THE SWOKOWSKI SERIES

FIFTH EDITION

FUNDAMENTALS OF ALGEBRA AND TRIGONOMETRY

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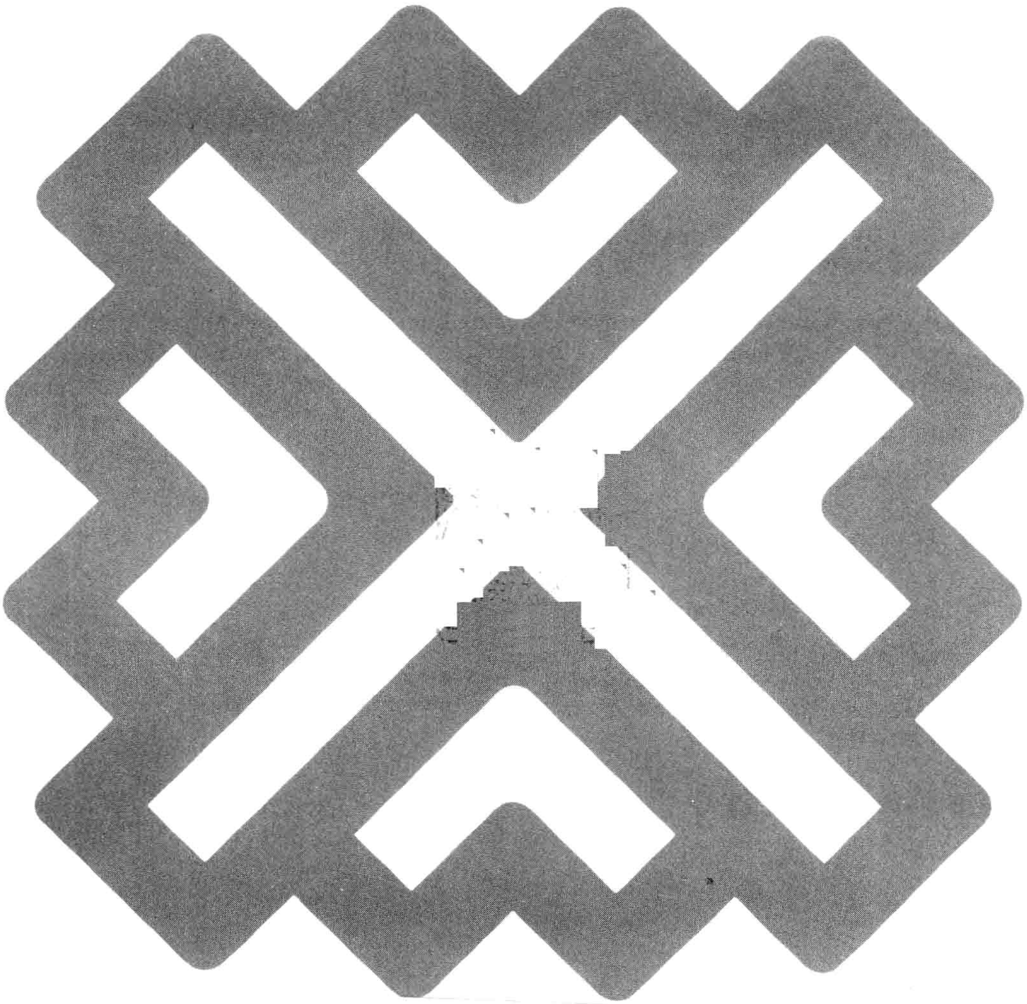

COMMENTS ABOUT HANDHELD CALCULATORS

There is disagreement among mathematics educators as to the extent that electronic handheld calculators should be employed in courses in algebra and trigonometry. Those who favor their use point out that, in addition to reducing the time spent on numerical computations, calculators are useful for illustrating and reinforcing mathematical concepts. It is unlikely that any teacher would recommend that a calculator be used as a tool for solving the quadratic equation $x^2 + 2x - 3 = 0$; however, the process of approximating the solutions of $241x^2 + 527x - 73 = 0$ *without* a calculator is pure drudgery, adding little to a student's mathematical knowledge. On the theoretical side, some claim that calculating $10^{\log x}$ for various values of x will help beginning students accept, and remember, the identity $10^{\log x} = x$. The extent to which this is true is, of course, difficult to measure.

Teachers who oppose the use of calculators in algebra and trigonometry courses argue that they distract from the mainstream of study, and that classroom time is better spent on theoretical and manipulative aspects which are needed in advanced courses such as calculus. Clearly, it is essential to understand basic results about equations before using a calculator to approximate solutions. As another illustration, working with logarithmic and trigonometric tables has some worthwhile side-effects concerning the variation of functions and intermediate values.

In this text an attempt is made to satisfy both those instructors who wish to use calculators as a teaching aid, and those who prefer not to use them. Specifically, optional problems labeled *Calculator Exercises* have been included in appropriate sections of the text, following the regular exercises. They may either be omitted or assigned, depending on course objectives. In order to make proper use of these exercises, it is necessary to have access to a scientific calculator which has the capability to deal with exponential, logarithmic, trigonometric, and inverse trigonometric expressions.

If calculators are used, students should be aware of the fact that inaccuracies may occur due to round off, truncation, or algorithms which the calculator employs to compute certain numerical values. For example, to use a calculator to demonstrate that $(1/3) \cdot 3 = 1$, we may enter 3, press the reciprocal key $\boxed{1/x}$, and then multiply by 3.

However, if the decimal approximation 0.33333333 is entered, then multiplying by 3 results in 0.99999999. Similarly, to demonstrate that $\sqrt{2}\sqrt{3} = \sqrt{6}$ we may, on the one hand, calculate $\sqrt{2}\sqrt{3}$ and $\sqrt{6}$ separately, and verify that the same decimal approximation is obtained in each case. On the other hand, calculating $\sqrt{6} - \sqrt{2}\sqrt{3}$ with a typical calculator we may obtain 0.000000000013 (this number may vary, depending on the type of calculator). If we agree to round off answers to eight decimal places, then this result may be considered as a calculator demonstration that $\sqrt{6} - \sqrt{2}\sqrt{3} = 0$, and hence that $\sqrt{6} = \sqrt{2}\sqrt{3}$.

Other discrepencies may occur. For example, given a calculator which displays a maximum of eight digits, if we enter 10,000,000 and then add 0.1, we obtain 10,000,000, a result not in keeping with laws of algebra! Similarly, laws such as $a(b + c) = ab + ac$ may not hold if certain values are entered for a, b, and c.

It should also be noted that if calculators are used to solve the *regular* exercises involving logarithmic or trigonometric functions, some answers may differ from those given in the text, since the latter were obtained using the four-place tables in the Appendix, whereas calculators produce a higher degree of accuracy.

Finally, instructions on how to operate a calculator are not provided in this text. The large variety of calculators on the market make it impractical to do so. Owner manuals should be consulted for information and illustrations.

PREFACE

The fifth edition of *Fundamentals of Algebra and Trigonometry* reflects the continuing change in the needs and abilities of students who enroll in precalculus mathematics courses. The goal of this new edition is to maintain the mathematical soundness of earlier editions, but to make some of the discussions less formal by rewriting, placing more emphasis on graphing and applications, and adding new examples, figures, and exercises.

Chapter 1 consists of topics that are fundamental for the study of algebra. These include properties of real numbers, exponents, radicals, and simplification of algebraic expressions. Equations and inequalities are considered in Chapter 2. The technique of solving inequalities involving polynomials has been simplified by introducing the notion of *test value.*

Basic properties of functions and graphs are discussed in Chapter 3. A noteworthy change from earlier editions is the additional emphasis on symmetry, horizontal or vertical shifts, and stretching of graphs.

Chapter 4 contains a rather detailed examination of polynomial and rational functions. Once again the use of test values has made it possible to simplify and unify this material. Section 4.4, on partial fractions, is new to this edition. The chapter ends with a discussion of conic sections.

Exponential and logarithmic functions are discussed in Chapter 5. Computational aspects of logarithms appear in the last two sections, and may be omitted without interrupting the continuity of the book.

In Chapter 6, an accessible introduction to trigonometric functions using a unit circle is managed by emphasizing examples and keeping the discussion relatively brief. Angles are introduced early in the chapter, and the unit circle approach is then supplemented by an angular description of the trigonometric functions. Later, examples are again emphasized to help students learn to sketch graphs rapidly. A new section on harmonic motion was added at the end of the chapter to provide some non-triangular applications of trigonometry.

Most of Chapter 7 consists of work with trigonometric identities, equations, and solutions of oblique triangles. The last section of this chapter contains an introduction to geometric and algebraic properties of vectors in two dimensions.

Matrices are introduced in Chapter 8 as a tool for solving systems of linear equations. The chapter concludes with a study of determinants and a brief introduction to linear programming.

Complex numbers are discussed in Chapter 9. Chapter 10 contains material on zeros of polynomials. Several additional theorems about the location and bounds of zeros were added to this edition. Chapter 11 has sections on mathematical induction, the Binomial Theorem, summation notation, sequences, and probability.

There is a review section at the end of each chapter consisting of a list of important topics and pertinent exercises. The review exercises are similar in scope to those which appear throughout the chapter and may be used to prepare for examinations. Answers to odd-numbered exercises are given at the end of the text. An answer booklet for the even-numbered exercises may be obtained from the publisher.

This revision has benefited from comments and suggestions of users of previous editions. I wish to thank the following individuals, who reviewed all or parts of the manuscript and offered many helpful suggestions: Ben P. Bockstege, Jr. (Broward Community College), Gary L. Ebert (University of Delaware), Leonard E. Fuller (Kansas State University), F. Cecilia Hakeem (Southern Illinois University), Douglas W. Hall (Michigan State University), Arthur M. Hobbs (Texas A&M University), Adam J. Hulin (University of New Orleans), William B. Jones (University of Colorado), Jimmie Lawson (Louisiana State University), Burnett Meyer (University of Colorado),Eldon L. Miller (University of Mississippi), Robert W. Owens (Lewis and Clark College), Anthony L. Peressini (University of Illinois), Jack W. Pope (University of San Diego), Larry H. Potter (Memphis State University), Sandra M. Powers (The College of Charleston), Billie Ann Rice (Dekalb Community College), John Riner (Ohio State University), Mary W. Scott (University of Florida), William F. Stearns (University of Maine), Janet T. Vasak (University of Wisconsin-Milwaukee), Richard G. Vinson (University of South Alabama), T. Perrin Wright, Jr. (Florida State University), and Paul M. Young (Kansas State University).

I am also grateful to the staff of Prindle, Weber & Schmidt, for their painstaking work in the production of this book. In particular, Mary Lu Walsh, John Martindale, and Nancy Blodget gave valuable assistance with the revision.

Special thanks are due to my wife Shirley and the members of our family: Mary, Mark, John, Steve, Paul, Tom, Bob, Nancy, Judy, and Jay. All have had an influence on the book—either directly, through working exercises, proofreading, or typing, or indirectly, through continued interest and moral support.

To all of the people named here, and to the many unnamed students and teachers who have helped shape my views about precalculus mathematics, I express my sincere appreciation.

Earl W. Swokowski

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1

FUNDAMENTAL CONCEPTS OF ALGEBRA

The material in this chapter is basic to the study of algebra. We begin by discussing properties of real numbers. Next we turn our attention to exponents and radicals, and how they may be used to simplify complicated algebraic expressions.

1.1 ALGEBRA: A POWERFUL LANGUAGE AND TOOL

A good foundation in algebra is essential for advanced courses in mathematics, the natural sciences, and engineering. It is also required for problems which arise in business, industry, statistics, and many other fields of endeavor. Indeed, every situation which makes use of numerical processes is a candidate for algebraic methods.

Algebra evolved from the operations and rules of arithmetic. The study of arithmetic begins with addition, multiplication, subtraction, and division of numbers, such as

$$4 + 7, \quad (37)(681) \quad 79 - 22, \quad \text{and} \quad 40 \div 8.$$

In *algebra* we introduce symbols or letters a, b, c, d, x, y, etc., to denote *arbitrary* numbers and, instead of special cases, we often consider *general* statements, such as

$$a + b, \quad cd, \quad x - y, \quad \text{and} \quad x \div a.$$

This *language of algebra* serves a two-fold purpose. First, it may be used as a shorthand, to abbreviate and simplify long or complicated statements. Second, it is a convenient means of generalizing many specific statements. To illustrate, at an early age, children learn that

$$2 + 3 = 3 + 2, \qquad 4 + 7 = 7 + 4, \qquad 5 + 9 = 9 + 5, \qquad 1 + 8 = 8 + 1,$$

and so on. In words, this property may be phrased "if two numbers are added, then the order of addition is immaterial; that is, the same result is obtained whether the second number is added to the first, or the first number is added to the second." This lengthy description can be shortened, and at the same time made easier to understand by means of the algebraic statement

$$a + b = b + a,$$

where a and b denote arbitrary numbers.

Illustrations of the generality of algebra may be found in formulas used in science and industry. For example, if an airplane flies at a constant rate of 300 mph (miles per hour) for two hours, then the distance it travels is given by

$$(300)(2), \quad \text{or} \quad 600 \text{ miles}.$$

If the rate is 250 mph and the elapsed time is 3 hours, then the distance traveled is

$$(250)(3), \quad \text{or} \quad 750 \text{ miles}.$$

If we introduce symbols and let r denote the constant rate, t the elapsed time, and d the distance traveled, then the two illustrations we have given are special cases of the general algebraic formula

$$d = rt.$$

When specific numerical values for r and t are given, the distance d may be found readily by an appropriate substitution in the formula. Moreover, the formula may also be used to solve related problems. For example, suppose the distance between two cities is 645 miles, and we wish to find the constant rate which would enable an airplane to cover that distance in 2 hours and 30 minutes. Thus, we are given

$$d = 645 \text{ miles}, \quad \text{and} \quad t = 2.5 \text{ hours},$$

and the problem is to find r. Since $d = rt$, it follows that

$$r = \frac{d}{t}$$

and hence, for our special case,

$$r = \frac{645}{2.5} = 258 \text{ mph}.$$

That is, if an airplane flies at a constant rate of 258 mph, then it will travel 645 miles in 2 hours and 30 minutes. In like manner, given r, the time t required to travel a distance d may be found by means of the formula

$$t = \frac{d}{r}.$$

Note how the introduction of a general algebraic formula not only allows us to solve special problems conveniently, but also to enlarge the scope of our knowledge by suggesting new problems that can be considered.

We have given several elementary illustrations of the value of algebraic methods. There are an unlimited number of situations where a symbolic approach may lead to insights and solutions that would be impossible to obtain using only numerical processes. As you proceed through this text and go on to either more advanced courses in mathematics or fields which employ mathematics, you will become further aware of the importance and the power of algebraic techniques.

1.2 REAL NUMBERS

Real numbers are used in all phases of mathematics and you are undoubtedly well acquainted with symbols which are used to represent them, such as

$$1, \quad 73, \quad -5, \quad \frac{49}{12}, \quad \sqrt{2}, \quad 0, \quad \sqrt[3]{-85}, \quad 0.33333\ldots, \quad \text{and} \quad 596.25.$$

The real numbers are said to be **closed** relative to operations of addition (denoted by $+$) and multiplication (denoted by $\cdot$). This means that, to every pair a, b of real numbers, there corresponds a real number $a + b$ called the **sum** of a and b and a unique real number $a \cdot b$ (also written ab) called the **product** of a and b. These operations have the following properties, where all lower-case letters denote arbitrary real numbers, and where 0 and 1 are special real numbers referred to as **zero** and **one**, respectively.

Commutative Properties

$$a + b = b + a, \qquad ab = ba$$

Associative Properties

$$a + (b + c) = (a + b) + c, \qquad a(bc) = (ab)c$$

Identities

$$a + 0 = a = 0 + a, \qquad a \cdot 1 = a = 1 \cdot a$$

Inverses For every real number a, there is a real number denoted by $-a$ such that

$$a + (-a) = 0 = (-a) + a.$$

For every real number $a \neq 0$, there is a real number denoted by $1/a$ such that

$$a\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right)a.$$

Distributive Properties

$$a(b + c) = ab + ac, \qquad (a + b)c = ac + bc$$

The equals sign, =, indicates that the expressions immediately to the right and left of the sign represent the same real number. The real numbers 0 and 1 are sometimes referred to as the **additive identity** and **multiplicative identity**, respectively. We call $-a$ the **additive inverse** of a (or the **negative** of a). If $a \neq 0$, then $1/a$ is called the **multiplicative inverse** of a (or the **reciprocal** of a). The symbol a^{-1} is often used in place of $1/a$. Thus, by definition,

Definition of a^{-1}

$$a^{-1} = \frac{1}{a}$$

Example 1 Verify the following special cases of the Associative and Distributive Properties.

(a) $2 + (3 + 4) = (2 + 3) + 4$

(b) $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$

(c) $2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4$

(d) $(2 + 3) \cdot 4 = 2 \cdot 4 + 3 \cdot 4$

Solutions To verify each of parts (a)–(d) we perform the operations indicated on opposite sides of the equals sign and observe that the resulting numbers are identical. Thus,

(a) $$2 + (3 + 4) = 2 + 7 = 9$$
$$(2 + 3) + 4 = 5 + 4 = 9$$

(b) $$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$$
$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$$

(c) $$2 \cdot (3 + 4) = 2 \cdot 7 = 14$$
$$2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14$$

(d) $$(2 + 3) \cdot 4 = 5 \cdot 4 = 20$$
$$2 \cdot 4 + 3 \cdot 4 = 8 + 12 = 20$$

■

Since $a + (b + c)$ and $(a + b) + c$ are always equal, we may, without ambiguity, use the symbol $a + b + c$ to denote this real number. Similarly, the notation abc is used for either $a(bc)$ or $(ab)c$. An analogous situation exists if four real numbers a, b, c, and d are added. For example, we could consider

$$(a + b) + (c + d), \qquad a + [(b + c) + d], \qquad [(a + b) + c] + d,$$

and so on. It can be shown that, regardless of how the four numbers are grouped, the same result is obtained, and consequently, it is customary to write $a + b + c + d$ for any of these expressions. Furthermore, it follows from the Commutative Property for addition that the numbers can be interchanged in any way. For example,

$$a + b + c + d = a + d + c + b = a + c + d + b.$$

A similar situation exists for multiplication, where the expression $abcd$ is used to denote the product of four real numbers.

The Distributive Properties are useful for finding products of many different types of expressions. The next example provides an illustration. Others will be found in the exercises.

Example 2 If a, b, c, and d denote real numbers, show that

$$(a + b)(c + d) = ac + bc + ad + bd.$$

Solution Using the two Distributive Properties,

$$\begin{aligned}(a + b)(c + d) &= (a + b)c + (a + b)d \\ &= (ac + bc) + (ad + bd) \\ &= ac + bc + ad + bd.\end{aligned}$$

■

If $a = b$ and $c = d$, then, since a and b are merely different names for the same real number, and likewise for c and d, it follows that $a + c = b + d$ and $ac = bd$. This is often called the **substitution principle**, since we may think of replacing a by b and c by d in the expressions $a + c$ and ac. As a special case, using the fact that $c = c$ gives us the following rules.

If $a = b$, then $a + c = b + c$.
If $a = b$, then $ac = bc$.

We sometimes refer to these rules by the statements "any number c may be added to both sides of an equality" and "both sides of an equality may be multiplied by the same number c." We shall make heavy use of them in Chapter 2, in conjunction with solving equations.

The following results can be proved. (See Exercises 34 and 35.)

$a \cdot 0 = 0$ for every real number a.
If $ab = 0$, then either $a = 0$ or $b = 0$.

The last two statements imply that $ab = 0$ *if and only if* either $a = 0$ or $b = 0$. The phrase "if and only if," which is used throughout mathematics, always has a two-fold character. Here it means that if $ab = 0$, then $a = 0$ or $b = 0$ and, *conversely*, if $a = 0$ or $b = 0$, then $ab = 0$. Consequently, if both $a \neq 0$ and $b \neq 0$, then $ab \neq 0$; that is, *the product of two nonzero real numbers is always nonzero*.

The following can also be proved. (See Exercise 39.)

Properties of Negatives

$$\begin{aligned}-(-a) &= a \\ (-a)b &= -(ab) = a(-b) \\ (-a)(-b) &= ab \\ (-1)a &= -a\end{aligned}$$

The operation of **subtraction** (denoted by $-$) is defined by

Definition of Subtraction

$$a - b = a + (-b)$$

The next example indicates that the Distributive Properties hold for subtraction.

Example 3 If a, b, and c are real numbers, show that

$$a(b - c) = ab - ac.$$

Solution We shall list reasons after each step, as follows.

$$\begin{aligned} a(b - c) &= a[b + (-c)] && \text{(Definition of Subtraction)} \\ &= ab + a(-c) && \text{(Distributive Property)} \\ &= ab + [-(ac)] && \text{(Property of Negatives)} \\ &= ab - ac && \text{(Definition of Subtraction)} \end{aligned}$$ ■

If $b \neq 0$, then **division** (denoted by $\div$) is defined by

Definition of Division

$$a \div b = a\left(\frac{1}{b}\right) = ab^{-1}$$

The symbol a/b is often used in place of $a \div b$, and we refer to it as the **quotient of *a* by *b*** or the **fraction *a* over *b***. The numbers a and b are called the **numerator** and **denominator**, respectively. It is important to note that, since 0 has no multiplicative inverse, a/b is not defined if $b = 0$; that is, *division by zero is not permissible*. Also note that

$$1 \div b = \frac{1}{b} = b^{-1}.$$

The following properties of quotients may be established, where all denominators are nonzero real numbers.

Properties of Quotients

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc$$

$$\frac{a}{b} = \frac{ad}{bd}, \qquad \frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$$

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}, \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 4 Find (a) $\frac{2}{3} + \frac{9}{5}$ (b) $\frac{2}{3} \cdot \frac{9}{5}$ (c) $\frac{2}{3} \div \frac{9}{5}$.

Solutions Using the Properties of Quotients we have

(a) $$\frac{2}{3} + \frac{9}{5} = \frac{(2 \cdot 5) + (3 \cdot 9)}{3 \cdot 5} = \frac{10 + 27}{15} = \frac{37}{15}$$

(b) $$\frac{2}{3} \cdot \frac{9}{5} = \frac{2 \cdot 9}{3 \cdot 5} = \frac{18}{15} = \frac{6 \cdot 3}{5 \cdot 3} = \frac{6}{5}$$

(c) $$\frac{2}{3} \div \frac{9}{5} = \frac{2}{3} \cdot \frac{5}{9} = \frac{2 \cdot 5}{3 \cdot 9} = \frac{10}{27}$$ ■

The **positive integers** 1, 2, 3, 4, . . . may be obtained by adding the real number 1 successively to itself. The negatives, $-1, -2, -3, -4, \ldots$, of the positive integers are referred to as **negative integers**. The **integers** consist of the totality of positive and negative integers together with the real number 0.

Observe that, by the Distributive Properties, if a is a real number, then

$$a + a = (1 + 1)a = 2a$$

and

$$a + a + a = (1 + 1 + 1)a = 3a.$$

Similarly, the sum of four a's is $4a$, the sum of five a's is $5a$, and so on.

If a, b, and c are integers and $c = ab$, then a and b are called **factors**, or **divisors**, of c. For example, the integer 6 may be written as

$$6 = 2 \cdot 3 = (-2)(-3) = 1 \cdot 6 = (-1)(-6).$$

Hence $1, -1, 2, -2, 3, -3, 6$, and -6 are factors of 6.

A positive integer p different from 1 is **prime** if its only positive factors are 1 and p. The first few primes are 2, 3, 5, 7, 11, 13, 17, and 19. One of the reasons for the importance of prime numbers is that every positive integer a different from 1 can be expressed in one and only one way (except for order of factors) as a product of primes. The proof of this result, which is called The **Fundamental Theorem of Arithmetic**, will not be given in this book. As examples, we have

$$12 = 2 \cdot 2 \cdot 3, \quad 126 = 2 \cdot 3 \cdot 3 \cdot 7, \quad \text{and} \quad 540 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5.$$

A real number is called a **rational number** if it can be written in the form a/b, where a and b are integers and $b \neq 0$. Real numbers that are not rational are called **irrational**. The ratio of the circumference of a circle to its diameter is an irrational real number and is denoted by π. It is often approximated by the decimal 3.1416 or by the rational number 22/7. We use the notation $\pi \approx 3.1416$ to indicate that π is