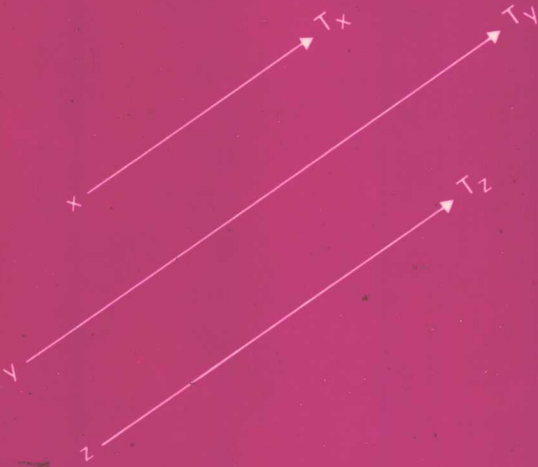


PURE AND APPLIED MATHEMATICS

A Series of Monographs and Textbooks



INTRODUCTION TO LINEAR OPERATOR THEORY

573

Vasile I. Istrăţescu

Introduction to Linear Operator Theory

VASILE I. ISTRĂȚESCU

MARCEL DEKKER, INC. New York and Basel

Library of Congress Cataloging in Publication Data

Istrăţescu, Vasile I.

Introduction to linear operator theory.

(Pure and applied mathematics ; 65)

Includes bibliographical references and indexes.

1. Linear operators. 2. Hilbert space. 3. Banach spaces. I. Title.

QA329.2.I8915

515.7'246

80-29511

ISBN 0-8247-6896-5

COPYRIGHT © 1981 by MARCEL DEKKER, INC. ALL RIGHTS RESERVED

Neither this book nor any part may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, microfilming, and recording, or by any information storage and retrieval system, without permission in writing from the publisher.

MARCEL DEKKER, INC.

270 Madison Avenue, New York, New York 10016

Current printing (last digit):

10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

*To my parents,
Paraschiva Istrăţescu and Ion Istrăţescu*

PREFACE

During the last 15 or 20 years much progress has been made in the theory of non-self-adjoint operators on Hilbert and Banach spaces. The present volume is intended to provide an introduction to the subject.

The first four chapters are devoted to standard material on linear functional analysis. However, whenever possible a unified approach is used, and this is the case, for example, for the three basic theorems of linear functional analysis which are treated consequences of a unique theorem. Chapter 5 is concerned with the general spectral representation of operators on Hilbert spaces and depends on the Banach algebra theory developed in Chapter 4.

Chapter 6 is concerned with the basic notion of this book: the numerical range. First this is considered for Hilbert spaces and next for Banach spaces. Various classes of operators connected with the numerical range are also discussed.

The classes of nonnormal operators have a long history, and the problem of deciding when an operator is normal (also hermitian, unitary) forms the content of Chapters 7 and 8.

As is well known, the class of hermitian operators has many important applications in various branches of mathematics and physics; thus related classes of operators for which many properties of hermitian operators are preserved are of great interest. Chapter 9 gives an account of results involving such classes as

well as some applications (for example, a simple proof of an interpolation theorem of Lions-Peetre).

In Chapter 10 the famous invariant subspace problem is discussed and some structure theorems are presented.

The Weyl spectrum of an operator is discussed in Chapter 11, as well as some applications. The elements of the von Neumann algebras are also given.

Chapter 12 is concerned with an important and useful notion: analytic and quasi-analytic vectors; also some applications are given. In Chapter 13 the Banach space version of the famous Schwarz theorem from complex function theory is presented. In Chapter 14 results on maximum theorems for operator-valued functions are given.

In the last chapter, I present some ergodic theorems for classes of operators containing the quasi-compact operators. The results presented are connected with the operator theoretic treatment of Markov processes as given by Kakutani-Yosida and refined by many authors. I have tried to indicate the origin of the various results, and the references (which in turn contain references to many earlier results) may be used to obtain further information. When I make no ascription, I am not claiming originality.

A part of this book has its origin in a course given by the author at the Centro Linceo Interdisciplinare di Scienze Matematiche e loro Applicazioni of the Accademia Nazionale dei Lincei (Rome, Italy).

For interesting and helpful conversations I am indebted to many friends. For discussions and constant encouragement I am indebted to Professor G. Köthe.

Vasile I. Istrăţescu

CONTENTS

PREFACE		v
1	PRELIMINARIES: SET THEORY AND GENERAL TOPOLOGY	1
1.1	The Algebra of Sets	1
1.2	Partially Ordered Sets	4
1.3	Topology and Topological Spaces	6
1.4	Baire's Theorem	16
2	BANACH SPACES	19
2.1	Linear Spaces	19
2.2	Linear Independence	21
2.3	Sets in Linear Spaces	22
2.4	Classes of Spaces: Isomorphic Spaces, Quotient Spaces, and Complementary Spaces	25
2.5	Seminorms and Norms on Linear Spaces	27
2.6	Linear Topological Spaces	29
2.7	Banach Spaces	30
2.8	Linear Operators on Banach Spaces	32
2.9	Uniformly Convex and Rotund Banach Spaces: Some Generalizations	35
2.10	The Hahn-Banach Extension Theorem	39
2.11	Extension Theorems for Complex Banach Spaces	43
2.12	Three Basic Theorems of Linear Analysis	45
2.13	Convergence in Banach Spaces	56
2.14	The Adjoint of an Operator	63
2.15	The Spectrum of an Operator	65
2.16	The Local Spectrum of an Operator	70
2.17	Analytic Representation of the Dual of Some Banach Spaces	76
2.18	Measures of Noncompactness and Classes of Mappings on Banach Spaces	84

3	HILBERT SPACES	109
3.1	Inner Products on Linear Spaces	109
3.2	Orthonormal Bases and the Bessel Inequality	
3.3	Separable Hilbert Spaces: Gram-Schmidt Orthogonalization Method	119
3.4	Orthogonal Subspaces of a Hilbert Space	125
3.5	The Dual of a Hilbert Space	126
3.6	Classes of Bounded Linear Operators on Hilbert Spaces	129
4	BANACH ALGEBRAS	142
4.1	Definitions and Some Examples	142
4.2	The Spectrum of an Element in a Banach Algebra with Unit	147
4.3	Representation Theorems for Commutative Banach Algebras	151
4.4	Structure Theorems for Commutative Banach Algebras	154
4.5	Representation Theorems for Noncommutative Banach Algebras	167
5	SPECTRAL REPRESENTATION OF OPERATORS ON HILBERT SPACES	174
5.1	Semispectral and Spectral Families of Radon Measures	174
5.2	Measurability and Integrability with Respect to Spectral Families	179
5.3	A Representation Theorem for L^∞	181
5.4	Spectral Decomposition of Some Classes of Operators	188
5.5	Some Remarks on the Spectral Mapping Theorem for Hermitian and Normal Operators	191
6	THE NUMERICAL RANGE	200
6.1	The Numerical Range for Bounded Linear Operators on Hilbert Spaces	200
6.2	The Numerical Range and the Spectrum	206
6.3	The Numerical Range and Its Closure	214
6.4	The Essential Numerical Range for Bounded Linear Operators on Hilbert Spaces	217
6.5	The Maximal Numerical Range of a Bounded Operator on a Hilbert Space	219
6.6	The Extreme Points of the Numerical Range for Hyponormal Operators and (WN) Operators	221
6.7	The Numerical Range and Some Classes of Operators	223
6.8	The Numerical Range and Tensor Products	224

6.9	The Numerical Range for Bounded Linear Operators on Banach Spaces	226
6.10	The Exponential Function on the Set of All Bounded Linear Operators on a Banach Space	233
6.11	The Numerical Radius, the Spectral Radius, and the Norm of a Bounded Linear Operator on a Banach Space	236
6.12	Hermitian and Normal Operators on Banach Spaces	238
6.13	Normal Operators on Banach Spaces	244
6.14	Classes of Elements in Banach Algebras with Unit: The Vidav-Palmer Theorem	246
6.15	Some Properties of Hermitian and Normal Elements of a Banach Algebra	255
6.16	The Numerical Radius and the Iterates of an Element	260
6.17	The Numerical Range of Elements of Locally m -Convex Algebras	262
7	NONNORMAL CLASSES OF OPERATORS	265
7.1	Classes of Nonnormal Operators	266
7.2	Spectral Sets and Dilations of Operators	275
7.3	Operators with G_1 Property and Some Generalizations	286
7.4	Operators with Property $\text{Re } \sigma(T) = \sigma(\text{Re } T)$	297
7.5	The Class \bar{R}_1	302
7.6	Other Classes of Bounded Operators	307
8	CONDITIONS IMPLYING NORMALITY	310
8.1	Conditions Implying Hermitianity	310
8.2	Conditions Implying Unitarity	317
8.3	Conditions Implying Normality	321
9	SYMMETRIZABLE OPERATORS: GENERALIZATIONS AND APPLICATIONS	344
9.1	Symmetrizable Operators on Hilbert Spaces	344
9.2	Symmetrizable Elements in Banach Algebras	349
9.3	Inner Products on Banach Spaces: Symmetrizable Operators and Some Generalizations	352
9.4	Some Applications of Symmetrizable Operators and Quasi-Normalizable Operators	369
9.5	Further Results on Symmetrizable Operators on Hilbert Spaces	374

10	INVARIANT SUBSPACES AND SOME STRUCTURE THEOREMS	380
	10.1 Invariant Subspaces: Some Existence Theorems	380
	10.2 Reducing Invariant Subspaces	398
	10.3 Some Structure Theorems	408
11	THE WEYL SPECTRUM OF AN OPERATOR	412
	11.1 Preliminaries and Some General Results	412
	11.2 Weyl's Theorem	420
	11.3 Weyl's Theorem for Some Classes of Operators	426
	11.4 The Weyl Spectrum of an Element in a von Neumann Algebra	433
	11.5 The von Neumann Theorem	439
12	ANALYTIC AND QUASI-ANALYTIC VECTORS	444
	12.0 Introduction	444
	12.1 Self-Adjoint Operators	446
	12.2 Classes of Vectors for an Operator	452
	12.3 Analytic and Quasi-Analytic Vectors and Essentially Self-Adjoint Operators	455
	12.4 Quasi-Analytic Vectors and Semigroups of Operators	463
	12.5 Analytic and Quasi-Analytic Elements in Commutative Banach Algebras	466
13	SCHWARZ NORMS	468
	13.1 Schwarz Norms	469
	13.2 A New Class of Schwarz Norms	476
	13.3 Schwarz Norms on Banach Spaces	478
14	MAXIMUM THEOREMS FOR OPERATOR-VALUED HOLOMORPHIC FUNCTIONS	483
	14.1 Holomorphic Functions	483
	14.2 Subharmonic Functions	485
	14.3 Maximum Theorems for the Norm	498
	14.4 Maximum Theorems for the Spectral Radius and for the Essential Spectral Radius	508
	14.5 Maximum Theorems for Other Operator-Valued Holomorphic Functions	528
15	UNIFORM ERGODIC THEOREMS FOR SOME CLASSES OF OPERATORS	537
	15.1 Classes of Operators	537
	15.2 Applications to Markov Processes	545

Contents	xi
APPENDIX. C _p CLASSES	549
REFERENCES	553
SYMBOL INDEX	573
SUBJECT INDEX	575

PRELIMINARIES: SET THEORY AND GENERAL TOPOLOGY

1.1 THE ALGEBRA OF SETS

In what follows we assume that the reader is familiar with the notion of a set. We give several examples of sets.

1.1.1 *EXAMPLE* The letters of the English alphabet form a set.

1.1.2 *EXAMPLE* The rational numbers in the interval $[0, 1] = \{x \mid 0 \leq x \leq 1\}$ form a set.

1.1.3 *EXAMPLE* The numbers of the form $2n$, where n is an integer, also form a set.

We consider a set A as being given when we can identify its elements; for example, the set E has the elements a, b, c, \dots , or, in brief,

$$E = \{a, b, c, \dots\}$$

Another way is to identify the elements of a set by a property P , or in brief,

$$E = \{a \mid a \text{ has the property } P\}$$

Generally speaking, we adhere to the standard notational conventions: we denote by lowercase letters a, b, c, \dots the elements of a set E and we write this as $a \in E, b \in E, c \in E, \dots$. Sets are denoted by uppercase letters A, B, C, \dots, X, Y, Z and families of sets by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ (script uppercase letters). We write \emptyset for

the empty set, and for any element a , $\{a\}$ denotes the set which has as an element only the element a .

1.1.4 DEFINITION Let A and B be two sets. We say that A is a subset of B if for any $a \in A$ we have $a \in B$. We write this as $A \subseteq B$. If there exists an element $b \in B$, b not in A , then we say that A is a *proper* subset of B .

If an element a is not in a set A , we write this as $a \notin A$.

1.1.5 DEFINITION If A and B are two sets, we say that $A = B$ if $A \subseteq B$ and $B \subseteq A$; in the contrary case, we say that A and B are distinct sets.

1.1.6 DEFINITION If A and B are two sets, then $A \cup B$ denotes the set of all elements which are in A or in B ; $A \cap B$ denotes the set of all elements which are in A and in B .

The set $A \cup B$ is called the *union* of the sets A and B ; $A \cap B$ is called the *intersection* of the sets A and B .

1.1.7 REMARK Similar definitions can be given for the case when we have a family of sets, $A = \{A_\alpha\}_{\alpha \in I}$ for $\bigcup_\alpha A_\alpha$ and $\bigcap_\alpha A_\alpha$.

1.1.8 EXAMPLE If $A = \{1, 2, 3\}$ and $B = \{2, 5\}$, then

$$A \cup B = \{1, 2, 3, 5\} \quad A \cap B = \{2\}$$

If $A = \{1, 2\}$ and $B = \{3, 5\}$, then

$$A \cup B = \{1, 2, 3, 5\} \quad A \cap B = \emptyset$$

1.1.9 DEFINITION For any set A we note $\mathcal{P}(A)$, the family of all subsets of A .

1.1.10 EXAMPLE For $A = \{2\}$ we have

$$\mathcal{P}(A) = \{\emptyset, \{2\}\}$$

1.1.11 PROPOSITION If A , B , and C are sets, then the following relations hold:

1. $A \cup B = B \cup A$
2. $A \cup A = A$
3. $A \cup \emptyset = A$

4. $A \cup (B \cup C) = (A \cup B) \cup C$
5. $A \subseteq A \cup B$
6. $A \subseteq B$ if and only if $A \cup B = B$

1.1.12 *REMARK* The reader can prove similar properties for the intersection; for example

- 1'. $A \cap B = B \cap A$
- 2'. $A \cap A = A$

1.1.13 *PROPOSITION* If A , B , and C are arbitrary sets, then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof. We prove only the first assertion; the second relation can be proved in a similar way.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and x is also in B or in C .

Suppose, for example, that $x \in B$. The case $x \in C$ is similar.

In this case $x \in A \cap B$ and thus $x \in (A \cap B) \cup (A \cap C)$. In this way, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let us now suppose that x is in $(A \cap B) \cup (A \cap C)$ and for simplicity x is in $A \cap B$. In this case it is clear that x is in $(A \cup B) \cap (A \cup C)$. The proposition is proved.

1.1.14 *DEFINITION* Two sets A and B are called *disjoint* if $A \cap B = \emptyset$; if \mathcal{A} is a family of sets such that for any sets A, B in \mathcal{A} we have $A \cap B = \emptyset$, we say that \mathcal{A} has *pairwise disjoint* sets.

Suppose that we have a set T and we consider $\mathcal{P}(T)$. For any $A \in \mathcal{P}(T)$ we can define a new set, called the *complement* of A , by the relation,

$$C_A = \{x \mid x \in T, x \notin A\}$$

It is easy to see that the following properties hold:

1. $C_{A \cup B} = C_A \cap C_B$
2. $C_{A \cap B} = C_A \cup C_B$

Also it is clear that these relations hold for the case of families of sets. Since the proof is easy we omit this.

The set $A \Delta B$ (the symmetric difference of the sets A and B) is defined by

$$A \Delta B = \{x \mid x \in A \cup B, x \notin A \cap B\}$$

It is obvious that $A \Delta B = B \Delta A$.

If T is a given set, then in $\mathcal{P}(T)$ we have several types of families of sets. Among these we mention two which are very useful in measure theory: the ring of sets and the σ -ring (σ -algebra).

1.1.15 DEFINITION A family of sets $\mathcal{R} \subseteq \mathcal{P}(T)$ is called a *ring* of sets if for any A and B in \mathcal{R} the following sets are also in \mathcal{R} :

$$A \cup B \quad A \cap B$$

1.1.16 DEFINITION A ring of sets \mathcal{B} is called a σ -algebra if for any $A_n \in \mathcal{B}$, $\bigcup_n A_n$ is again an element of \mathcal{B} .

1.2 PARTIALLY ORDERED SETS

Let A be a nonempty set.

1.2.1 DEFINITION A *relation* on A is a collection of ordered pairs (x, y) of elements of A .

1.2.2 DEFINITION A *partially ordered set* is a nonempty set with a relation denoted " \leq " such that the following properties hold:

1. If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).
2. $a \leq a$ for all $a \in A$ (reflexivity).
3. If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).

1.2.3 DEFINITION A *totally ordered set* is any partially ordered set A (with the relation \leq) with the property that for any pair a, b of elements of A , we have $a \leq b$ or $b \leq a$.

1.2.4 EXAMPLE The set of all real numbers \mathbb{R} is a totally ordered set. The relation " \leq " is defined as follows: We say that two real numbers are in the relation $a \leq b$ if the difference $b - a$ is a positive number.

The set of all complex numbers is a partially ordered set.

Here we have several possible ways to define the relation \leq ; for example, we can define the relation " \leq " as follows: Two complex numbers z_1 and z_2 are in the relation $z_1 \leq z_2$ if $\text{Re } z_1 \leq \text{Re } z_2$, where $\text{Re } z$ is the real part of z .

If T is any set, then we can define in $\mathcal{P}(T)$ a relation in the following way: If A and B are elements of $\mathcal{P}(T)$, then we say that $A \leq B$ if A is a subset of B . In this case $\mathcal{P}(T)$ is a partially ordered set. In this case we say also that we have an ordering by *inclusion*.

1.2.5 DEFINITION If a is a partially ordered set and $A_1 \subset A$, then an element $a \in A$ is said to be an *upper bound* for A_1 if $a_1 \leq a$ for all $a_1 \in A_1$.

The element \tilde{a} is called a *least upper bound* of A_1

1. If \tilde{a} is an upper bound of A_1 .
2. If a_1 is another upper bound of A_1 , then $\tilde{a} \leq a_1$.

The least upper bound is denoted, generally, by $\text{lub } A_1$.

The element $b \in A$ is called a *lower bound* of A_1 if $b \leq a_1$ for all $a_1 \in A_1$ and an element $\tilde{b} \in A$ is called the *greatest lower bound*

1. If \tilde{b} is a lower bound.
2. If b_1 is another lower bound, then $b_1 \leq \tilde{b}$.

The greatest lower bound is denoted, generally, by $\text{glb } A_1$.

1.2.6 DEFINITION An element x of a partially ordered set A is called *maximal* if $x \leq y$ implies $y \leq x$.

Similarly we can define the notion of *minimal* element.

1.2.7 DEFINITION A *chain* in a partially ordered set is any subset C of A such that the relation order " \leq " of A , restricted to C , gives that C with this relation is a totally ordered set.

One of the most important axioms in set theory is the axiom of E. Zermelo and is called the *axiom of choice*. There exist several equivalent formulations of this axiom and here we quote without proof only two. First we define the notion of cartesian product. For I , any set, we suppose that for each $i \in I$ there exists a set

A_i . The cartesian product $\prod_{i \in I} A_i$ is the set of all functions f defined on I such that,

$$f(i) = f_i \in A_i$$

for each $i \in I$; f_i is called the i coordinate of f .

Now we are ready to give the two formulations of the choice axiom.

1.2.8 CHOICE AXIOM The cartesian product of any nonvoid family of nonvoid sets is a nonvoid set.

1.2.9 ZORN'S LEMMA If A is a partially ordered set such that for every chain there exists an upper bound, then A has a maximal element.

In several sections of this book we apply these assertions; we mention here the application in the proof of the Hahn-Banach theorem.

1.3 TOPOLOGY AND TOPOLOGICAL SPACES

In analysis we use intensively the notion of convergence. For example, we say that a sequence of complex numbers or real numbers (or a sequence of real- or complex-valued functions) converge.

Also the notion of convergence is used to characterize certain important classes of functions. For example, a function $F: [0, 1] \rightarrow \mathbb{R}$ is continuous if and only if for each $s \in [0, 1]$ and any $s_n \rightarrow s$, $f(s_n) \rightarrow f(s)$.

These situations and others lead to an axiomatic treatment of the notion of convergence, and one of the basic settings in which this is best realized is in metric spaces.

1.3.1 DEFINITION A metric space (X, ρ) is a pair, where X is a nonempty set and ρ is a real-valued function on $X \times X$ with the following properties:

1. $\rho(x, y) = 0$ if and only if $x = y$
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$