Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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# Bifurcation and Nonlinear Eigenvalue Problems

Proceedings, Villetaneuse, France 1978

Edited by
C. Bardos, J. M. Lasry, and M. Schatzman

Springer-Verlag
Berlin Heidelberg New York

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AMS Subject Classifications (1980): 35 G 20

ISBN 3-540-09758-9 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09758-9 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Main entry under title: Bifurcation and nonlinear eigenvalue problems. (Lecture notes in mathematics; 782) Bibliography: p. Includes index. 1. Differential equations, Partial--Congresses. 2. Differential equations, Nonlinear--Congresses. 3. Eigenvalues--Congresses. 4. Bifurcation theory--Congresses. I. Bardos, Claude, 1940- II. Lasry, J. M. III. Schatzman, M. IV. Series: Lecture notes in mathematics (Berlin; 782. QA3.L28 no. 782 [QA374] 510s [515.3'53] 80-11358
ISBN 0-387-09758-9



Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

# PREFACE

This volume contains the notes of a session⁺ organized on October 2,3 and 4,1978 at the Departement of Mathematics of the University Paris 13,Centre Scientifique et Polytechnique,Villetaneuse.

The aim of this session was to gather mathematicians and scientists of other fields:chemistry,biology,physics and astrophysics,and to let them exchange information and methods.

The common points to all the lectures are partial differential equations,non linear phenomena,study of the dependence with respect to a parameter,and the methods used are very diverse.

The lectures can be classified into three groups according to their relation to applied science : papers belonging to the first cluster deal with a phenomenological approach;in this case,a complete system of equations describing the experimental phenomenon is either too complicated or not entirely known and understood;therefore,a simpler system is studied which mimicks the behavior of the complete system, and one expects qualitative results.Here belong the talks of J.Heyvaerts,J.M.Lasry, M.Schatzman & P.Witomski,of G.Iooss,of J.P.Kernevez,G.Joly,D.Thomas & B.Bunow,and of P.Ortoleva.

The second group is made of mathematical and numerical studies of more complete modelizations:here,the model is better understood,and the study is more precise,so that it may give quantitative results;this group contains the contributions of C.M. Brauner & B.Nicolaenko,of C.Guillopé,of G.Iooss & R.Lozi,and of J.Mossino.

Though the papers of the third group are not directly concerned with natural phenomena,they develop theoretical tools and an understanding of non-linear phenomena,which are intended to meet the needs and preoccupations of the applied scientists. We include here the papers of H.Berestycki & P.L.Lions,of C.Bolley,M.Barnsley, F.Mignot,F.Murat & J.P.Puel,of J.C.Saut,and of D.Serre.

Striking observational data were brought by M.Dupeyrat who showed a beautiful dynamic periodic chemical phenomenon with a movie film.

We added a paper of J.P.Chollet and M.Lesieur.They show how the nonlinearity of the Navier-Stokes equation can create turbulence and give a phenomenological model that fits with the Kolmogorov law.

We thank all the participants for their active presence and interesting contributions

C.Bardos,J.M.Lasry,M.Schatzman.

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⁺Supported by the University Paris 13,and the Centre National de la Recherche Scientifique,A.T.P. Mathématiques pour les Sciences de l'Ingénieur.

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CONTENTS



The contributions of J.Mossino,A non Linear Eigenvalue Problem and a Variational Problem connected with Increasing Rearrangement in Plasma Physics,and of G.Iooss and R.Lozi,Convection between two Rotating Plates and Dynamo Effect occuring from a Secondary Bifurcation will appear elsewhere.

# PARAMETER DEPENDENCE OF SOLUTIONS OF CLASSES OF QUASI-LINEAR ELLIPTIC AND PARABOLIC DIFFERENTIAL EQUATIONS

by

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ABSTRACT

Earlier work, on the dependence of solutions of certain classes of quasi-linear elliptic and parabolic differential equations on embedded parameters, is extended and generalized. In particular, generic classes of linearly perturbed, and inhomogeneously perturbed, quasi-linear elliptic and parabolic boundary values problems whose stable positive solutions are Laplace transforms of positive measures, are identified. For a particular class of such problems the conjecture that the solution is a Stieltjes transform of a positive measure is explored. It is shown that low order rational fraction Padé approximants provide useful bounds, independently of whether or not the conjecture itself is true.

## 1. INTRODUCTION

We consider some extensions and generalizations of earlier work[1,2] concerning the dependence of solutions of certain quasi-linear elliptic and parabolic differential equations on an external parameter. Our interest is in those cases where the solution, as a function of the external parameters, can be expressed as a transform of a positive measure. In such cases one can use moment theory to yield convergent sequences of upper and lower bounds on the solution throughout the range of the parameter, as described in[2]. To construct the bounds one needs to know either an initial sequence of terms in a perturbation expansion of the solution in the parameter (in some cases these can be obtained by solving a set of linear equations), or a set of experimental points corresponding to different values of the parameter. The latter possibility is attractive because the resulting bounds are to some extent "model-independent" as described in[1].

In II and III we describe two generic situations which, in the elliptic case, have positive stable solutions which are Laplace transforms of positive measures. As such, they are amenable to analysis using generalized Padé approximants.

In IV we consider a class of nonlinearly perturbed elliptic boundary value problems for which it is conjectured that the positive stable solution is a Stieltjes transform of a positive measure in the perturbation parameter. In certain cases this conjecture has been established, and then rational fraction Padé approximants provide not only convergent sequences of bounds on the solution but also they yield bounds on the associated turning point. It is shown that, for low order Padé approximants, similar results pertain in general, whether on not the conjecture itself is true.

## II. LINEARLY PERTURBED NONLINEAR EQUATIONS

We consider quasi-linear differential equations of the form

$$\left.\begin{aligned} &L\phi + F(\phi) + \lambda p\phi = f \quad \text{in} \quad D, \\ &B\phi = 0 \quad \text{on } \partial D \end{aligned}\right\} \tag{2.1}$$

Here D denotes a bounded domain of real N-space $\mathbb{R}^N$ with boundary $\partial D$ and closure $\bar{D}$. We assume that $\partial D$ belongs to the class $C^{2+\alpha}$, where $\alpha \in (0,1)$ is fixed. L is the uniformly elliptic differential operator

$$L\phi = -\sum_{i,j=1}^{N} a_{i,j}(x)\,\frac{\partial^2\phi(x)}{\partial x_i\,\partial x_j} + \sum_{k=1}^{N} a_k(x)\,\frac{\partial\phi(x)}{\partial x_k} + a(x)\phi(x) \tag{2.2}$$

with real coefficients $a_{i,j} \in C^{2+\alpha}(\bar{D})$, $a_k \in C^{1+\alpha}(D)$, and $a \in C^{\alpha}(\bar{D})$ where we assume that, for all $x = (x_1, x_2, \ldots, x_N) \in \bar{D}$

$$a(x) \geq 0 \qquad (2.3)$$

The matrix $(a_{ij})$ is supposed to be uniformly positive definite over $\bar{D}$. B is either of the boundary operators

$$B\phi \equiv \phi(x) \quad \text{on } \partial D, \qquad (2.4)$$
$$B\phi \equiv \beta(x)\phi(x) + \partial\phi(x)/\partial\nu \quad \text{on } \partial D, \qquad (2.5)$$

where $\beta(x) \in C^{1+\alpha}(\partial D)$ and satisfies

$$\beta(x) \geq 0 \quad \text{for all } x \in \partial D. \qquad (2.6)$$

$\partial/\partial\nu$ denotes the outward conormal derivative. In the case where (2.5) applies we assume that $a(x)$ and $\beta(x)$ do not both vanish identically. The functions p and f in (2.1) belong to $C^{\alpha}(\bar{D})$ and satisfy

$$p(x) > 0, \quad \text{and } f(x) \geq 0 \quad \text{for all } x \in \bar{D}, \qquad (2.7)$$

with $f \neq 0$.

The real valued function $F(\phi) = F(x,\phi)$ may depend explicitly on both $x \in \bar{D}$ and $\phi$. We suppose that it has the following properties

(i) $F(0) - f \leq 0$ for all $x \in \bar{D}$.

(ii) There exists a constant $C > 0$ such that

$F(C) - f \geq 0$ for all $x \in \bar{D}$.

(iii) $F(\phi)$ is $C^{\infty}$ in $\phi$ for all $\phi \in [0,C]$, each of its derivatives in this range belonging to $C^{\alpha}(\bar{D})$ in x, and such that uniformly in $x \in \bar{D}$ and $n = 2,3,4,\ldots$

$$-\hat{F} \leq F^{(n)}(\phi) \leq 0 \quad \text{for all } \phi \in [0,C]$$

for some constant $\hat{F} > 0$.

(iv) $F^{(1)}(\phi) \geq 0$ for all $x \in \bar{D}$ and $\phi \in [0,C]$. This means that the linear operator in $\mathscr{L}^2(\bar{D})$ corresponding to $L + F^{(1)}(\phi)$ together with the boundary condition in (2.1) has strictly positive least eigenvalue for all smooth $\phi(x) \in [0,C]$.

The conditions above on (L,B) are such that are such that the *Maximum Principle* [3] and the *Positivity Lemma* [4] apply. Moreover, the smoothness conditions of $F(\phi)$ mean that *Amann's Theorem* [5], on the existence of solutions via sandwiching between upper and lower solutions, applies to (2.1). The key condition on $F(\phi)$ which ensures the establishment of the Laplace transform property (Proposition 1.2) is (iii).

*Proposition (2.1). The problem (2.1) with* $\lambda \geq 0$ *possesses exactly one solution* $\phi \in C^{2+\alpha}(\bar{D})$ *which satisfies* $0 \leq \phi(x) \leq C$ *for all* $x \in \bar{D}$.

Proof : The existence of at least one solution in the desired range is provided by conditions (i) and (ii), upon application of Amann's Theorem. Zero is a lower solution while the constant C is an upper solution, for all $\lambda \geq 0$.

To establish uniqueness let $\phi_1$ and $\phi_2$ be two solutions. Then Taylor's Theorem with remainder provides

$$F(\phi_1) - F(\phi_2) = F^{(1)}(\phi_3)(\phi_1 - \phi_2) \tag{2.8}$$

for some $\phi_3$ lying between $\phi_1$ and $\phi_2$, so that $\phi_3 \in [0,C]$. Hence

$$\left.\begin{aligned} &[L + \lambda p + F^{(1)}(\phi_3)](\phi_1 - \phi_2) = 0 \quad \text{in } D \\ &B(\phi_1 - \phi_2) = 0 \quad \text{on } \partial D \end{aligned}\right\}, \tag{2.9}$$

and condition (iv) now yields $\phi_1 = \phi_2$. Q.E.D.

We will denote the solution referred to above by $\phi(\lambda)$. In order to examine its analytic nature let $\lambda_0 \geq 0$ be held fixed, let $\lambda \in \mathbb{C}$ be given, and set

$$\rho = \lambda - \lambda_0 \tag{2.10}$$

Then we will investigate the formal series

$$\Psi = \Psi[\lambda_0,\rho] = \sum_{n=0}^{\infty} \frac{1}{n!} \psi_n[\lambda_0]\rho^n \tag{2.11}$$

where the $\rho$-independent functions $\psi_n = \psi_n[\lambda_0]$ are supposed to satisfy the set of equations obtained by equating the coefficients of the different powers of $\rho$ which occur in the formal expansion of

$$\left.\begin{aligned} &L\Psi + F(\Psi) + (\lambda_0 + \rho)p\Psi = f \text{ in } D, \\ &B\Psi = 0 \quad \text{on } \partial D, \end{aligned}\right\} \tag{2.12}$$

and where $\psi_0$ is constrained by

$$0 \leq \psi_0 \leq C, \qquad \text{for all } x \in \bar{D} \tag{2.13}$$

The equations to be satisfied by the $\psi_n$'s are found to be

$$\left.\begin{aligned} &L\psi_0 + F(\psi_0) + \lambda_0 p\psi_0 = f \quad \text{in } D \\ &B\psi_0 = 0 \quad \text{on } \partial D, \ \psi_0 \in [0,C] \end{aligned}\right\} \tag{2.14.0}$$

$$\left.\begin{aligned} &[L + F^{(1)}(\psi_0) + \lambda_0 p]\psi_1 + p\psi_0 = 0 \text{ in } D \\ &B\psi_1 = 0 \quad \text{on } \partial D \end{aligned}\right\} \tag{2.14.1}$$

$$\left.\begin{aligned} &[L + F^{(1)}(\psi_0) + \lambda_0 p]\psi_2 + F^{(2)}(\psi_0)\psi_1^2 + 2p\psi_1 = 0 \quad \text{in } D, \\ &B\psi_2 = 0 \quad \text{on } \partial D, \end{aligned}\right\} \tag{2.14.2}$$

$$\left.\begin{aligned} &[L + F^{(1)}(\psi_0) + \lambda_0 p]\psi_n + np\psi_{n-1} \\ &+ \sum_{m=2}^{n} F^{(m)}(\psi_0)\Big\{\sum_{0<k_1<k_2<\cdots<k_{m-1}<n} C(n;k_1,\ldots,k_{m-1})\psi_{(n-k_{m-1})} \\ &\psi_{(k_{m-1}-k_{m-2})}\cdots\psi_{(k_2-k_1)}\psi_{k_1}\Big\} = 0 \quad \text{in } D \\ &B\psi_n = 0 \quad \text{on } \partial D, \ n = 2,3,4,\ldots \end{aligned}\right\} \tag{2.14.n}$$

where $C(n,k_1,\ldots,k_{m-1}) = \binom{n-1}{k_{m-1}}\binom{k_{m-1}-1}{k_{m-2}}\cdots\binom{k_2-1}{k_1}$.

The existence of a unique set of functions $\{\psi_n \in C^{2+\alpha}(\bar{D}) : n = 0,1,2,\ldots\}$ satisfying the above set of equations is readily seen. By Proposition 1, (2.14.0) possesses a unique solution

$$\psi_0 = \phi(\lambda_0) \in C^{2+\alpha}(\bar{D}) \tag{2.15}$$

One now proceeds inductively for $n = 1,2,3,\ldots$ The $n^{th}$ equation is linear in $\psi_n$, involving it only in the term $[L + F^{(1)}(\psi_0) + \lambda_0 p]\psi_n$, and the inhomogeneous part is a function of $(\psi_0, \psi_1, \ldots, \psi_{n-1})$ which, through inductive hypothesis, belongs to $C^{\alpha}(\bar{D})$. Thus, since $(L + F^{(1)}(\psi_0) + \lambda_0 p, B)$ is positive and has smooth coefficients, the $n^{th}$ equation has a unique solution $\psi_n \in C^{2+\alpha}(\bar{D})$, which completes the induction. One sees, moreover, with the aid of the Positivity Lemma and by proceeding inductively using (iii), that

$$(-1)^n \psi_n(x) \geq 0 \qquad \text{for all} \quad x \in \bar{D} \tag{2.16}$$

Having shown that the formal series (2.11) is well defined, our next objective is to show that it has a finite radius of converge $R(\lambda_0)$ and that for $|\rho| \leq R(\lambda_0)$ it converges to a solution of (2.1). This solution is then readily identified to be $\phi(\lambda)$. To achieve these ends we consider the algebraic functional equation, defining $\xi(\rho)$,

$$\{\sigma(\lambda_0) + \rho\hat{p} + \hat{F}\}\xi - \hat{F}e^{\xi-\hat{\phi}_0} = \{\sigma(\lambda_0) + \hat{F}\}\hat{\phi}_0 - \hat{F} \tag{2.17}$$

Here

$$\hat{\phi}_0 = \text{Max}\left\{\underset{x\in\bar{D}}{\text{Max}}\left\{\phi_0(x)\right\} ; 1\right\}, \quad \hat{p} = \underset{x\in\bar{D}}{\text{Max}}\left\{p(x)\right\}, \tag{2.18}$$

and $\sigma(\lambda_0)^{-1} = \underset{x\in\bar{D}}{\text{Max}}\{\theta(x)\} > 0$ where $\theta \in C^{2+\alpha}(\bar{D})$ is the unique positive solution of

$$\left.\begin{aligned} &\{L + F^{(1)}(\phi_0) + \lambda_0 p\}\theta = 1 \quad \text{for all} \quad x \in D, \\ &B\phi = 0 \quad \text{on } \partial D \end{aligned}\right\} \tag{2.19}$$

The motivation behind (2.17) will become clear shortly.

Equation (2.17) possesses the solution $\xi = \hat{\phi}_0$ when $\rho = 0$. Using standard analytical techniques we find that this solution $\xi(\rho)$ exists and is regular in a neighbourhood of $\rho = 0$, say for $|\rho| \leq R(\lambda_0)$ where $R(\lambda_0) > 0$. Indeed, with the aid of Bernstein's theorem[6] one finds that $\xi(\rho)$ is a regular analytic function throughout $\text{Re}\,\rho > -R(\lambda_0)$, being expressible in the form

$$\xi(\rho) = \int_0^\infty \exp\{-(R(\lambda_0) + \rho)\gamma\}\, d\mu(\gamma) \tag{2.20}$$

where $\mu(\gamma)$ is a bounded monotone nondecreasing function over the range $0 \leq u < \infty$.

Writing the Taylor series expansion of $\xi(\rho)$ about $\rho = 0$ as

$$\xi(\rho) = \sum_{n=0}^{\infty} \frac{1}{n!}\xi_p \rho^n, \qquad |\rho| \leq R(\lambda_0) \tag{2.21}$$

we find that the $\xi_n$'s are given recursively by

$$\xi_0 = \hat{\phi}_0, \tag{2.22.0}$$

$$\sigma(\lambda_0)\xi_1 + \hat{p}\,\xi_0 = 0, \tag{2.22.1}$$

$$\sigma(\lambda_0)\xi_2 - \hat{F}\,\xi_1^2 + 2\,\hat{p}\,\xi_1 = 0, \tag{2.22.2}$$

and

$$\sigma(\lambda_0)\xi_n + n\,\hat{p}\,\xi_{n-1} - \hat{F}\sum_{m=2}^{n}\left\{\sum_{0<k_1<k_2<\dots<k_{m-1}<n} C(n;k_1,\dots k_{m-1})\xi_{(n-k_{m-1})}\right.$$
$$\left.\xi_{(k_{m-1}-k_{m-2})}\cdots\xi_{(k_2-k_1)}\xi_{k_1}\right\} = 0 \tag{2.22.n}$$

for $n = 2,3,4\dots$

We now show that

$$0 \le (-1)^n\,\psi_n \le (-1)^n\,\xi_n \qquad \text{for all } x\in D,\ n = 0,1,2,\dots \tag{2.23}$$

These inequalities are clearly true for $n = 0$. Now consider (2.14.1). An upper solution is provided by zero, and a lower solution is provided by $-\hat{p}\hat{\phi}_0\theta(x)$ since

$$\left.\begin{aligned}&[L + F^{(1)}(\psi_0) + \lambda_0 p](-\hat{p}\hat{\phi}_0\theta) + p\phi_0 = -p\hat{\phi}_0 + p\phi_0 \le 0 \quad \text{in } D\\ &B(-\hat{p}\hat{\phi}_0\theta) = 0 \quad \text{on } \partial D\end{aligned}\right\} \tag{2.24}$$

Hence $0 \le (-1)\psi_1 \le +\hat{p}\hat{\phi}_0\theta \le \hat{p}\hat{\phi}_0/\sigma(\lambda_0) = (-1)\xi_1$ which proves (2.23) when $n = 1$. Now assume that (2.23) is true for all $n = 0,1,2,\dots,K$, and let us first suppose K is odd. Then it is readily seen that zero is a lower solution for (2.14. K + 1), while an upper solution is

$$\left\{-(K+1)\hat{p}\xi_K + \hat{F}\sum_{m=2}^{K+1}\left\{\sum_{0<k_1<k_2<\dots<k_{m-1}<K+1} C(K+1;k_1,\dots,k_{m-1})\xi_{(K+1-k_{m-1})}\right.\right.$$
$$\left.\left.\xi_{(k_{m-1}-k_{m-2})}\cdots\xi_{(k_2-k_1)}\xi_{k_1}\right\}\theta = \bar{\psi}_{K+1}, \tag{2.25}$$

Since replacing $\psi_{K+1}$ by $\bar{\psi}_{K+1}$ in the left-hand-side of (2.14. K+1) provides

$$\left\{(K+1)(p\psi_K - \hat{p}\xi_K) + \sum_{m=2}^{K+1}\left\{\sum_{0<k_1<k_2<\dots<k_{m-1}<K+1} C(K+1;k_1,\dots,k_K)\right.\right.$$
$$\left.\left.\left[\hat{F}\,\xi_{(K+1-k_{m-1})}\cdots\xi_{(k_2-k_1)}\xi_{k_1} + F^{(m)}(\psi_0)\psi_{(K+1-k_{m-1})}\cdots\psi_{(k_2-k_1)}\psi_{k_1}\right]\right\}\right\} \tag{2.26}$$

which is positive via the inductive hypothesis together with assumption (iii). Moreover $B\bar{\psi}_{K+1} = 0$ on $\partial D$. Hence

$$0 \le \psi_{K+1} \le \bar{\psi}_{K+1} \le \xi_{K+1} \tag{2.27}$$

The induction is completed after similar treatment of the case K even, and the relations (2.23) are proved.

In particular, the series (2.21) being absolutely convergent for $|\rho| \leq R(\lambda_0)$ provides, on using (2.23), that $\Psi[\lambda_0,\rho]$ is absolutely convergent for $|\rho| \leq R(\lambda_0)$ uniformly for $x \in \bar{D}$. We will use the same notation $\Psi[\lambda_0,\rho]$ to denote its sum, where it converges.

Let $\Psi_N[\lambda_0,\rho]$ denote the $N^{th}$ partial sum of $\Psi[\lambda_0,\rho]$ and let $\rho$ be fixed such that $|\rho| \leq R(\lambda_0)$. Then we show that $\Psi[\lambda_0,\rho] \in C^{2+\alpha}(\bar{D})$ and satisfies the differential equation (2.12) by applying the Compactness Theorem[11]. Provided with the conditions (a) through (d) which follow, the desired result is assured. We already know that (a) the sequence of functions $\{\Psi_N[\lambda_0,\rho]\}_{N=0}^{\infty}$ is uniformly convergent to $\Psi[\lambda_0,\rho]$; (b) $\Psi_N[\lambda_0,\rho] \in C^{2+\alpha}(\bar{D})$ for each N; (c) $B\Psi_N[\lambda_0,\rho] = 0$ on $\partial D$ for each N; and we need only to prove that (d) the sequence of functions $\{L\Psi_N[\lambda_0,\rho]\}$ is uniformly convergent to $f - \lambda p\Psi[\lambda_0,\rho] - F(\Psi[\lambda_0,\rho])$. Since F and its derivatives are continuous it suffices to prove that

$$[L + F^{(1)}(\Phi_0) + \lambda_0 p](\Psi_M - \Psi_N) \to 0 \quad \text{uniformly in D,}$$

as N tends to infinity, with M > N. But for N > 2,

$$[L + F^{(1)}(\Phi_0) + \lambda_0 p](\Psi_M - \Psi_N)| = \Bigg| \sum_{\ell=N+1}^{M} \rho^{\ell} \frac{1}{\ell!} \Big[ \ell p \psi_{\ell-1} + \sum_{m=2}^{\ell} F^{(\ell)}(\psi_0) \Big\{ \sum_{0<k_1<k_2<\ldots<k_{m-1}<\ell} C(\ell;k_1,\ldots,k_{m-1}) \psi_{(\ell-k_{m-1})} \cdots \psi_{(k_2-k_1)} \psi_{k_1} \Big\}\Big]\Bigg|$$

$$\leq \sum_{\ell=N+1}^{N} (-1)^{\ell} |\rho|^{\ell} \frac{1}{\ell!} \Big[ -\ell\hat{p}\,\xi_{\ell-1} + \hat{F} \sum_{m=2}^{\ell} \Big\{ \sum_{0<k_1<k_2<\ldots<k_{m-1}<\ell} C(\ell;k_1,\ldots,k_{m-1}) \xi_{(\ell-k_{m-1})} \xi_{(k_{m-1}-k_{m-2})} \cdots \xi_{(k_2-k_1)} \xi_{k_1} \Big\}\Big] = \sigma(\lambda_0) \sum_{\ell=N+1}^{M} (-1)^{\ell} |\rho|^{\ell} \xi_{\ell} \frac{1}{\ell!} \tag{2.29}$$

where we have used the inequalities (2.23) and the definitive equations (2.22.n) (d) is now proved because the last expression in (2.29) tends to zero as N tends to infinity, uniformly for M > N, and independently of $x \in \bar{D}$, because the series $\sum_{n=0}^{\infty} \rho^n \frac{1}{n!} \xi_n$ is absolutely convergent for $|\rho| \leq R(\lambda_0)$.

The identification of $\Psi[\lambda_0,\rho]$ with $\phi(\lambda) = \phi(\lambda_0 + \rho)$ for $\lambda_0$ and $\rho$ real with $|\rho| \leq R(\lambda_0)$ follows immediately from the fact that $\Psi[\lambda_0,\rho]$ is positive in some neighbourhood of $\lambda_0$, together with the uniqueness part of proposition (2.1). We have in particular that $\phi(\lambda)$ is analytic and regular in some neighbourhood of the real axis $0 \leq \lambda \leq \infty$, and that

$$(-1)^n \frac{d^n\phi}{d\lambda^n} \geq 0 \qquad \text{for all } \lambda \in [0,\infty] \tag{2.30}$$

Thus, Bernstein's Theorem[6] completes the proof of the following proposition.

*Proposition (2.2). The function* $\phi(\lambda)$ *defined by Proposition (2.1) can be analytically continued throughout* $\mathrm{Re}\,\lambda \geq 0$, *where it can be expressed in the form*

$$\phi(\lambda) = \int_0^\infty e^{-\lambda s}\, d\mu_x(s), \tag{2.31}$$

*the function* $\mu_x(s)$ *being uniformly bounded and monotone non-decreasing for* $0 \leq s < \infty$, *for all* $x \in \bar{D}$. $\phi(\lambda)$ *satisfies (2.1) at least throughout some open neighbourhood of the real axis* $0 \leq \lambda < \infty$.

A similar result also applies in the case of the associated parabolic differential equation obtained by the adjunction of $\partial/\partial t$ to L, and adjoining an initial positive boundary condition to B, as in[2].

## 3. INHOMOGENEOUSLY PERTURBED NON LINEAR EQUATIONS

We consider equations of the form

$$\left.\begin{aligned} &[L - \lambda p]\phi + G(\phi) = \gamma f \quad \text{in } D \\ &B\phi = 0 \quad \text{on } \partial D \end{aligned}\right\} \tag{3.1}$$

Everything here is defined as in §II except that now $\lambda \in (-\infty + \infty)$ and $\gamma$ is the parameter of interest. The real valued function $G(\phi) = G(x,\phi)$ is assumed to have the following properties :

(i) $\min\limits_{x\in\bar{D}} \{G(u)/u\} \to \infty$ as $u \to \infty$

(ii) $G(0) \equiv 0$

(iii) $G(u)$ and all of its derivatives exist and belong to $C^\alpha(\bar{D})$ for all $u \in [0,\infty)$.

(iv) $[G(u)/u]^{(1)}$ exists and is $\geq 0$ for all $u \in [0,\infty)$, all $x \in \bar{D}$.

(v) $\exists$ a constant $\hat{G}$ such that $\hat{G} \geq (-1)^n G^{(n)}(u) \geq 0$ for all $u \in [0,\infty)$, $x \in \bar{D}$, $n = 2,3,\ldots$

Using Bernstein's Theorem we find that the most general form of G is

$$G(x,\phi) = A(x)\phi^2 + B(x)\phi + \int_0^\infty \left\{\exp\{-\gamma\phi\} - 1\right\} d\rho_x(\gamma) \tag{3.2}$$

where $A(x) > 0$ for all $x \in \bar{D}$, and $\rho_x(\gamma)$ is a bounded monotone non-decreasing function on $0 \leq \gamma < \infty$, for each $x \in D$.

*Proposition 3.1. For each* $\gamma \geq 0$ *and* $\lambda \in (-\infty,+\infty)$ *the problem* (3.1) *possesses exactly one non negative solution.* $\phi \in C^{2+\alpha}(\bar{D})$. *This solution is regular in* $\gamma$ *and can be analytically continued throughout* $\mathrm{Re}\,\gamma \geq 0$, *where it satisfies a representation of the form*

$$\frac{\partial\phi}{\partial\gamma}[\gamma] = \int_0^\infty e^{-\gamma s}\, d\,\Pi_{x,\lambda}(s) \quad , \quad \textit{for each fixed } x,\lambda \tag{3.2}$$

*where* $\Pi_{x,\lambda}$ *is bounded monotone nondecreasing function on* $0 \le s < \infty$ . *The continued solution satisfied* (3.1) *for all* $\gamma$ *in some neighbourhood of* $[0,\infty)$.

We omit the proof of this proposition as it follows somewhat similar lines to the demonstrations en §2. A key point is that the operator $([L - \lambda p + G^{(1)}(\phi[\gamma])],B)$ has strictly positive least eigenvalue for all $\gamma \ge 0$.

## 4. PADE APPROXIMANTS AND THE PROBLEM $[L-\lambda p]\phi + \gamma q\,\phi^N = f$

We consider the $\gamma$-dependence of the positive stable solution $\phi[\gamma]$ of the problem

$$\left.\begin{array}{l} L\phi + \gamma q\phi^N = f \quad \text{in } D, \\ B\phi = 0 \quad \text{on} \quad \partial D; \; N = 2,3,\ldots \end{array}\right\} \tag{4.1}$$

where $q \in C^{\alpha}(\bar{D})$, $q(x) > 0$ for all $x \in \bar{D}$, and all other quantities are defined as in §2. In particular, we are interested in the location of the *turning point* $\gamma^*$, which corresponds to the first singularity in $\phi[\gamma]$ on the real axis as $\gamma$ goes towards minus infinity starting from zero $\phi[\gamma]$ is a positive stable solution of (4.1) for all $\gamma > \gamma^*$.

We are also interested in the possibility of using classical rational fraction Padé approximants (P.A.'s) to provide upper and lower bounds on $\phi[\gamma]$ for all $\gamma > \gamma^*$, starting from the Taylor series expansion of $\phi[\gamma]$ about $\gamma = 0$ which we write

$$\phi[\gamma] = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma^n \phi_n \tag{4.2}$$

Over and above the practical utility of such bounds, the motivation here is the *conjecture* that $\phi[\gamma]$ is a Stieltjes transform of positive measure, namely

$$\phi[\gamma] = \int_0^{-1/\gamma^*} \frac{d\mu(s)}{1+s\gamma} \quad \text{for all } \gamma \in \mathbb{C}-(-\infty,\gamma^*] \quad , \tag{4.3}$$

where $\mu(s)$ is a bounded monotone nondecreasing function on $-1/\gamma^* \le s < \infty$ . This conjecture was shown indeed to be true in the case $N = 2$, in dimension $\le 5$, when $L = -\Delta$, $q = 1$, and f is sufficiently smooth,[5].

We begin by saying why a representation of the form (4.1) is suggested in the first place. In the case where the boundary condition in (4.1) is replaced by $\partial\phi/\partial\nu = 0$ on $\partial D$, and a, q, and f, are constants we have that $\phi[\gamma]$ is itself a constant, being the positive solution of the algebraic equation

$$a\phi[\gamma] + \gamma q\phi[\gamma]^N - f = 0 \tag{4.4}$$

The solution of the latter which is regular around $\gamma = 0$ is expressible in the form (4.3) with

$$\gamma^* = -\left[\frac{a(N-1)}{f\,N}\right]^N \frac{f}{(N-1)} \,, \tag{4.5}$$

see[8], example 1. In view of the close relationship which often exists between solutions of uniform elliptic equations and the analogous algebraic equations, the conjecture (4.3) is suggested.

Let us now examine some consequences of (4.3) when we suppose that it *is* true [e.g. $N=2$]. The first and perhaps most important consequence is that the [M/M] and [(M-1)/M] sequences of PA.'s, constructed from initial sets of coefficients occuring in the expansion (4.2), provide convergent bounds on $\phi[\gamma]$ according to

$$[M/M] \geq [(M+1)/(M+1)] \geq \ldots \geq \phi[\gamma] \geq \ldots \geq [M/(M+1)] \geq [(M-1)/M] \quad \text{for all } \gamma > 0,$$
$$\phi[\gamma] \geq \ldots \geq [(M+1)/(M+1)] \geq [M/(M+1)] \geq [M/M] \geq [(M-1)/M] \quad \text{for } \gamma^* < \gamma \leq 0 \ ; \quad M=1,2,3,\ldots \tag{4.6}$$

The [R/S] P.A. is defined as follows :

$$[R/S] = \frac{P_R(\gamma)}{Q_s(\gamma)} = \frac{P_0 + P_1\gamma + \ldots + P_R\gamma^R}{1 + q_1\gamma + \ldots + q_s\gamma^s} \tag{4.7}$$

where the (R+S+1) unknowns, the p's and q's, are determined by the requirement

$$Q_s(\gamma)\left(\sum_{n=0}^{R+S} \frac{1}{n!}\gamma^n\phi_n\right) - P_R(\gamma) = \text{terms of order } \gamma^{R+S+1} \text{ and higher.} \tag{4.8}$$

Certain subtle modifications of this definition are needed in special cases, but the above suffices in general. For full details see[9]. There also exist complementary P.A.'s, denoted $[R/S]^c$, whose bounding properties complement those in (4.6). These are defined similarly to the above except that the polynomial occuring in the denominator is required to have a zero at a point $\bar{\gamma}^* \geq \gamma^*$, while the number of agreements demanded in (4.8) is decreased by one see[10]. An elementary pair of approximants is

$$[0/1] = \phi_0/(1-\gamma\phi_1/\phi_0) \quad \text{and} \quad [0/1]^c = \phi_0/(1-\gamma/\bar{\gamma}^*) \tag{4.9}$$

and these display in particular the bounds

$$[0/1] \leq \phi[\gamma] \leq [0/1]^c \quad \text{for} \quad \bar{\gamma}^* < \gamma \leq 0 \tag{4.10}$$

provided that (4.3) is true.

We note that the $\phi_n$'s needed for the construction of the P.A.'s can be obtained by successive solution of the set of linear equations

$$\left.\begin{aligned} L\,\phi_0 &= f \text{ in } D, \\ B\,\phi_0 &= 0 \text{ on } \partial D; \end{aligned}\right\} \tag{4.11.0}$$

$$\left.\begin{aligned} L\,\phi_1 + q\,\phi_0^2 &= 0 \text{ in } D, \\ B\,\phi_1 &= 0 \text{ on } \partial D; \end{aligned}\right\} \tag{4.11.1}$$