LECTURES ON DIFFERENTIAL AND INTEGRAL EQUATIONS

KÔSAKU YOSIDA
UNIVERSITY OF TOKYO
TOKYO, JAPAN

1960

INTERSCIENCE PUBLISHERS, INC., NEW YORK
INTERSCIENCE PUBLISHERS LTD., LONDON

LIBRARY OF CONGRESS CATALOG CARD NUMBER
60-53007

INTERSCIENCE PUBLISHERS, INC.
250 Fifth Avenue, New York, 1, N.Y.

For Great Britain and Northern Ireland:
INTERSCIENCE PUBLISHERS LTD.
88/90, Chancery Lane, London W.C. 2

Lithoprinted in U.S.A.
EDWARDS BROTHERS, INC.
Ann Arbor, Michigan

PURE AND APPLIED MATHEMATICS

A Series of Texts and Monographs

Edited by: R. COURANT • L. BERS • J. J. STOKER

Vol. I: **Supersonic Flow and Shock Waves**
By R. Courant and K. O. Friedrichs

Vol. II: **Nonlinear Vibrations in Mechanical and Electrical Systems**
By J. J. Stoker

Vol. III: **Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces**
By R. Courant

Vol. IV: **Water Waves**
By J. J. Stoker

Vol. V: **Integral Equations**
By F. G. Tricomi

Vol. VI: **Differential Equations: Geometric Theory**
By Solomon Lefschetz

Vol. VII: **Linear Operators**
By Nelson Dunford and Jacob T. Schwartz

Vol. VIII: **Modern Geometrical Optics**
By Max Herzberger

Vol. IX: **Orthogonal Functions**
By G. Sansone

Vol. X: **Lectures on Differential and Integral Equations**
By Kôsaku Yosida

Additional volumes in preparation

PURE AND APPLIED MATHEMATICS

A Series of Texts and Monographs

VOLUME X

Foreword

The present book is the English edition of my book published originally, in Japanese, by the Iwanami Shoten in the series "Iwanami Zensho." It was intended to be a self-contained exposition of the theory of ordinary differential equations and integral equations. It especially gives a fairly detailed treatment of the boundary value problem of second order linear ordinary differential equations, and includes an elementary exposition of the theory of Weyl-Stone's eigenfunction expansions in the form completed by Titchmarsh-Kodaira's formula concerning the density matrix of the expansion.

The author wishes to express his sincere thanks to Professor Shigeharu Harada of Chiba Institute of Technology for his time and effort in preparing the English translation of this book, and to Professor Lipman Bers of New York University for his kind suggestion to include this book in his series.

The author also extends his cordial thanks to the Iwanami Shoten and the Interscience Publishers, Inc., who kindly agreed to the publication of the English edition, and to the International Academic Printing Company for the painstaking work of printing the book.

August, 1960 Kôsaku Yosida

Contents

CHAPTER 1

THE INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

An nth *order ordinary differential equation* is a functional relation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

between a variable x, an unknown function y, and its derivatives

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}$$

A function $y(x)$ which satisfies this relation is called a solution of the differential equation. For example, for any constant C, the function

$$y = \sin (x - C)$$

is a solution of the first order equation

$$\left(\frac{dy}{dx}\right)^2 + y^2 - 1 = 0$$

In general, the *general solution* of an nth order equation contains n arbitrary constants $C_1, C_2, \cdots, C_n$. These constants may be determined by so-called *initial conditions* at a point $x = x_0$, which are the prescribed values of $y, dy/dx, \cdots, d^{n-1}y/dx^{n-1}$ at the point $x = x_0$. The solution thus obtained is called a *particular solution.*

In this chapter, we shall show that the *initial value problem* for differential equations can be reduced to a system of integral equations and solved by the method of successive approximations. This chapter is intended as a prerequisite for the theory of the differential equations as considered in the following chapters.

§1. Successive approximations

1. *Existence and uniqueness of the solution of the ordinary differential equation of the first order*

An ordinary differential equation of the first order is generally written in the form

$$F(x, y, dy/dx) = 0 \tag{1.1}$$

In the following, we shall restrict ourselves to those cases in which (1.1) can be solved for dy/dx and written in the form

$$dy/dx = f(x, y) \tag{1.2}$$

The simplest case of (1.2) is of the form

$$dy/dx = f(x) \tag{1.3}$$

The solution of (1.3) is given by

$$y(x) = \int_{x_0}^{x} f(t)\,dt + C \tag{1.4}$$

in the region where $f(x)$ is continuous. The integration constant C is determined by the value of $y(x)$ at $x = x_0$, that is,

$$y_0 = y(x_0) = C \tag{1.5}$$

Accordingly, the solution of (1.3) satisfying the condition $y = y_0$ at $x = x_0$ is given by

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt \tag{1.6}$$

The condition,

$$y = y_0 \quad \text{at} \quad x = x_0 \tag{1.5'}$$

is called the *initial condition* for the solution of the differential equation (1.3).

Our purpose is to find a solution of the general equation (1.2) subject to the initial condition (1.5′). To formulate the problem precisely, we make the following assumptions concerning $f(x, y)$.

Assumption 1. The function $f(x, y)$ is real-valued and continuous on a *domain*[1] D of the (x, y)-plane given by

$$x_0 - a \leqq x \leqq x_0 + a\,, \qquad y_0 - b \leqq y \leqq y_0 + b \tag{1.7}$$

where a, b are positive numbers.

Assumption 2. $f(x, y)$ satisfies the *Lipschitz condition* with respect to y in D, that is, there exists a positive constant K such that

$$|f(x, y_1) - f(x, y_2)| \leqq K|y_1 - y_2| \tag{1.8}$$

for every pair of points (x, y_1), (x, y_2) of D.

[1] By a domain, we mean a nonempty open connected set, and by a closed domain, the closure of a domain. However, we shall use the word "domain" to denote "closed domain," when the domain is explicitly defined as (1.7).

REMARK. The second assumption would seem to be less natural than the first, but its importance will become clearer in the following proof.

It is worth while to show that if $f(x, y)$ has a continuous partial derivative $\partial f(x, y)/\partial y$ on D, then $f(x, y)$ satisfies the second assumption. This is seen as follows. Since $|\partial f(x, y)/\partial y|$ is continuous on the bounded closed domain D of the (x, y)-plane, $|\partial f(x, y)/\partial y|$ is bounded on D. Put

$$K = \sup_{(x,y)\in D} \left| \frac{\partial f(x, y)}{\partial y} \right| \tag{1.9}$$

Then the mean-value theorem implies that (1.8) holds for $f(x, y)$.

By Assumption 1, $|f(x, y)|$ is continuous on the bounded closed domain D, therefore $|f(x, y)|$ is bounded on D, that is,

$$\sup_{(x,y)\in D} |f(x, y)| = M < \infty \tag{1.10}$$

Set

$$h = \min(a, b/M) \tag{1.11}$$

Let us define a sequence of functions $\{y_n(x)\}$ for $|x - x_0| \leqq h$, successively, by

$$\begin{aligned} y_0(x) &= y_0 \\ y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt \end{aligned} \tag{1.12}$$

The theorem to be proved is that $\{y_n(x)\}$ converges uniformly on the interval $|x - x_0| \leqq h$, and the limit $y(x)$ of the sequence is a solution of (1.2) which satisfies (1.5′).

Proof. According to (1.10) and (1.11), we obtain

$$|y_1(x) - y_0| \leqq hM \leqq b$$

for $|x - x_0| \leqq h$. Therefore $\int_{x_0}^{x} f(t, y_1(t))\,dt$ can be defined for $|x - x_0| \leqq h$, and

$$|y_2(x) - y_0| \leqq hM \leqq b$$

In the same manner, we can define $y_3(x), \cdots, y_n(x)$ for $|x - x_0| \leqq h$, and obtain

$$|y_k(x) - y_0| \leqq hM \leqq b\,, \qquad k = 1, 2, \cdots, n$$

Using Assumption 2, we have

$$|y_{k+1}(x) - y_k(x)| \leqq K\left|\int_{x_0}^{x} |y_k(t) - y_{k-1}(t)|\,dt\right|$$

for $|x - x_0| \leqq h$. Therefore, if we assume that, for $k = 1, 2, \cdots, n$,

$$(1.13) \qquad |y_k(x) - y_{k-1}(x)| \leqq \frac{b(K|x - x_0|)^{k-1}}{(k-1)!} \qquad \text{for } |x - x_0| \leqq h$$

we obtain, for $k = n + 1$,

$$(1.14) \qquad |y_{n+1}(x) - y_n(x)| \leqq \frac{b(K|x - x_0|)^n}{n!} \qquad \text{for } |x - x_0| \leqq h$$

Since (1.13) holds for $k = 1$ as mentioned above, we see, by mathematical induction, that (1.14) holds for every n. Thus, for $m > n$ we obtain

$$(1.15) \qquad |y_m(x) - y_n(x)| \leqq \sum_{k=n}^{m-1} |y_{k+1}(x) - y_k(x)| \leqq b\sum_{k=n}^{m-1} \frac{(Kh)^k}{k!}$$

Since the right side of (1.15) tends to zero as $n \to \infty$, $\{y_n(x)\}$ converges uniformly to a function $y(x)$ on the interval $|x - x_0| \leqq h$. Since the convergence is uniform, $y(x)$ is continuous and moreover, evidently, $y(x_0) = y_0$. To prove that $y(x)$ is a solution, we use the following known fact: If the sequence of functions $\{y_n(x)\}$ converges uniformly and $y_n(x)$ is continuous on the interval $|x - x_0| \leqq h$, then

$$\lim_{n\to\infty} \int_{x_0}^{x} y_n(x)\,dx = \int_{x_0}^{x} \{\lim_{n\to\infty} y_n(x)\}\,dx$$

Hence we obtain

$$\begin{aligned} y(x) &= \lim_{n\to\infty} y_{n+1}(x) \\ &= y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(t, y_n(t))\,dt \\ &= y_0 + \int_{x_0}^{x} \{\lim_{n\to\infty} f(t, y_n(t))\}\,dt \\ &= y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \end{aligned}$$

that is,

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt\,, \qquad |x - x_0| \leqq h \tag{1.16}$$

The integrand $f(t, y(t))$ on the right side of (1.16) is a continuous function, hence $y(x)$ is differentiable with respect to x, and its derivative is equal to $f(x, y(x))$, q.e.d.

The method of the above proof is called the *method of successive approximations*, or Picard's method.

Integrating from x_0 to x, we see that a solution $y(x)$ of (1.2) satisfying the initial condition (1.5′) must satisfy the *integral equation* (1.16). The above proof shows that this integral equation can be solved by the method of successive approximations.

Uniqueness of the solution. We have obtained, by the method of successive approximations, a solution $y(x)$ of (1.2) satisfying the initial condition (1.5′). However, there remains another important problem, the problem of uniqueness: Is there any other solution satisfying the same initial condition? If the solution is not unique and there are solutions other than the $y(x)$ obtained above, we must find other methods to obtain them. Fortunately, under our two assumptions, we can prove the uniqueness of the solution. To see this let $z(x)$ be another solution of (1.2) such that $z(x_0) = y_0$. Then

$$z(x) = y_0 + \int_{x_0}^{x} f(t, z(t))\, dt$$

By Assumption 2, we obtain

$$|y(x) - z(x)| \leqq K \left| \int_{x_0}^{x} |y(t) - z(t)|\, dt \right| \tag{1.17}$$

Therefore we also obtain for $|x - x_0| \leqq h$

$$|y(x) - z(x)| \leqq KN|x - x_0|$$

where

$$N = \sup_{|x-x_0|\leqq h} |y(x) - z(x)|$$

Substituting the above estimate for $|y(t) - z(t)|$ on the right side of (1.17), we obtain further

$$|y(x) - z(x)| \leqq N(K|x - x_0|)^2/2!$$

for $|x - x_0| \leqq h$. Substituting this estimate for $|y(t) - z(t)|$ once more on the right side of (1.17), we have

$$|y(x) - z(x)| \leqq N(K|x - x_0|)^3/3!$$

for $|x - x_0| \leqq h$. Repeating this substitution, we obtain

$$|y(x) - z(x)| \leqq N(K|x - x_0|)^m/m! \qquad m = 1, 2, \cdots$$

for $|x - x_0| \leqq h$. The right side of the above inequality tends to zero as $m \to \infty$. This means that

$$N = \sup_{|x-x_0|\leqq h} |y(x) - z(x)|$$

is equal to zero, q.e.d.

EXAMPLE. To illustrate the general procedure, we shall solve the differential equation

$$dy/dx = y$$

under the initial condition $y(0) = 1$ by the method of successive approximations. We have from (1.12)

$$y_1(x) = 1 + \int_0^x dt = 1 + x$$

$$y_2(x) = 1 + \int_0^x (1 + t)\, dt = 1 + x + \frac{x^2}{2!}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$y_n(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{n-1}}{(n-1)!}\right) dt$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

In this way we obtain the well-known formula

$$y(x) = \lim_{n\to\infty} y_n(x) = \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. *Remark on approximate solutions*

Letting $m \to \infty$ in (1.15), we obtain

$$|y(x) - y_n(x)| \leqq b \sum_{k=n}^{\infty} \frac{(Kh)^k}{k!} \tag{2.1}$$

for $|x - x_0| \leqq h$. The equation (2.1) is an estimate of the error of the nth approximate solution $y_n(x)$. In view of (2.1), the method of successive approximations may be used, in principle, to obtain an approximate solution to any degree of accuracy. This method, however, is not always practical because it requires one to repeat the evaluation of indefinite integrals many times.

We shall now consider another method which is sometimes rather useful. Suppose that $g(x, y)$ is a suitable approximation to $f(x, y)$

such that we can find the solution $z(x)$ of the differential equation

$$dz/dx = g(x, y) \tag{2.2}$$

on the interval $|x - x_0| \leqq h$ satisfying the initial condition $z(x_0) = y_0$. We put

$$\sup_{(x,y)\in D} |f(x, y) - g(x, y)| \leqq \varepsilon \tag{2.3}$$

Let $y(x)$ be the unique solution of the differential equation

$$dy/dx = f(x, y) \tag{2.4}$$

on the interval $|x - x_0| \leqq h$ satisfying the initial condition $y(x_0) = y_0$. Then from (2.2) it follows that

$$y(x) - z(x) = \int_{x_0}^{x} \{f(t, y(t)) - g(t, z(t))\}\, dt$$

We obtain by Assumption 2

$$\begin{aligned} |y(x) - z(x)| &= \left| \int_{x_0}^{x} \{f(t, z(t)) - g(t, z(t))\}\, dt \right. \\ &\qquad \left. + \int_{x_0}^{x} \{f(t, y(t)) - f(t, z(t))\}\, dt \right| \\ &\leqq \left| \int_{x_0}^{x} |f(t, z(t)) - g(t, z(t))|\, dt \right| \\ &\qquad + K \left| \int_{x_0}^{x} |y(t) - z(t)|\, dt \right| \\ &\leqq \varepsilon |x - x_0| + K \left| \int_{x_0}^{x} |y(t) - z(t)|\, dt \right| \end{aligned} \tag{2.5}$$

Therefore, setting

$$\sup_{|x - x_0| \leqq h} |y(x) - z(x)| = M'$$

we have

$$|y(x) - z(x)| \leqq \varepsilon |x - x_0| + KM' |x - x_0|$$

for $|x - x_0| \leqq h$. Substituting this estimate for $|y(t) - z(t)|$ on the right side of (2.5), we obtain

$$|y(x) - z(x)| \leqq M' \frac{K^2 |x - x_0|^2}{2!} + \varepsilon \sum_{m=1}^{2} \frac{K^{m-1} |x - x_0|^m}{m!}$$

for $|x - x_0| \leqq h$. Repeating this substitution, we obtain, for each $n = 1, 2, 3, \cdots,$

$$|y(x) - z(x)| \leqq M' \frac{K^n |x - x_0|^n}{n!} + \varepsilon \sum_{m=1}^{n} \frac{K^{m-1} |x - x_0|^m}{m!}$$

for $|x - x_0| \leqq h$. As $n \to \infty$, the first term on the right side converges to zero uniformly on the interval $|x - x_0| \leqq h$. The second term is less than

$$\varepsilon K^{-1}\{\exp(K|x - x_0|) - 1\}$$

Accordingly, the estimate of the error of the approximate solution $z(x)$ in the interval $|x - x_0| \leqq h$ is given by

$$|y(x) - z(x)| \leqq (\varepsilon/K)(\exp(K|x - x_0|) - 1) \tag{2.6}$$

EXAMPLE. Consider the initial value problem

$$dy/dx = \sin(xy)$$

with the initial condition $y(0) = 0.1$. We shall calculate the estimate of the error of the approximate solution $z(x) = 0.1 \exp(\frac{1}{2}x^2)$ which is the solution of the equation

$$dy/dx = xy$$

satisfying the initial condition $z(0) = 0.1$. We take the domain

$$|x| \leqq \tfrac{1}{2}, \qquad |y - 0.1| \leqq \tfrac{1}{2}$$

as D, so that $a = b = \frac{1}{2}$. Then we have $M = \sup_{(x,y)\in D} |\sin(xy)| < 1$. Moreover, by the mean-value theorem, we have

$$|\sin(xy_1) - \sin(xy_2)| \leqq |xy_1 - xy_2|$$

Hence we may set the Lipschitz constant $K = \frac{1}{2}$. Since $a = b = \frac{1}{2}$, and $M < 1$, we have $h = \min(a, b/M) = \frac{1}{2}$. Moreover $|z(x)| \leqq \frac{1}{2}$ for $|x| \leqq h = \frac{1}{2}$. Hence by the Taylor expansion theorem we obtain

$$|f(x, z(x)) - g(x, z(x))| \leqq |\sin(x\,z(x)) - x\,z(x)| \leqq \frac{|x\,z(x)|^3}{6} \leqq \frac{1}{384}$$

for $|x| \leqq \frac{1}{2}$. Setting $\varepsilon = \frac{1}{384}$ and $K = \frac{1}{2}$ in (2.6), we obtain

$$|y(x) - (0.1)\exp(\tfrac{1}{2}x^2)| \leqq \frac{2}{384}\left\{\exp\left(\frac{|x|}{2}\right) - 1\right\} \leqq \frac{0.6}{192}|x|$$

as the estimate of the error of the approximate solution $z(x)$.

3. *Integration constants*

As was shown in Part 1, a particular initial condition (1.5′) deter-