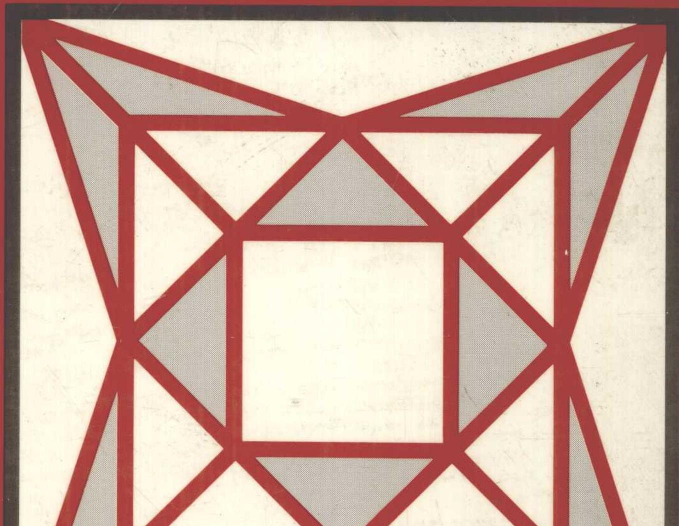
MATRIX STRUCTURAL ANALYSIS



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PREFACE

This book was written for the student and practicing engineer who wish to use matrix methods of structural analysis to predict the static response of structures. The text is introductory, emphasizes the stiffness method, and contains the fundamentals of the flexibility method.

The general theory of the stiffness method is initially derived from the intuitive concepts of the direct solution of the basic equations of equilibrium, compatibility, and material properties. The theory is presented and explained using truss behavior. Thus, at the beginning of the book the reader has the opportunity to observe the method unencumbered by generalized arguments. Subsequently, the principle of virtual work is explained and offered as an alternative theoretical basis for the stiffness method. The flexibility method is similarly derived by direct solution of the basic equations and from the principle of complementary virtual work. Applications of the stiffness method are given for beams, planar frames, space trusses, beam grid works, and space frames. Miscellaneous topics required to complete our coverage of the stiffness method are also described.

The three principal aspects of analyzing structures using matrix methods are: (a) understanding the method (i.e., the theory plus its limitations and applications); (b) developing appropriate computer programs; and (c) solving actual structures on the computer—this involves idealizing the problem, preparing and investigating the data, information processing, and the numerical methods necessary to obtain the solution. Each component is a vital link in implementing and using matrix methods of structural analysis for routine production problems. The material in this text is devoted to an understanding of the method with an appreciation for writing computer programs and using production-level programs to solve actual structures. The stiffness and flexibility methods are cast in a form appropriate for use on the computer, but the details of computer implementation are not discussed. It is occasionally useful for the student to understand the power of the method by investigating structures that require a computer; a sufficient number of exercise problems of this type are provided. We urge the reader to use available software to carry out long tedious computations. For example, programs such as CAL86,[1] used in the mode where only matrices are manipulated, require the user to load the basic matrices and write the appropriate code in the metalanguage of the program to attain the required solution.

The contents of the main body of the book are divided into three categories. The materials in Chapter 1 serve to orient the student to matrix structural analysis and provide a basic introduction and appreciation for the history and scope of the various theorems and methods. The development of the stiffness method in Chapters 2 and 3 using the basic equations and energy meth-

[1]Wilson, E. L. *CAL86 Computer Assisted Learning of Structural Analysis and the CAL/SAP Development System,* Report No. UCB/SESM-86/05, Berkeley, Calif., 1986.

ods, respectively, is strongly tied to practical structures. Chapters 4 and 5 reinforce the theory and give definite applications of the method for various types of structures. The special topics of the stiffness method have been assembled in Chapter 6; thus, the orderly flow of the development is not disrupted, and these important aspects of solving problems are treated together. Chapter 7 on the flexibility method using the solution of the basic equations and the principle of complementary virtual work parallels the stiffness method derivation in Chapters 2 and 3. The theory in this chapter is strongly connected to actual structural problems. The appendix materials complement the main body of the book and give the reader opportunities to study the solution of linear algebraic equations and review elementary matrix operations.

A broad collection of examples demonstrates the principles and assists the reader in developing an active understanding of the concepts. An unlimited number of exercise problems can be obtained through the program shell PANDORAS BOX (available from the author). This program contains a menu of structural groups and subgroups that are illustrated in the text (i.e., appropriate configurations of trusses, beams, and frames will appear at the end of each chapter). The program also contains a random number generator, an executable segment to solve for the structural deflections and forces, plus supporting graphics. An instructor equipped with a disk of PANDORAS BOX can select the category of exercise problem, and the program responds with a graphics display of the structure, the problem data, and the solution. Problem dimensions, loads, etc., are randomly generated; therefore, a different problem can be assigned to each student, and problems need never be repeated from year to year.

The book is designed to be used in aerospace engineering, civil engineering, mechanical engineering, and engineering science curricula. A logical progression of topics with a uniform and continuous flow of information can be obtained by exercising modest discretion in selecting individual chapters or sections. An introductory course in matrix structural analysis with an emphasis on problem-solving skills can be constructed using material from Chapters 1, 2, 4, and 5, with topics such as matrix condensation, release of generalized member nodal forces, and nodal coordinates selected from Chapter 6. By omitting Chapter 5 and including Chapter 3, one obtains a more theoretically oriented presentation. In contrast, an energy orientation toward the subject is obtained by choosing Chapters 1 and 3, plus selections from Chapters 5, 6, and 7. These skeletal outlines can be expanded with additional topics for multicourse sequences, and the book is also arranged so that a cover-to-cover study is possible. It is also envisaged that the book can be used effectively in self-study programs.

Many people have played a role in writing this book. I express my sincere thanks for the valuable suggestions of the reviewers: Dr. Dan Frangopol, University of Colorado; Dr. Daniel L. Garber, University of Maryland; Dr. James K. Nelson, Texas A&M University; and Dr. Jay A. Puckett, University of Wyoming. Finally, I extend special thanks to the students at the University of Idaho and the University of Wyoming for enduring the inconvenience of studying from the manuscript form of this book.

Ronald L. Sack

SYMBOLS AND NOTATION

Symbols are generally defined where they first appear. Some symbols have been used in different contexts to define several quantities. In general we have used lower-case symbols to indicate quantities associated with element coordinates and capital letters for global quantities. We use **p** (**P**) to indicate nodal applied forces (both loads and reactions) and **u** (**U**) to denote nodal displacements; **k** (**K**) contains the stiffness elements and **f** (**F**) denotes the flexibility matrix. Matrices are shown in bold print, and the elements of a column matrix are written within brackets, { }, to conserve space in the text.

$\mathbf{a}$	Kinematics matrix (partitioned into $\mathbf{a}_0$, and $\mathbf{a}_1$); matrix of coefficients for polynomial
A	Cross-sectional area of a member
b	Member width
$\mathbf{b}$	Statics matrix (partitioned into $\mathbf{b}_0$ and $\mathbf{b}_1$)
$\mathbf{B}$	Matrix relating nodal displacements to element strains
$\mathbf{d}$	Column matrix of element deformations ($\mathbf{d} = \mathbf{aU}$)
$\mathbf{d}^o$	Column matrix of initial element deformations
$\mathbf{e}$	Element force transformation matrix for global displacements
$\bar{\mathbf{e}}$	Element force transformation matrix for local displacements
E	Modulus of elasticity (i.e., Young's modulus)
$\mathbf{E}$	Matrix of elastic constants
$\mathbf{f}$	Element flexibility matrix
$\mathbf{F}$	Global flexibility matrix
G	Modulus of elasticity in shear
I	Moment of inertia
$\mathbf{I}$	Identity (unit) matrix
J	St. Venant's torsion constant
$\mathbf{k}$	Element stiffness matrix with elements k_{ij} expressed in global coordinates
$\bar{\mathbf{k}}$	Element stiffness matrix with elements $\bar{k}_{ij}$ expressed in local coordinates
$\mathbf{K}$	Structural stiffness matrix with elements K_{ij} expressed in global coordinates
L	Length
M	Bending moment
NDOF, NE, NN, NR, NOK, NOS	Number of: degrees of freedom; elements; nodes; reactions; kinematic indeterminacies; static indeterminacies
N	Element axial force

N Column matrix of shape functions

O Null matrix

p Column matrix of nodal element forces in global coordinates

$\bar{\mathbf{p}}$ Column matrix of nodal element forces in local coordinates

$\mathbf{p}^o$ Column matrix of initial nodal element forces in global coordinates

$\bar{\mathbf{p}}^o$ Column matrix of initial nodal element forces in local coordinates

P Column matrix of applied nodal forces in global coordinates ($\mathbf{s} = \mathbf{bP} = \mathbf{b}_0\mathbf{P} + \mathbf{b}_1\mathbf{X}$)

$\mathbf{P}_f$ Column matrix of known applied nodal forces in global coordinates

$\mathbf{P}_s$ Column matrix of unknown applied nodal forces in global coordinates

$\mathbf{P}^o$ Column matrix of initial forces in global coordinates

q Distributed load magnitude

R Column matrix of reaction forces

s Column matrix of element forces and reactions

T Temperature

T Transformation matrix

U_i, V_i, W_i Displacements at node i for a structure in the x, y, and z directions, respectively

u, v, w Continuous functions expressing displacements in the x, y, and z directions, respectively

u_i, v_i, w_i Displacements at node i for an element in the x, y, and z directions, respectively

$\bar{u}_i, \bar{v}_i, \bar{w}_i$ Displacements at node i for an element in the $\bar{x}$, $\bar{y}$, and $\bar{z}$ directions, respectively

u Column matrix of nodal displacements in global coordinates

$\bar{\mathbf{u}}$ Column matrix of nodal displacements in local coordinates

U Column matrix of nodal displacements for the entire structure in global coordinates

$\mathbf{U}_f$ Column matrix of unknown displacements in global coordinates

$\mathbf{U}_s$ Column matrix of known displacements in global coordinates

V Shear force

W_e, W_e^* Work and complementary work done by external forces

W_i, W_i^* Strain energy and complementary strain energy

$\bar{x}, \bar{y}, \bar{z}$ Orthogonal cartesian global (structural) coordinates

$\bar{x}, \bar{y}, \bar{z}$ Orthogonal cartesian local coordinates

X Column matrix of redundant forces

SUBSCRIPTS

i	The node (point) associated with the quantity
f	Degrees of freedom with known forces and unknown displacements
s	Degrees of freedom with unknown forces and known displacements
0	Quantity associated with the primary structure; used in the flexibility method
1	Quantity associated with the structural redundants; used in the flexibility method

SUPERSCRIPTS

E	Force (or moment) that is equivalent in an energy sense to a distributed loading
F	Force (or moment) required to give zero displacement at the point (i.e., a fixed-end force)
ij	The interval (element) associated with the quantity
o	Quantity initially introduced by temperature, fabrication error, precambering, etc.
T	Transpose of a matrix
-1	Inverse of a matrix

GREEK SYMBOLS

α (alpha)	Coefficient of linear thermal expansion
α,β,γ	Angles measure to a vector from the positive x, y, and z axes, respectively
γ (gamma)	Shear strain
Γ (gamma)	Matrix of direction cosines; an orthogonal transformation
δ (delta)	Deflection; increment of a quantity; first variation of a quantity (a virtual quantity)
Δ (delta)	Deflection; total change in a quantity
ϵ (epsilon)	Translational strain
θ (theta)	Angle; rotation of node with respect to local coordinates
Θ (theta)	Angle; rotation of node with respect to global coordinates
κ (kappa)	Curvature of a beam
κ_v (kappa)	Shear constant
λ,μ,ν	Direction cosines (i.e., cos α, cos β, cos γ, respectively)
ν (nu)	Poisson's ratio
σ (sigma)	Normal stress
Σ (sigma)	Summation of quantities

CONTENTS

1 INTRODUCTION

Engineered structures must ensure the safety and welfare of the occupants and general public by performing in a prescribed manner. Strength requirements are accompanied by stiffness constraints to prevent excessive deflections, bouncy floors, outward-tilting walls, uncomfortable structural oscillations, and the like. Thus structural analysis and design are intertwined since behavior is affected by the arrangement of members and distribution of materials. New complex systems require more precise engineering; many major contemporary structures, such as the Boeing 747 aircraft, the Swiss Flesenau Bridge, and the Sears Tower, owe their existence to computer-oriented structural analysis and design. This chapter examines the origins and utility of matrix structural analysis, and its relationship to classical methods.

1.1 HISTORICAL CONTEXT OF MATRIX STRUCTURAL ANALYSIS

The airplane and digital computer are responsible for revolutionizing structural analysis. In the 1940s and 1950s structural engineers were confronted with two highly statically indeterminate systems: the swept-wing and delta-wing aircraft. The governing equations were cast *ab initio* (from the beginning) in matrix format, but this approach required solution of large sets of simultaneous linear algebraic equations. At the time, relaxation methods were used extensively to solve the governing equations of structural behavior; therefore, the requirement to deal with great numbers of algebraic equations was an anathema to the engineer. Fortuitously, the University of Pennsylvania unveiled the 30-ton ENIAC digital computer in 1946. The invention of the transistor in 1947 and the silicon chip in 1959 were pivotal discoveries that accelerated the development of the digital computer and gave impetus to the structural analysis revolution. By embracing this new computing technology, the structural engineers of those two decades completely changed structural analysis. Trusses, beams, and frames were initially investigated, but in the mid-1950s a group at the Boeing Company demonstrated that the procedure could be extended to continua. Common usage now dictates that *matrix structural analysis* designates investigations of structures composed of articulated or discrete components, whereas the *finite element method* denotes analysis of continua.

Structural analysis of the early swept-wing aircraft in 1947 depended upon work from the 1800s by James Clerk Maxwell and Otto Mohr. Their method of *consistent displacements* is an example of a classical *compatibility method* yielding sets of simultaneous linear algebraic equations, with the structural flexibilities as the coefficients and the forces as the unknowns. A new method, called the *flexibility* or *force method*, was formulated and distinguished from traditional compatibility procedures by the fact that all quantities and equations were formulated initially as matrices and manipulated using the associated algebra; therefore, the operations are computer oriented.

In 1953 the delta-wing aircraft was the impetus for a second computer-oriented approach to structural analysis. By broadening the scope of traditional *equilibrium methods* and formulating the equations from inception using matrices, the structural engineers of the 1950s obtained a set of linear algebraic equations with the structural stiffnesses as the coefficients and the displacements as the unknowns. Thus the *stiffness* or *displacement method* was conceived.

Interest in energy methods was also stimulated during this time, but structural mechanics has historically relied upon energy principles. Archimedes (287–212 B.C.), Leonardo da Vinci (1452–1519), and Galileo (1564–1642) each used some form of the work expression to substitute for the equations of equilibrium in lever and pulley systems. Johann Bernoulli (1717) was the first to suggest virtual displacement, and Maupertuis (1740) introduced the concept of measuring equilibrium of rigid bodies by minimizing the total system potential. Leonhard Euler (1744) recognized that energy methods are an alternative approach for solving problems of structural mechanics and used minimization principles to investigate stable equilibrium for deformable bodies; he used expressions for strain energy suggested by Daniel Bernoulli. Lame (1852) derived the principle of conservation of energy and named it for his friend Clapeyron; he used actual forces, stresses, displacements, and strains. James Clerk Maxwell (1864) and Otto Mohr (1874) independently took the results of Lame, and, using a dummy load, investigated statically indeterminate trusses. Thus, the principle of virtual forces is also known as the Maxwell-Mohr method. Castigliano (1873) published the extremum version of Lame's work. Since Lame used actual quantities, Castigliano's theorem, part II, is valid only for linear elastic systems. Crotti (1878) and Engesser (1889) subsequently extended this result, thereby making the minimization principle conform to the principle of complementary virtual work for nonlinear elastic systems.

In 1954 J. Argyris and S. Kelsey formulated matrix structural analysis using energy principles. Matrix structural analysis emanated from physically directed thinking and was derived by satisfying the fundamental equations of structural mechanics; therefore, the application of energy principles was the next logical step in the evolution. Because of the initial popularity of the flexibility method during the 1950s, the corresponding *principle of complementary virtual work* was emphasized. In contrast, the stiffness method arises from the *principle of virtual work*, but early derivations represented this simply as an alternate choice of variables (i.e., displacements as unknowns instead of forces). Subsequent work revealed that the stiffness method, based upon the principle of virtual work, is a numerically efficient procedure for implementing the classical Rayleigh-Ritz method, which was conceived in 1909. The finite element method owes its existence and wide appeal to this fact.

1.2 MATRIX STRUCTURAL ANALYSIS AND CLASSICAL METHODS

Thus matrix structural analysis has come to fruition since the 1940s, but its roots are in classical structural mechanics. Since the computer formulates and solves the equations, large structures can be investigated. We can use either compatibility or equilibrium methods and formulate the method using the fundamental equations of structural mechanics or energy principles. Therefore, it is instructive to recall the basic principles and classical methods of structural analysis and observe their relationship to matrix structural analysis.

Structures must be in equilibrium, with their displacements in a compatible state and material laws satisfied. The structural engineer can investigate these primary behavioral tenets by either: (a) solving the fundamental equations or (b) employing energy principles.

Double integration, the method of elastic weights, and the moment-area method yield structural displacements from the fundamental equations. We use force-displacement relationships to assemble equations of structural response, thereby satisfying equilibrium, compatibility, and material laws. *Compatibility methods* mandate identifying statically indeterminate elements and imposing compatibility requirements, thus producing sets of equations with the structural flexibilities as coefficients and forces as unknowns. The method of consistent displacements and the three-moment equation are examples of the compatibility method.

Alternatively, by invoking equilibrium at points connecting structural elements we can formulate sets of simultaneous linear algebraic equations with the structural stiffnesses as coefficients and displacements as unknowns. This approach begets *equilibrium methods*; the slope-deflection and the moment-distribution methods are two classical procedures in this category.

Energy principles present an alternative approach for investigating structural behavior. The principle of virtual displacements, the unit displacement theorem, and Castigliano's theorem, part I, are examples of energy methods, wherein equilibrium is satisfied implicitly. In contrast, the principle of virtual forces, the unit-load theorem, and Castigliano's theorem, part II, are complementary virtual work theorems that satisfy compatibility implicitly.

Virtual work theorems call for varying the displacement and corresponding strains, whereas *complementary virtual work* theorems require the forces and stresses to undergo variations. The former approach produces equilibrium methods, while the latter gives rise to compatibility procedures. For example, recall the method of least work for linearly elastic systems. By Castigliano's theorem, part II, the partial derivative of the strain energy with respect to a force gives the corresponding displacement. If that displacement is zero (e.g., for a redundant reaction), we obtain what appears to be a minimum principle.

This, of course, is a compatibility method obtained from a complementary virtual work principle.

For systems with many unknowns it is convenient to formulate the equations from the beginning in matrix form; thus, subsequent manipulations are executed using matrix operations, which can be conveniently programmed in a computer language. If we use a compatibility method (either by solving the fundamental equations or invoking a complementary virtual work principle) we obtain sets of simultaneous linear algebraic equations involving structural flexibilities. In contrast, by formulating the solution using either the principle of virtual work or the fundamental equations of structural behavior, we mandate nodal equilibrium and obtain sets of simultaneous linear algebraic equations embodying the structural stiffnesses. The former approach is the *flexibility* or *force method*, whereas the latter is the *stiffness* or *displacement method*.

We can program the force method to automatically identify statically indeterminate components or systems. We formulate the coefficient matrix of system flexibilities using a matrix triple product; one of the basic matrices required for this process expresses system equilibrium, while another simply contains element flexibilities. The computer executes a large number of operations and consumes a great amount of time in formulating the global flexibility matrix. In the formative stages, the stiffness method suffered from engineers obsessed with the duality of the two methods. That is, since the basic equilibrium matrix of the flexibility method can be shown to define the system compatibility equations, we can formulate the global stiffness matrix using a matrix triple product in a fashion resembling that employed in constructing the global flexibility matrix. Equations can elegantly express this duality, but the approach is computationally inefficient. The global stiffness matrix is most efficiently formulated using list processing. By clinging to the duality of equations, the early pioneers nearly rang the death knell for the stiffness method. We now recognize duality for what it is: an interesting fact with few useful computational implications. Today, the stiffness method has almost totally supplanted the flexibility method.

1.3 DISCUSSION

Perhaps in a few years we will not be required to distinguish computer-oriented structural analysis by appending the adjective, matrix. This speculation is strengthened by the capabilities of the available computing hardware and software. In the early days of computers, the user was required to prepare punched cards, learn elaborate access protocol, and struggle with "turn-around time"; the microcomputer has eliminated all of this. In addition, many classical methods can be implemented on a standard spreadsheet, thus qualifying them to be called computer methods. Larger problems simply require more computing power. The supercomputer offers solutions for mammoth systems, whereas intermediate-sized problems can be solved using some component of the array of available equipment between the micro- and supercomputer such as the mini- or mainframe computer.

The spectrum of approaches to structural analysis includes classical, approximate, and computer-oriented methods, and each has its function. By interpreting computer solutions using approximate analysis, the structural engineer can avoid the computer siren that lures acceptance of dubious machine-generated results with the implication that computer output is above question. Since nodes can be misplaced, members inadvertently omitted, and entire systems incorrectly modeled, it is wise to remember the old computer maxim: "garbage in, garbage out."