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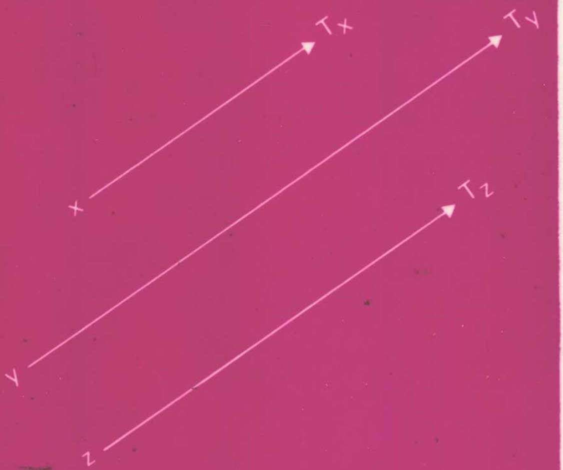


# INTRODUCTION TO LINEAR OPERATOR THEORY



Vasile I. Istrăţescu

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*To my parents,*
*Paraschiva Istrăţescu and Ion Istrăţescu*

# PREFACE

During the last 15 or 20 years much progress has been made in the theory of non-self-adjoint operators on Hilbert and Banach spaces. The present volume is intended to provide an introduction to the subject.

The first four chapters are devoted to standard material on linear functional analysis. However, whenever possible a unified approach is used, and this is the case, for example, for the three basic theorems of linear functional analysis which are treated consequences of a unique theorem. Chapter 5 is concerned with the general spectral representation of operators on Hilbert spaces and depends on the Banach algebra theory developed in Chapter 4.

Chapter 6 is concerned with the basic notion of this book: the numerical range. First this is considered for Hilbert spaces and next for Banach spaces. Various classes of operators connected with the numerical range are also discussed.

The classes of nonnormal operators have a long history, and the problem of deciding when an operator is normal (also hermitian, unitary) forms the content of Chapters 7 and 8.

As is well known, the class of hermitian operators has many important applications in various branches of mathematics and physics; thus related classes of operators for which many properties of hermitian operators are preserved are of great interest. Chapter 9 gives an account of results involving such classes as well as some applications (for example, a simple proof of an interpolation theorem of Lions-Peetre).

In Chapter 10 the famous invariant subspace problem is discussed and some structure theorems are presented.

The Weyl spectrum of an operator is discussed in Chapter 11, as well as some applications. The elements of the von Neumann algebras are also given.

Chapter 12 is concerned with an important and useful notion: analytic and quasi-analytic vectors; also some applications are given. In Chapter 13 the Banach space version of the famous Schwarz theorem from complex function theory is presented. In Chapter 14 results on maximum theorems for operator-valued functions are given.

In the last chapter, I present some ergodic theorems for classes of operators containing the quasi-compact operators. The results presented are connected with the operator theoretic treatment of Markov processes as given by Kakutani-Yosida and refined by many authors. I have tried to indicate the origin of the various results, and the references (which in turn contain references to many earlier results) may be used to obtain further information. When I make no ascription, I am not claiming originality.

A part of this book has its origin in a course given by the author at the Centro Linceo Interdisciplinare di Scienze Matematiche e loro Applicazioni of the Accademia Nazionale dei Lincei (Rome, Italy).

For interesting and helpful conversations I am indebted to many friends. For discussions and constant encouragement I am indebted to Professor G. Köthe.

*Vasile I. Istrăţescu*

CONTENTS

Chapter 1

# PRELIMINARIES: SET THEORY AND GENERAL TOPOLOGY

## 1.1 THE ALGEBRA OF SETS

In what follows we assume that the reader is familiar with the notion of a set. We give several examples of sets.

*1.1.1 EXAMPLE* The letters of the English alphabet form a set.

*1.1.2 EXAMPLE* The rational numbers in the interval $[0, 1] = \{x \mid 0 \leq x \leq 1\}$ form a set.

*1.1.3 EXAMPLE* The numbers of the form 2n, where n is an integer, also form a set.

We consider a set A as being given when we can identify its elements; for example, the set E has the elements a, b, c, ..., or, in brief,

$$E = \{a, b, c, \ldots\}$$

Another way is to identify the elements of a set by a property P, or in brief,

$$E = \{a \mid a \text{ has the property } P\}$$

Generally speaking, we adhere to the standard notational conventions: we denote by lowercase letters a, b, c, ... the elements of a set E and we write this as $a \in E$, $b \in E$, $c \in E$, ... . Sets are denoted by uppercase letters A, B, C, ..., X, Y, Z and families of sets by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... (script uppercase letters). We write $\emptyset$ for the empty set, and for any element a, $\{a\}$ denotes the set which has as an element only the element a.

*1.1.4 DEFINITION* Let A and B be two sets. We say that A is a *subset* of B if for any $a \in A$ we have $a \in B$. We write this as $A \subseteq B$. If there exists an element $b \in B$, b not in A, then we say that A is a *proper* subset of B.

If an element a is not in a set A, we write this as $a \notin A$.

*1.1.5 DEFINITION* If A and B are two sets, we say that $A = B$ if $A \subseteq B$ and $B \subseteq A$; in the contrary case, we say that A and B are distinct sets.

*1.1.6 DEFINITION* If A and B are two sets, then $A \cup B$ denotes the set of all elements which are in A or in B; $A \cap B$ denotes the set of all elements which are in A and in B.

The set $A \cup B$ is called the *union* of the sets A and B; $A \cap B$ is called the *intersection* of the sets A and B.

*1.1.7 REMARK* Similar definitions can be given for the case when we have a family of sets, $A = \{A_\alpha\}_{\alpha \in I}$ for $\bigcup_\alpha A_\alpha$ and $\bigcap_\alpha A_\alpha$.

*1.1.8 EXAMPLE* If $A = \{1, 2, 3\}$ and $B = \{2, 5\}$, then

$A \cup B = \{1, 2, 3, 5\}$ $\quad$ $A \cap B = \{2\}$

If $A = \{1, 2\}$ and $B = \{3, 5\}$, then

$A \cup B = \{1, 2, 3, 5\}$ $\quad$ $A \cap B = \emptyset$

*1.1.9 DEFINITION* For any set A we note $\mathcal{P}(A)$, the family of all subsets of A.

*1.1.10 EXAMPLE* For $A = \{2\}$ we have

$\mathcal{P}(A) = \{\emptyset, \{2\}\}$

*1.1.11 PROPOSITION* If A, B, and C are sets, then the following relations hold:

1. $A \cup B = B \cup A$
2. $A \cup A = A$
3. $A \cup \emptyset = A$

4. $A \cup (B \cup C) = (A \cup B) \cup C$

5. $A \subseteq A \cup B$

6. $A \subseteq B$ if and only if $A \cup B = B$

*1.1.12 REMARK* The reader can prove similar properties for the intersection; for example

1'. $A \cap B = B \cap A$

2'. $A \cap A = A$

*1.1.13 PROPOSITION* If A, B, and C are arbitrary sets, then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

*Proof.* We prove only the first assertion; the second relation can be proved in a similar way.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and x is also in B or in C. Suppose, for example, that $x \in B$. The case $x \in C$ is similar. In this case $x \in A \cap B$ and thus $x \in (A \cap B) \cup (A \cap C)$. In this way, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let us now suppose that x is in $(A \cap B) \cup (A \cap C)$ and for simplicity x is in $A \cap B$. In this case it is clear that x is in $(A \cup B) \cap (A \cup C)$. The proposition is proved.

*1.1.14 DEFINITION* Two sets A and B are called *disjoint* if $A \cap B = \emptyset$; if $\mathcal{A}$ is a family of sets such that for any sets A, B in $\mathcal{A}$ we have $A \cap B = \emptyset$, we say that $\mathcal{A}$ has *pairwise disjoint* sets.

Suppose that we have a set T and we consider $\mathcal{P}(T)$. For any $A \in \mathcal{P}(T)$ we can define a new set, called the *complement* of A, by the relation,

$$C_A = \{x \mid x \in T, x \notin A\}$$

It is easy to see that the following properties hold:

1. $C_{A \cup B} = C_A \cap C_B$

2. $C_{A \cap B} = C_A \cup C_B$

Also it is clear that these relations hold for the case of families of sets. Since the proof is easy we omit this.

The set $A \Delta B$ (the symmetric difference of the sets A and B) is defined by

$$A \Delta B = \{x \mid x \in A \cup B,\ x \notin A \cap B\}$$

It is obvious that $A \Delta B = B \Delta A$.

If T is a given set, then in $\mathcal{P}(T)$ we have several types of families of sets. Among these we mention two which are very useful in measure theory: the ring of sets and the σ-ring (σ-algebra).

*1.1.15 DEFINITION* A family of sets $\mathcal{R} \subseteq \mathcal{P}(T)$ is called a *ring* of sets if for any A and B in $\mathcal{R}$ the following sets are also in $\mathcal{R}$:

$$A \cup B \qquad A \cap C_B$$

*1.1.16 DEFINITION* A ring of sets $\mathcal{B}$ is called a *σ-algebra* if for any $A_n \in \mathcal{B}$, $\cup_n A_n$ is again an element of $\mathcal{B}$.

## 1.2 PARTIALLY ORDERED SETS

Let A be a nonempty set.

*1.2.1 DEFINITION* A *relation* on A is a collection of ordered pairs (x, y) of elements of A.

*1.2.2 DEFINITION* A *partially ordered set* is a nonempty set with a relation denoted "$\leq$" such that the following properties hold:

1. If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).
2. $a \leq a$ for all $a \in A$ (reflexivity).
3. If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).

*1.2.3 DEFINITION* A *totally ordered set* is any partially ordered set $A$ (with the relation $\leq$) with the property that for any pair a, b of elements of A, we have $a \leq b$ or $b \leq a$.

*1.2.4 EXAMPLE* The set of all real numbers R is a totally ordered set. The relation "$\leq$" is defined as follows: We say that two real numbers are in the relation $a \leq b$ if the difference $b - a$ is a positive number.

The set of all complex numbers is a partially ordered set.

Here we have several possible ways to define the relation $\leq$; for example, we can define the relation "$\leq$" as follows: Two complex numbers $z_1$ and $z_2$ are in the relation $z_1 \leq z_2$ if $\text{Re } z_1 \leq \text{Re } z_2$, where Re z is the real part of z.

If T is any set, then we can define in $P(T)$ a relation in the following way: If A and B are elements of $P(T)$, then we say that $A \leq B$ if A is a subset of B. In this case $P(T)$ is a partially ordered set. In this case we say also that we have an ordering by *inclusion*.

*1.2.5 DEFINITION* If a is a partially ordered set and $A_1 \subset A$, then an element $a \in A$ is said to be an *upper bound* for $A_1$ if $a_1 \leq a$ for all $a_1 \in A_1$.

The element $\tilde{a}$ is called a *least upper bound* of $A_1$

1. If $\tilde{a}$ is an upper bound of $A_1$.
2. If $a_1$ is another upper bound of $A_1$, then $\tilde{a} \leq a_1$.

The least upper bound is denoted, generally, by lub $A_1$.

The element $b \in A$ is called a *lower bound* of $A_1$ if $b \leq a_1$ for all $a_1 \in A_1$ and an element $\tilde{b} \in A$ is called the *greatest lower bound*

1. If $\tilde{b}$ is a lower bound.
2. If $b_1$ is another lower bound, then $b_1 \leq \tilde{b}$.

The greatest lower bound is denoted, generally, by glb $A_1$.

*1.2.6 DEFINITION* An element x of a partially ordered set A is called *maximal* if $x \leq y$ implies $y \leq x$.

Similarly we can define the notion of *minimal* element.

*1.2.7 DEFINITION* A *chain* in a partially ordered set is any subset C of A such that the relation order "$\leq$" of A, restricted to C, gives that C with this relation is a totally ordered set.

One of the most important axioms in set theory is the axiom of E. Zermelo and is called the *axiom of choice*. There exist several equivalent formulations of this axiom and here we quote without proof only two. First we define the notion of cartesian product. For I, any set, we suppose that for each $i \in I$ there exists a set

$A_i$. The cartesian product $\Pi_{i \in I} A_i$ is the set of all functions f defined on I such that,

$$f(i) = f_i \in A_i$$

for each $i \in I$; $f_i$ is called the i coordinate of f.

Now we are ready to give the two formulations of the choice axiom.

*1.2.8 CHOICE AXIOM* The cartesian product of any nonvoid family of nonvoid sets is a nonvoid set.

*1.2.9 ZORN'S LEMMA* If A is a partially ordered set such that for every chain there exists an upper bound, then A has a maximal element.

In several sections of this book we apply these assertions; we mention here the application in the proof of the Hahn-Banach theorem.

## 1.3 TOPOLOGY AND TOPOLOGICAL SPACES

In analysis we use intensively the notion of convergence. For example, we say that a sequence of complex numbers or real numbers (or a sequence of real- or complex-valued functions) converge.

Also the notion of convergence is used to characterize certain important classes of functions. For example, a function F: $[0, 1] \to R$ is continuous if and only if for each $s \in [0, 1]$ and any $s_n \to s$, $f(s_n) \to f(s)$.

These situations and others lead to an axiomatic treatment of the notion of convergence, and one of the basic settings in which this is best realized is in metric spaces.

*1.3.1 DEFINITION* A metric space $(X, \rho)$ is a pair, where X is a nonempty set and $\rho$ is a real-valued function on $X \times X$ with the following properties:

1. $\rho(x, y) = 0$ if and only if $x = y$
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$