most workers in orthogonal polynomials tend to avoid continued fractions whenever possible."

A very detailed survey of Gauss-Christoffel quadrature formulae was recently given by Gautschi [5] .

We now turn to other major topics in our constellation. One of these is the expansion of functions (both real-valued as well as analytic) in series of orthogonal polynomials (or as Blumenthal in 1898 still said, "continued fraction denominators").

Expansions of this kind are important for many reasons. One of these where continued fractions enter "naturally" is an interpolation problem of Tchebycheff of 1858 which involves determination of the finite J-fraction equal to the sum

$$\sum_{\nu=1}^{n} \frac{\lambda_\nu}{z+\tau_\nu} .$$

The interpolation problem can then be answered in terms of a finite sum $\sum c_\nu L_\nu(z)$, where the $L_\nu(z)$ are the "continued fraction denominators". Tchebycheff also considered the limiting situation where one determines the infinite J-fraction which is equal to an integral of the form

$$\int_a^b \frac{\phi(t)dt}{z+t} ,$$

at least in certain special cases. Other mathematicians who worked on this problem during the 19th century were Heine, Pincherle, Darboux and Blumenthal.

R. Murphy in 1833-35 was probably the first to study a moment problem. He referred to it as "the inverse method of definite integrals". In this context he encountered polynomials satisfying the orthogonality condition. Recognizing the importance of the condition, he used the term "reciprocal functions" for what we call today orthogonal functions. He was not as we mistakenly stated in [11, p. 6] the originator of the name "orthogonal". According to Gautschi [5, p. 78] , "The name 'orthogonal' for function systems came into use only later, probably first in E. Schmidt's 1905 Göttingen dissertation;..." Murphy was interested, among others, in the following problem. If

$$\int_0^1 t^k f(t)dt = 0 \quad , \quad k = 0,1,2,\ldots,n-1 \quad ,$$

what can be said about $f(t)$? Clearly this is the question, to what extent a finite set of moments determines the function $f(t)$. Tchebycheff starting in 1855 took an interest in moment problems. (Some of our information is taken from the brief historical review in the book, The Problem of Moments by Shohat and Tamarkin.) Among others, he was interested in the question whether from

$$\int_{-\infty}^{\infty} x^n p(x)dx = \int_{-\infty}^{\infty} x^n e^{-x^2} dx \quad , \quad n = 0,1,2,\ldots \quad ,$$

one could conclude that $p(x) = e^{-x^2}$. According to Shohat and Tamarkin, "Tchebycheff's main tool is the theory of continued fractions which he uses with extreme ingenuity." He also obtained the approximation for

$$\int_a^x f(t)dt \quad , \quad a < x < b$$

given the moments

$$\int_a^b t^n f(t)dt \quad .$$

Stieltjes in 1894 was able to pull together the work on moments of his predecessors in a very satisfying manner at least as far as the interval $[0,\infty)$ was concerned. He used and refined the tools which had been introduced by the mathematicians we have mentioned here. Stieltjes found necessary and sufficient conditions for the existence of a solution $\psi \in \Phi^c(0,\infty)$. He also described a way to obtain the solution by first obtaining a continued fraction expansion of

$$G(z) = \int_0^\infty \frac{d\psi(t)}{z+t} \quad ,$$

valid for all z not on the negative real axis. By an inversion process he then arrived at $\psi(t)$. It was not until 1920 that the general moment problem was solved by Hamburger using J-fractions. Further work on the moment problem by M. Riesz, R. Nevanlinna, Carleman, and Hausdorff in the early nineteen twenties does not make use of continued fractions.

Another motivation of Stieltjes' work was the "summing" of the divergent series to which the continued fraction corresponds. The divergent series then becomes an asymptotic expansion of the function (represented by the integral) to which the continued fraction converges. Stieltjes had written his thesis in 1886 on asymptotic series (he called them semi-convergent) the same year in which Poincaré wrote a fundamental paper on the subject.

The history of integral representation of continued fractions is sketched in Section 4 .

3. Semi-positive T-fractions. In [13] we gave references for the history of general T-fractions. In that article we also proved the general theorem for correspondence, which, in terms of L_0 and L_∞ of (1.3) , and without loss of generality, can be stated as follows.

Theorem 3.1. *Let*

$$L_0 = \sum_{m=1}^{\infty} - c_{-m} z^m \quad \text{and} \quad L_\infty = \sum_{m=0}^{\infty} c_m z^{-m}$$

be given. Then there exists a general T-fraction