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ABSTRACT

The well known Selberg trace formula reduces the problem of calculating the dimension of cusp forms of Siegel upper-half plane, when the fundamental domain is not compact but has finite volume, to the evaluation of certain integrals combining with special values of certain zeta functions. In this paper, we shall obtain explicit dimension formulae for cusp forms of degree three with respect to the full modular group $Sp(3, \mathbf{Z})$ and its principal congruence subgroups by a long computation.

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NOTATION

1. $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$: ring of rational integers and the fields of rational numbers, real numbers and complex numbers respectively.
2. $M_n(Z)$, $M_n(R)$, $M_n(\mathbf{C})$: rings of $n \times n$ matrices over Z, $\mathbf{R}$, $\mathbf{C}$ respectively.
3. $GL_n(Z)$, $GL_n(R)$: general linear groups over $\mathbf{Z}$, $\mathbf{R}$ respectively.
4. $SL_n(Z)$, $SL_n(R)$: special linear groups over $\mathbf{Z}$, $\mathbf{R}$ respectively.
5. $U(n)$: group of $n \times n$ unitary matrices;

$$U(n) = \{U \in M_n(\mathbf{C}) \mid U^{-1} = {}^t\bar{U}\} .$$

6. $Sp(n, \mathbf{R})$: the real symplectic matrices of degree n; specifically,

$$Sp(n, \mathbf{R}) = \left\{ M \in M_{2n}(\mathbf{R}) \mid {}^tMJM = J,\ J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix} \right\}.$$

Here E_n is the identity of matrices ring $M_n(\mathbf{C})$.

7. $Sp(n, \mathbf{Z}) = Sp(n, \mathbf{R}) \cap M_{2n}(Z)$: the discrete modular subgroup of degree n .
8. $\Gamma_n(N)$: the principal congruence sungroup of level N of $Sp(n, \mathbf{Z})$, specifically,

$$\Gamma_n(N) = \{M \in Sp(n, \mathbf{Z}) \mid M \equiv E_{2n} (\text{mod } N)\} .$$

9. H_n: Siegel upper-half space of degree n ; specifically,

$$H_n = \{Z \in M_n(\mathbf{C}) \mid {}^tZ = Z,\ \operatorname{Im} Z > 0\} .$$

10. D_n: the generalized disc of degree n ; specifically,

$$D_n = \{W \in M_n(\mathbf{C}) \mid {}^tW = W,\ E - W{}^t\bar{W} > 0\} .$$

11. $[S, U]$: element of $Sp(n, \mathbf{R})$ of the form $\begin{bmatrix} E & S \\ 0 & E \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{bmatrix}$.

12. diag $[a_1, a_2, \ldots, a_n]$ or $[a_1, a_2, \ldots, a_n]$: the diagonal matrix

$$\begin{bmatrix} a_1 & & & & \\ & a_2 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & a_n \end{bmatrix}.$$

13. $\Gamma(s)$: gamma-function; it is defined by

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \quad \text{for} \quad \operatorname{Re} s > 0 .$$

14. $\alpha(k) = [a_0, a_2, \ldots, a_{m-1}] \pmod{2m}$: $\alpha(k) = a_j$ if $k \equiv 2j$ for $j = 0, 1, \ldots, m-1$.

TABLE OF CONTENTS

INTRODUCTION

Let H_n be the generalized upper-half plane of degree n and Γ be a subgroup of the symplectic group $Sp(n, \mathbf{R})$, which acts on H_n properly discontinuous (i.e., given two compact subset A and B of H_n, the set $\Gamma_{A,B} = \{\gamma \in \Gamma \mid \gamma(A) \cap B \neq \phi\}$ is finite) on H_n. Denote by $S(k;\Gamma)$ be the vector space of Seigel cusp forms of weight k and degree n with respect to Γ. In other words, $S(k;\Gamma)$ consists of holomorphic function f on H_n satisfying the following conditions:

(1) $f(\gamma(Z)) = \det(CZ+D)^k f(Z)$ for all $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma$.

(2) Suppose that $\Sigma\, a(T)[\exp 2\pi i\ (TZ)]$ is the Fourier expansion of f; then $a(T) = 0$ if rank $T < n$. Here the summation is over all half integral matrices T such that $T \geqslant 0$ and $\sigma(TZ) =$ trace of TZ.

The second condition can be replaced by the growth condition as follows:

(2') $(\det \operatorname{Im} Z)^{k/2} |f(Z)|$ is bounded on H_n.

It is well known that $S(k;\Gamma)$ is a finite dimensional vector space. Furthermore, the dimension of $S(k;\Gamma)$ over $\mathbf{C}$ is given by Selberg trace formula as follows [12]:

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$$\dim_{\mathbf{C}} S(k;\ \Gamma) = C(k,\ n)\int_F \sum_M K_M(Z,\ \overline{Z})^k dZ$$

when $k \geqslant 2n+3$.

1. $C(k,\ n) = 2^{-n}(2\pi)^{-n(n+1)/2} \prod_{i=0}^{n-1}\Gamma(k - \frac{n-i-1}{2}) \prod_{i=0}^{n-1}[\Gamma(k-n+\frac{i}{2})]^{-1}$,

2. F is a fudamental domain on H_n for Γ ,

3. In the summation M ranges over all matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\Gamma/\{\pm 1\}$,

4. $K_M(Z,\ \overline{Z}) = (\det \operatorname{Im} Z)\det(\frac{Z-M(\overline{Z})}{2i})^{-1} \det(C\overline{Z}+D)^{-1}$ for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma$,

5. dZ is the symplectic volume defined by

$$dZ = (\det Y)^{-(n+1)} dXdY \quad \text{if} \quad Z = X+iY .$$

Our main interest in this paper is to compute explicitly $\dim_{\mathbf{C}} S(k;\ \Gamma)$ via Selberg trace formula when $\Gamma = Sp(3,\ \mathbf{Z})$.

As claimed in my previous paper [11], a dimension formula for the vector space of Siegel cusp forms of degree three with respect to $Sp(3,\ \mathbf{Z})$ can be obtained once the conjugacy classes of $Sp(3,\ \mathbf{Z})$ are given explicitly. However, the number of conjugacy classes in $Sp(3,\ \mathbf{Z})$ are so large that one cannot expect to get a correct formula without making any mistake in the computation of more than 300 contributions. Fortunately, we observe that $\dim_{\mathbf{C}} S(k; Sp(3, \mathbf{Z}))$ is a finite sum of $P(k)C(k)$ with $P(k)$ being an integral divisor of $(2k-2)(2k-3)(2k-4)^2 \times (2k-5)(2k-6)$ such as $(2k-2)(2k-4)^2(2k-6), (2k-3)(2k-5)(2k-4)$ and $C(k)$ being a constant or a period function in k such as $(-1)^k$, $\cos(2k-2)\pi/3$, $\sin(k-2)\pi/3$. After selected [14] concributions (we call these contributions the main terms) from the dimension formula, we found that the sum of the remaining terms appears to be the form

$$C_1(k)(2k-4)^2 + C_2(k)(2k-4) + C_3(k)$$

with $C_j(k) = C_j(k+12)$, $j = 1, 2, 3$.

Note that the sum of the main terms and $C_1(k)(2k-4)^2 + C_2(k)(2k-4) + C_3(k)$ is an integer. It forces that $C_j(k)$ $(j = 1, 2, 3)$ must satisfy certains conditions. More precisely, if we let $P(k)$ denote the sum of the main terms, then we have

$$C_1(k)(2k-4)^2 + C_2(k)(2k-4) + C_3(k) = \dim_{\mathbf{C}} S(k; Sp(3, \mathbf{Z})) - P(k),$$

$$C_1(k)(2k+20)^2 + C_2(k)(2k+20) + C_3(k) = \dim_{\mathbf{C}} S(k+12; Sp(3, \mathbf{Z})) - P(k+12),$$

$$C_1(k)(2k+44)^2 + C_2(k)(2k+44) + C_3(k) = \dim_{\mathbf{C}} S(k+24; Sp(3, \mathbf{Z})) - P(k+24).$$

This tells us that $C_j(k)$ $(j = 1, 2, 3)$ can be determined by three consecutive integers $\dim_{\mathbf{C}} S(k; Sp(3, \mathbf{Z}))$, $\dim_{\mathbf{C}} S(k+12; Sp(3, \mathbf{Z}))$, $\dim_{\mathbf{C}} S(k+24; Sp(3, \mathbf{Z}))$ and the sum of the main terms $P(k)$. Now a direct computation with the help of the above observation, we are able to write down the explicit expression of $\dim_{\mathbf{C}} S(k; Sp(3, \mathbf{Z}))$ correctly.

MAIN THEOREM I. <u>For even integer</u> $k \geqslant 10$, <u>the dimension formula for the vector space of Siegel cusp forms of degree three and weight</u> k <u>is given by</u>

$$\dim_{\mathbb{C}} S(k, Sp(3, \mathbb{Z})) = \underline{\text{Sum of Main Terms}}$$

$$+ C_1(k)(2k-4)^2 + C_2(k)(2k-4) + C_3(k)$$

<u>where the main terms and the values of</u> $C_j(k)$ $(j = 1, 2, 3)$ <u>are given by TABLE I as follows</u>:

TABLE I Main Terms in the Dimension Formula

No.	Contribution	Conjugacy Classes
1	$2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6)$	E_6
2	$2^{-15}3^{-4}5^{-1}31(2k-2)(2k-4)^2(2k-6)$	$[1, 1, -1]$
3	$-2^{-13}3^{-3}5^{-1}16(2k-3)(2k-4)(2k-5)$	$E_4 \times \begin{bmatrix}-1 & s\\ 0 & -1\end{bmatrix}$, $(s \neq 0)$
4	$2^{-10}3^{-5}5^{-1}(2k-3)(2k-4)(2k-5) \times [-2, 0, 2]$	$[1, 1, e^{i\theta}]$, $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (nos. 4–6)
5	$2^{-10}3^{-5}5^{-1}(2k-4)(2k-5) \times [-10, 20, -10]$	
6	$2^{-9}3^{-5}5^{-1}(2k-4) \times [8, 10, -18]$	
7	$-(-1)^{k/2}2^{-12}3^{-2}5^{-1}(2k-4)^2$	$[1, 1, \pm i]$
8	$-2^{-9}3^{-2}5^{-1}(2k-4)$	$[S, E_3]$, rank $S = 2$
9	$2^{-3}3^{-1}5^{-2}(2k-4) \times [1, 0, -1, 3, -3]$	Elements with characteristic polynomial $(X \pm 1)^2(X^4 \pm X^3 + X^2 \pm X + 1)$ (nos. 9–10)
10	$2^{-3}3^{-1}5^{-2} \times [-66, 0, 54, -54, 66]$	
11	$\frac{1}{7}$ [1, 0, 1, 0, 0, 0, 0]	Elements of order 7
12	$\frac{1}{9}$ [1, 0, 1, 0, -1, 0, 0, -1, 0]	Elements of order 9
13	$\frac{1}{20}$ [1, 0, 1, 1, -1, -1, 0, -1, -1, 1]	Elements of order 20
14	$\frac{1}{15}$ [1,0,1,0,0,-1,0,0,0,0,0,0,-1,0,0]	Elements of order 30

15 The remaining term is $C_1(k)(2k-4)^2 + C_2(k)(2k-4) + C_3(k)$, where

$$C_1(k) = 2^{-7}3^{-2}[4,2,4,3,3,3]$$

$$+ 2^{-12}3^{-6}[451,1249,451,937,763,937]$$

(TABLE I CONTINUED)

$C_2(k) = -2^{-3}3^{-1} + 2^{-8}3^{-6}[-3010, 783, -4496, -1714, -1161, -1904]$

$C_3(k) = 2^{-4}3^{-6}[5314, 0, 8770, 2560, 2916, 2128]$.

* Here $C(k) = [a_0, a_1, \ldots, a_{m-1}]$ means $C(k) = a_j$ if $k \equiv 2j \pmod{2m}$ for $0 \leqslant j \leqslant m-1$.

MAIN THEOREM II. *The dimension formula for the vector space of Siegel cusp forms of degree three with respect to the congruence subgroup* $\Gamma_3(2)$ *of* $\Gamma_3 = Sp(3, Z)$ *is given by*

$$\begin{aligned}
&\dim_{\mathbf{C}} S(k;\ \Gamma_3(2)) \\
&= [\Gamma_3 : \Gamma_3(2)] \times [2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6) \\
&\quad + 2^{-15}3^{-4}5^{-1}(2k-2)(2k-4)^2(2k-6) \\
&\quad - 2^{-14}3^{-4}5^{-1}(2k-3)(2k-4)(2k-5) - 2^{-13}3^{-3}(2k-3)(2k-5) \\
&\quad - 2^{-14}3^{-2}5^{-1}(2k-4) + 2^{-13}3^{-1}(2k-4) - 2^{-13}3^{-1} + 2^{-13}3^{-3}]
\end{aligned}$$

for an even integer $k \geqslant 10$, *where* $[\Gamma_3 : \Gamma_3(2)] = 2^9 3^4 \cdot 35$.

MAIN THEOREM III. *The dimension formula for the vector space of Siegel cusp forms of degree three with respect to the principal congruence subgroup* $\Gamma_3(N)$ $(N \geqslant 3)$ *of* $\Gamma_3 = Sp(3, \mathbf{Z})$ *is given by*

$$\begin{aligned}
&\dim_{\mathbf{C}} S(k;\ \Gamma_3(N)) \\
&= [\Gamma_3 : \Gamma_3(N)] \times [2^{-15}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6) \\
&\quad - 2^{-9}3^{-2}5^{-1}(2k-4)N^{-5} + 2^{-7}3^{-3}N^{-6}],
\end{aligned}$$

where k *is an even integer greater than* 9 *and*

$$[\Gamma_3 : \Gamma_3(N)] = \frac{1}{2} N^{21} \prod_{p \mid N \ p:\ \text{prime}} (1 - p^{-2})(1 - p^{-4})(1 - p^{-6}).$$

The method we employed here applies to cases of higher degrees. Indeed, we did reduce the problem of finding $\dim_{\mathbf{C}} S(k; Sp(n, Z))$, at least for the case $n = 1, 2, 3$; to the problem of

(1) finding conjugacy classes of $Sp(n, Z)$,

(2) calculating contributions from certain conjugacy classes or families of conjugacy classes

and

(3) determining values of certain constants.

Part of the problem in (1) is treated in [22, 30] . Thus we can write down conjugacy classes of elements whose characteristic polynomials are products of cyclotomic polynomials by an iduction on the degree n . The problem in (2) is treated in [19] in a more general context though not so explicitly. The problem in (3) can be treated by our knowledge of modular forms of lower weight instead of direct computation. In our determination of $\dim_{\mathbf{C}} S(k; Sp(3, Z))$, the constants $C_j(k)$ $(j = 1, 2, 3)$ can be determined uniqued by $\dim_{\mathbf{C}} S(k; Sp(3, Z))$ when $10 \leqslant k \leqslant 44$ and the sum of main terms as shown in TABLE I.

In CHAPTER 1 and 2, we shall determine all conjugacy classes of $Sp(3, Z)$ explicitly for further usage. We began to compute contributions by Theorems in [11] concerning evaluation of integrals involving in Selberg trace formula and conjugacy classes given in CHAPTER 1 and 2. In the final CHAPTER, we shall combine all contributions by the method we mentioned to obtain MAIN THEOREMS in this paper.

This is a continuation of my previous work [11] on the dimension formula of Siegel cusp forms of degree three. I would like to thank my advisor Professor W. L. Baily Jr. at the University of Chicago. Without his constant encouragement, I may give up in the middle owning to the complication of computation.

CHAPTER I

FIXED POINTS AND CONJUGACY CLASSES OF REGULAR ELLIPTIC ELEMENTS IN Sp(3, Z)

1.1. Introduction

In [13] and [14], E. Gottschling studied the fixed points and their isotropy groups of finite order elements in Sp(2, Z). He finally obtained six Sp(2, Z)-inequivalent isolated fixed points as follows:

(1) $Z_1 = \operatorname{diag}[i, i]$, (2) $Z_2 = \operatorname{diag}[\rho, \rho],\ \rho = e^{\pi i/3}$,

(2) $Z_3 = \operatorname{diag}[i, \rho]$, (4) $Z_4 = \frac{i}{\sqrt{3}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

(5) $Z_5 = \begin{bmatrix} \eta & (\eta-1)/2 \\ (\eta-1)/2 & \eta \end{bmatrix},\quad \eta = \frac{1}{3} + \frac{2\sqrt{2}i}{3}$,

(6) $Z_6 = \begin{bmatrix} \omega & \omega+\omega^{-2} \\ \omega+\omega^{-2} & -\omega^{-1} \end{bmatrix},\quad \omega = e^{2\pi i/5}$.

The isotropy subgroups at Z_i $(i = 1, 2, 3, 4, 5, 6)$ are groups of order 16, 36, 12, 24, 5 respectively.

By the arguement of [30], these fixed points can be obtained from symplectic embeddings of

$$Q(i) \oplus Q(i),\quad Q(\rho) \oplus Q(\rho),\quad Q(i) \oplus Q(\rho),\quad Q(e^{\pi i/6}),$$

$$Q(e^{\pi i/4}),\quad Q(e^{2\pi i/5})$$

into $M_4(Q)$. In this CHAPTER, we shall combine the reduction theory of symplectic matrices [5, 6] with the arguements of [22, 30] and obtain all Sp(3, **Z**)-inequivalent isolated fixed points and conjugacy classes of regular elliptic elements in Sp(3, **Z**). A table for all representatives and their centralizer in Sp(3, **Z**)/{±1} of regular elliptic conjugacy classes in Sp(3, Z) is given in 1.4.

1.2 Notations and Basic Results

Let **Z**, **Q**, **R** and **C** denote the ring of integers, the fields of rational, real and complex numbers respectively. The real symplectic matrices of degree n ,

$$Sp(n, \mathbf{R}) = \left\{ M \in M_{2n}(\mathbf{R}) \mid {}^tMJM = J,\ J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix} \right\},$$

acts on the generalized half space H_n defined by

$$H_n = \{Z \in M_n(\mathbf{C}) \mid Z = {}^tZ,\quad \mathrm{Im}\, Z > 0\} .$$

Here $M_{2n}(\mathbf{R})$ is the $2n \times 2n$ matrix ring over **R**, $M_n(\mathbf{C})$ is the $n \times n$ matrix ring over **C**, E_n is the identity of $M_n(\mathbf{C})$ and ${}^tZ$ is the transpose of Z .

A point Z_0 in H_n is called an isolated fixed point of Sp(3, **Z**) if there exists $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in Sp(3, **Z**) such that Z_0 is the unique solution of the equation,

$$AZ + B = Z(CZ + D),\ Z \in H_n .$$

An element M of Sp(3, Z) is regular elliptic if M has an isolated fixed point [see 10]. Now suppose M is a regular elliptic element of Sp(3, Z), then by the discreteness of Sp(3, **Z**) and the property that Sp(3, **R**) acts transitively on H_3, we concluded that

(1) M is an element of finite order,

(2) M is conjugate in Sp(3, **R**) to $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $A + Bi = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$, λ_i $(i = 1, 2, 3)$ root of unity and $\lambda_i\lambda_j \neq 1$ for all i, j,

(3) the centralizer of M in Sp(3, **Z**) is a group of finite order.

By the property (1), we see that the minimal polynomial of M is a product of different cyclotomic polynomials of degree at most 6 as follows:

$$X^2+1,\ X^2-X+1,\ X^2+X+1,\ X^4+1,\ X^4-X^2+1,\ X^4+X^3+X^2+X+1,$$
$$X^4-X^3+X^2-X+1,\ X^6-X^3+1,\ X^6+X^3+1,\ X^6+X^5+X^4+X^3+X^2+X+1,$$
$$X^6-X^5+X^4-X^3+X^2-X+1 .$$

For our convenience, we identify $Sp(n_1, \mathbf{R}) \times Sp(n_2, \mathbf{R})$ as a subgroup of $Sp(n_1+n_2, \mathbf{R})$ via the embedding

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \to \begin{bmatrix} A & 0 & B & 0 \\ 0 & P & 0 & Q \\ C & 0 & C & 0 \\ 0 & R & 0 & S \end{bmatrix}$$