Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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E. Odell H. Rosenthal (Eds.)

Functional Analysis

Proceedings, The University of Texas at Austin 1986–87

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LONGHORN NOTES

Preface

This is the fifth annual proceedings of our Functional Analysis Seminar at The University of Texas. It is the first issue to be published in the Springer-Verlag Lecture Notes. All the articles that appear are based on talks given in the seminar. Some of the articles contain expositions of known results; some of them present fresh discoveries, perhaps not yet formulated in the final style they would assume in a journal article. Other articles may contain both ingredients and are written in complete, final form. The purpose of the Notes is to provide an outlet for all of these kinds of mathematical exposition. We thank the participants in our seminar for sharing their mathematical ideas with us throughout the year, and for contributing to the Longhorn Notes.

This entire issue was again typeset by Margaret Combs on a Sun Computer, using the TEX text formatting system. We are deeply appreciative of her considerable patience and remarkable craftsmanship. We also wish to thank The University of Texas for supporting the publication of the Longhorn Notes.

Ted Odell
Haskell Rosenthal
December 1987

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On the Choquet representation theorem

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Introduction

We give here a self-contained elementary proof of the Choquet representation theorem (both existence and uniqueness), as well as an exposition of the Choquet-Kendall simplex-characterization theorem. The existence part of Choquet's theorem goes as follows.

Theorem 1. *Let K be a compact metrizable convex subset of a locally convex topological space, and let $x \in K$. There exists a Borel probability measure μ on K, supported on the extreme points of K, so that*

$$x = \int_K k \, d\mu(k) \; . \tag{1}$$

The integral in (1) may be interpreted as the Bochner-integral of the identity function $I(k) = k$ on K, since I is strongly measurable. Of course (1) holds if and only if $f(x) = \int_K f(k)\, d\mu(k)$ for all $f \in X^*$, where X is the locally convex space containing K.

We prove Theorem 1 in Section 1, as follows: as is well-known, K is affinely homeomorphic to a compact convex subset K' of Hilbert space (see the last remark in Section 1 for a review of the standard proof). Assume $K = K'$ and $x \in K$. It follows by the weak*-compactness of $\mathcal{P}(K)$, the probability measures on K, that there exists a $\nu \in \mathcal{P}(K)$ maximizing the integral of the norm-squared function on K over all $\mu \in \mathcal{P}(K)$ representing x, *i.e.*, satisfying (1). We then show by a direct elementary argument, using the uniform convexity of Hilbert space, that ν is supported on the extreme points of K.

Our argument makes no explicit use of the classical treatment as given in *e.g.*, [**1**], [**2**] or [**17**]. Of course it is strongly motivated by the previous classical proofs and some more recent developments, which we now indicate. (Phelps has also given an exposition of Theorem 1 by considering the Hilbert space case; see [**18**].)

Following Choquet's seminal work ([**8**], [**9**], [**10**]) and important further developments by Bishop and de Leeuw [**5**], Hervé gave a direct proof of Theorem 1 in [**15**], as follows: Let K

* Research was partially supported by NSF DMS-8601752.

be a compact subset of a locally convex space. For f a continuous function on K, define $\hat{f}$ by

$$\hat{f} = \inf \{ \ a \geq f : a \ \text{ is an affine continuous function on } \ K \ \} .$$

($\hat{f}$ is called the upper semi-continuous concave envelope of f). Now fix f a continuous convex function on K and $x \in K$. Hervé then proved that there exists a regular Borel probability measure μ on K representing x and satisfying

$$(*) \qquad \int_K f \, d\mu = \hat{f}(x) .$$

Now if ν is another such measure on K representing x, it is easily seen that $\int_K f \, d\nu \leq \int_K \hat{f} \, d\nu \leq \hat{f}(x)$. Hence the μ satisfying $(*)$ maximizes $\int_K f \, d\nu$ over such ν representing x, and moreover $\int_K f \, d\mu = \int_K \hat{f} \, d\mu$ so $\hat{f} = f$ μ-a.e. Now Hervé showed that K is metrizable if and only if there exists a strictly convex continuous function f on K; moreover if f is such a function and $y \in K$, he proved that $\hat{f}(y) = f(y)$ if and only if y is an extreme point of K. It thus follows that if K is as in Theorem 1 and f is strictly convex continuous, then any such μ representing x and satisfying $(*)$ is supported on the extreme points of K, thus proving Theorem 1.

Subsequent to Hervé's work, Bonsall gave a simple deduction of the existence of such a μ satisfying $(*)$ using the analytic form of the Hahn-Banach theorem ([**6**]; see also [**7**]). Edgar studied the representation theorem in the more general context of separable closed bounded convex subsets of a Banach space, and proved its validity here for sets C having the Radon-Nikodým property, using a transfinite-martingale argument [**11**]. Ghoussoub and Maurey subsequently gave a martingale proof of Edgar's result in [**14**], by using their previous theorem that such sets C admit "H_δ-embeddings" onto H_δ subsets of Hilbert space. The Ghoussoub-Maurey argument uses the parallelogram identity on Hilbert space; as we show elsewhere [**23**], a variation of their argument holds in arbitrary Banach spaces, also for a more general class of convex RNP subsets than the H_δ ones. Thus the uniform convexity of Hilbert space is not really needed for a martingale proof of Theorem 1.

These martingale proofs of Choquet type representation theorems are quite penetrating but not "elementary". The same may be said of the previous classical development discussed above. The latter is certainly beautiful and deep, and immediately yields the validity of our procedure. Indeed, if K is a compact convex subset of Hilbert space, $f(k) = \|k\|^2$ is strictly convex and continuous on K, so if $\mu \in \mathcal{P}(K)$ maximizes the integral of f over all $\mu' \in \mathcal{P}(K)$ representing x, then μ satisfies $(*)$ and hence the conclusion of Theorem 1, by Hervé's results. As suggested by the work of Ghoussoub-Maurey and the expository article of Phelps [**18**], it is our point of view here that rather than constructing such functions and carrying out arguments in the general setting, it is preferable to place K in Hilbert space (or a uniformly convex Banach space). We then have a very natural strictly convex continuous function already defined, and moreover we do not need to work with the *class* of all continuous convex functions and the theory of their (usually discontinuous) upper semi-continuous concave envelopes, as required in the classical development of Choquet theory.

In Section 2, we prove the uniqueness part of Choquet's theorem, which is formulated as follows.

Theorem 2. *Let K be as in Theorem 1. Then K is a simplex if and only if every point of K is represented by a unique probability measure supported on the extreme points of K.*

A non-empty convex subset K of a real linear space X is called a *simplex* provided $K \times \{1\}$ is the base of a lattice cone in $X \times \mathbb{R}$. (The terms "base" and "lattice cone" are defined in Section 2.) The non-trivial part of Theorem 2 is the "only if" assertion; we prove this by first developing some elementary properties of vector lattices and ordered vector spaces with a given cone-base. The main part of the proof then consists in showing that if μ and ν are probability measures on the extreme points of K with disjoint supports, then their barycenters are orthogonal as elements of the vector lattice generated by K (assuming K is in "general position" as defined in Section 2). This is precisely the approach taken by Choquet in his original treatment [**8**], and our proof is a variation of his arguments. In particular, we formulate a geometric version of a result in [**8**] as Theorem 2.5. This fundamental discovery of Choquet's shows that for K an arbitrary compact convex subset of a locally convex space and A a compact subset of Ext K, $\overline{co}A$ is contained in the affine closure of the convex hull of the set of all points x in K with $F_x \subset V$, where V is a prescribed neighborhood of A in K and F_x denotes the smallest face of K containing x. (See the paragraph following the statement of 2.5 for the definition of "affine closure".) We first deduce Theorem 2 from Theorem 2.5, then conclude Section 2 with the proof of the latter. We use a new lemma for this purpose, Lemma 2.8. Its formulation is motivated by Lemma 3 of [**8**]; however our proof of 2.8 is rather different than the discussion in [**8**].

Suppose K is as in Theorem 1 except that K is not assumed to be metrizable. Our discussion also yields Choquet's result in [**8**] that again if K is a simplex, every point of K is represented by at most one regular Borel probability measure supported on the extreme points of K. However as is well known, it can happen that there are points of K which have no such representing measure. We refer the reader to the standard references [**1**], [**2**] and [**17**], for the proper formulations and proofs of Theorem 1 and 2 in the non-metrizable setting.

Section 3 is devoted to the proof of the following result.

Theorem 3. *Let K be a convex subset of a linear space X. The following conditions are equivalent:*

(1) K is a simplex.

(2) K is line-compact and the non-empty intersection of two homothets of K is a (possibly degenerate) homothet of K.

Moreover if X is a linear topological space and K is σ-convex, we have a third equivalent condition:

(3) K is line-compact and the non-empty intersection of two translates of K is a (possibly degenerate) homothet of K.

These notions are defined as follows:

For A, B non-empty subsets of X, B is a *translate* of A if $B = A + x$ for some $x \in X$; B is a *homothet* of A if B is a translate of a positive multiple of A; B is a degenerate homothet of A if B is a singleton. K is said to be *line-compact* (resp. *line-closed*) if $L \cap K$ is a compact (resp. closed) subset of L for every line L in X. A subset K of a linear topological space X is *σ-convex* provided for all sequences (k_j) in K and (λ_j) in $\mathbb{R}^+$ with $\sum_{j=1}^{\infty} \lambda_j = 1$, $\sum_{j=1}^{\infty} \lambda_j k_j$ converges to an element of K.

Now fix K and X as in Theorem 3. Let us say that K is a *classical simplex* provided K is the convex hull of an affinely-independent finite set. In his fundamental work in [**8**] and [**10**], Choquet defined K to be what is now called a "simplex of Choquet" provided K satisfies condition (2) of Theorem 3, with the "line-compactness" condition deleted, and he stated the equivalence of conditions (1) and (2) of Theorem 3 for the case where K is compact and X is a locally convex space. Kendall introduced the "line-compactness" condition and formulated and proved the equivalence (1) $\Leftrightarrow$ (2) in general, in his remarkable article [**16**]. An alternate proof of this equivalence, due to Choquet, may also be found in the nice expository paper by Goullet de Rugy [**13**]. As remarked by Kendall in [**16**], if K is a line-compact convex subset of a linear space X, then $K \cap Y$ is a compact subset of Y for any finite-dimensional subspace Y of X (endowed with its unique linear topology). Thus if K is a finite-dimensional convex set, then if K is a simplex as defined above, K is compact; it followed easily from Theorem 2 that then K is a classical simplex; of course the converse is obvious. Independently of Choquet's work, Rogers and Shephard proved in [**19**] that if K is finite-dimensional, then K is a classical simplex if and only if K satisfies (3) of Theorem 3. Another treatment of the Rogers-Shephard result was given by Eggleston, Grünbaum and Klee in [**12**]. It does not seem to be known if condition (3) of Theorem 3 implies K is a simplex in general, without any topological assumptions. (It seems worth pointing out, however, that the σ-convexity condition is very general and occurs in most natural convex sets encountered in analysis. For example, it's easily seen that K is σ-convex provided K is a sequentially complete bounded convex subset of a linear topological space.)

The proof we give of (1) $\Leftrightarrow$ (2) is essentially that of Kendall's; we "localize" the conditions in Theorem 3.2 in order to obtain (3) $\Rightarrow$ (1). Our proof of this shows and is motivated by the following elementary result (see Lemma 3.7): *Let f be an integrable function on* $[0,1]$ *with* $\int_0^1 f\,dt > 0$. *Then f^+ belongs to W, where W is the smallest class of integrable functions satisfying the following properties for all u, v in W and scalars $\lambda > 0$:*

(a) f and $\mathbf{1} \in W$

(b) $\lambda u \in W$

(c) $\max\{u,v\} \in W$ whenever $\int_0^1 u\,dt = \int_0^1 v\,dt$

(d) If $w_1 \geq w_2 \geq \cdots$ are non-negative elements of W and w is such that $w_n \to w$ a.e., then $w \in W$.

To complete our argument, we also use a result from [**22**] involving the notion of L^1-convexity first introduced by the author in [**20**].

We conclude Section 3 with the following simple application of Theorem 3 (whose formulation and proof are motivated by the elegant discussion for the compact case given in [**12**]): *Let $K_1 \supset K_2 \supset \cdots$ be bounded sequentially complete simplexes in a linear topological space, and assume $K = \bigcap_{j=1}^{\infty} K_j$ is non-empty. Then K is a simplex.*

I have attempted to keep the exposition as elementary as possible. The first section in particular should be accessible to anyone familiar with some functional analysis. For some further references to Choquet's theorem and related material in addition to that already mentioned, the reader is referred to [**3**], [**4**], [**7**], [**24**] and [**26**].

1. The existence theorem

We first give some notation and review some standard elementary facts. For M a compact metric space, $\mathcal{P}(M)$ denotes the set of all Borel probability measures on M, endowed with the weak*-topology with respect to $C(M)$, the space of real-valued continuous functions on M. For x in M, P_x denotes the measure with mass one at x. $\mathcal{P}_f(M)$ denotes the set of finitely supported members of $\mathcal{P}(M)$; *i.e.*, $\mathcal{P}_f(M) = co\{P_x : x \in M\}$.

For A a Borel subset of M and μ a finite non-negative Borel measure on M, $\mu|_A$ denotes the measure defined by $(\mu|_A)(B) = \mu(A \cap B)$ for all Borel subsets B of M. μ is said to be *supported* on A if $\mu = \mu|_A$.

Fact 1. *For M as above, $\mathcal{P}_f(M)$ is weak*-dense in $\mathcal{P}(M)$ and $\mathcal{P}(M)$ is weak*-compact.*

Now let K be as in the statement of Choquet's theorem. For $x \in K$ and $\mu \in \mathcal{P}(K)$, we say that μ *represents* x if (1) holds.

Fact 2. *Every $\mu \in \mathcal{P}(K)$ represents a unique $x \in K$.*

Fact 3. *Let (μ_n), μ in $\mathcal{P}(K)$ with $\mu_n \to \mu$ weak* and let (x_n), x in K with μ representing x and μ_n representing x_n for all n. Then $x_n \to x$.*

Fact 4. *K is affinely homeomorphic to a norm-compact convex subset of Hilbert space.*

(See Remark 2 below for a review of the standard proof.)

Proof of the existence theorem. By Fact 4, we may assume that K is a compact convex subset of Hilbert space. Fix $x \in K$ and define λ by

$$\lambda = \sup\left\{ \int \|k\|^2 \, d\mu(k) : \mu \in \mathcal{P}(K) \text{ and } \mu \text{ represents } x \right\} . \tag{2}$$

It then follows that there exists a $\mu \in \mathcal{P}(K)$ so that

$$\mu \text{ represents } x \text{ and } \lambda = \int \|k\|^2 \, d\mu(k) . \tag{3}$$

Indeed, choose (μ_n) in $\mathcal{P}(K)$ with $\int \|k\|^2 \, d\mu_n(k) \to \lambda$ and μ_n representing x for all n. By passing to a subsequence if necessary, we may assume $\mu_n \to \mu$ weak* for some $\mu \in \mathcal{P}(K)$ (by Fact 1). Then μ satisfies (3) (by Fact 3). (Of course we are using the fact that $f(k) = \|k\|^2 \in C(K)$, which incidentially also shows that $\lambda < \infty$.)

Now let $\mu \in \mathcal{P}(K)$ satisfy (3). We shall prove that μ is supported on $\operatorname{Ext} K$, the set of extreme points of K. *Suppose this is not the case.* For $\delta > 0$, define F_δ by

$$F_\delta = \left\{ x \in K : \text{There are } y \text{ and } z \text{ in } K \text{ with } x = \frac{y+z}{2} \text{ and } \left\|\frac{y-z}{2}\right\| \geq \delta \right\} . \tag{4}$$

We have that F_δ is closed and $F_\delta \subset F_{\delta'}$ if $\delta > \delta' > 0$, hence $K \sim \operatorname{Ext} K = \bigcup_{n=1}^{\infty} F_{1/n}$ (which incidentally shows $\operatorname{Ext} K$ is a G_δ subset of K). It follows that there exists a $\delta > 0$ so that

$$\alpha \overset{\text{df}}{=} \mu(F_\delta) > 0 . \tag{5}$$

It follows (by Fact 1) that we may choose a sequence (μ'_n) in $\mathcal{P}_f(F_\delta)$ so that

$$\mu'_n \longrightarrow \frac{1}{\alpha}\mu \mid F_\delta \text{ weak* in } \mathcal{P}(F_\delta) . \tag{6}$$

(By the Tietze extension theorem, convergence in $\mathcal{P}(F_\delta)$ is the same as convergence in $\mathcal{P}(K)$.) For each n, let $\mu_n = \alpha\mu'_n$, choose $(x_i^n)_{i=1}^{m_n}$ in F_δ and $\lambda_i^n \geq 0$ with $\sum_i \lambda_i^n = \alpha$ so that

$$\mu_n = \sum_i \lambda_i^n P_{x_i^n} \tag{7}$$

(the summations in (7) and below extend over all i with $1 \le i \le m_n$). Now for each n and i, choose y_i^n and z_i^n in K with

$$x_i^n = \frac{y_i^n + z_i^n}{2} \quad \text{and} \quad \|\frac{y_i^n - z_i^n}{2}\| \ge \delta \ . \tag{8}$$

Define ν_n by

$$\nu_n = \sum_i \lambda_i^n \frac{P_{y_i^n} + P_{z_i^n}}{2} \ .$$

By passing to a subsequence if necessary, we may assume that there is a ν supported on F_δ with $\frac{1}{\alpha}\nu \in \mathcal{P}(K)$ and $\nu_n \to \nu$ weak*.

Then defining $\underline{\mu}$ by $\underline{\mu} = \nu + \mu|_{\sim F_\delta}$, we have that $\underline{\mu} \in \mathcal{P}(K)$ and $\underline{\mu}$ represents x. Indeed, letting X be our Hilbert space and $f \in X^*$, we have that for each n,

$$\int f\,d\nu_n = \sum_i \lambda_i^n \frac{f(y_i^n) + f(z_i^n)}{2} = \sum_i \lambda_i^n f\left(\frac{y_i^n + z_i^n}{2}\right)$$
$$= \sum_i \lambda_i^n f(x_i^n) = \int f\,d\mu_n \ .$$

Hence since $\nu_n \to \nu$ and $\mu_n \to \mu|_{F_\delta}$ weak*, $\int f\,d\nu = \int_{F_\delta} f\,d\mu$. Since ν is supported on F_δ,

$$\int_K f\,d\underline{\mu} = \int_{F_\delta} f\,d\mu + \int_{\sim F_\delta} f\,d\mu = \int_K f\,d\mu = f(x)$$

(and also $\nu(F_\delta) = \alpha$ so $\underline{\mu} \in \mathcal{P}(K)$).

So far we have not really used that X is a Hilbert space (or even a Banach space, for that matter). We do so now. Fix n and i; setting $\underline{x} = x_i^n$, $y = y_i^n$ and $z = z_i^n$, we have by (8) and the parallelogram identity that

$$\begin{aligned}\frac{\|y\|^2 + \|z\|^2}{2} &= \|\frac{y+z}{2}\|^2 + \|\frac{y-z}{2}\|^2 \\ &\ge \|\underline{x}\|^2 + \delta^2 \ .\end{aligned} \tag{9}$$

Hence we obtain that

$$\begin{aligned}\int \|k\|^2\,d\nu_n(k) &= \sum_i \lambda_i^n \frac{\|y_i^n\|^2 + \|z_i^n\|^2}{2} \\ &\ge \sum_i \lambda_i^n \|x_i^n\|^2 + \sum_i \lambda_i^n \delta^2 \qquad \text{(by 9)} \\ &= \int \|k\|^2\,d\mu_n(k) + \alpha\delta^2 \ .\end{aligned} \tag{10}$$

Again since $\nu_n \to \nu$ and $\mu_n \to \mu|_{F_\delta}$ weak*, we obtain immediately from (10) that $\int \|k\|^2\,d\nu(k) \ge \int_{F_\delta} \|k\|^2\,d\mu(k) + \alpha\delta^2$, whence also by (2) and (3),

$$\lambda \ge \int \|k\|^2\,d\underline{\mu}(k) \ge \int \|k\|^2\,d\mu(k) + \alpha\delta^2 = \lambda + \alpha\delta^2 > \lambda \ . \tag{11}$$

This contradiction completes the proof.

The argument actually gives direct quantitative information, which we summarize as follows:

Theorem 1.1. *Let K be a compact convex subset of Hilbert space, $x \in K$, λ as in (2) and $\mu \in \mathcal{P}(K)$ with μ representing x. Then for $\delta > 0$ and F_δ as in (4),*

$$\lambda \geq \int \|k\|^2 \, d\mu(k) + \delta^2 \mu(F_\delta) \,. \tag{12}$$

Evidently if $\mu \in \mathcal{P}(K)$ satisfies (3), we obtain immediately from (12) that $\mu(F_\delta) = 0$ for all $\delta > 0$, thus recapturing our assertion that μ is supported on $\operatorname{Ext} K$.

Remarks.

1. This argument holds for compact convex subsets of uniformly convex Banach spaces. Indeed, if X is such a Banach space, it is easily seen that if K is a bounded subset of X, then for all $\delta > 0$, there is an $\eta(\delta) > 0$ so that for all y and z in K,

$$\frac{\|y\|^2 + \|z\|^2}{2} \geq \left\|\frac{y+z}{2}\right\|^2 + \eta^2(\delta) \text{ if } \left\|\frac{y-z}{2}\right\| \geq \delta \,.$$

(Of course the function η depends only on the modulus of convexity of X and $\sup\{\|k\| : k \in K\}$.) We then obtain the following analogue of the above result.

Theorem. *Let X be a uniformly convex Banach space, K a compact convex subset of X, $x \in K$, λ as in (2) and $\mu \in \mathcal{P}(K)$ with μ representing x. Then for $\delta > 0$, F_δ as in (4) and $\eta(\delta)$ as above,*

$$\lambda \geq \int \|k\|^2 \, d\mu(k) + \eta^2(\delta)\mu(F_\delta) \,.$$

2. We deduce the standard Fact 4 as follows: let K be a compact convex metrizable subset of a locally convex space X and let $A(K)$ denote the space of all affine continuous functions on K under the supremum norm. Then $A(K)$ is a closed subspace of $C(K)$ and hence $A(K)$ is a separable Banach space; since $X^* \mid K \subset A(K)$, the members of $A(K)$ separate the points of K. Now defining $T_1 : K \to A(K)^*$ by $(T_1 k)(\varphi) = \varphi(k)$ for all $k \in K$ and $\varphi \in A(K)$, we have that T_1 is a one–one affine map, continous from K into $A(K)^*$ in its weak*-topology. Thus setting $K_1 = T_1(K)$, T_1 is an affine homeomorphism from K onto K_1. Let now $f_1, f_2, \ldots$ be a countable dense subset of the ball of $A(K) = \{f \in A(K) : \sup_{k \in K} |f(k)| \leq 1\}$ and define $S : \ell^2 \to A(K)$ by $Sg = \sum_{j=1}^{\infty} 2^{-j} g(j) f_j$ for all $g \in \ell^2$. It follows that S is a compact operator with range dense in $A(K)$ and hence setting $T_2 = S^*$, then $T_2 : A(K)^* \to \ell^2$ is a one–one weak*-continuous compact operator; hence letting $K_2 = T_2(K_1)$, K_2 is norm-compact and $T_2 \mid K_1$ is an affine homeomorphism from K_1 onto K_2. Thus $T = T_2 T_1$ is an affine homeomorphism between K and K_2.

2. The uniqueness result

We begin with recalling some standard algebraic notions. We then review several elementary propositions, beginning with the proof of Theorem 2 itself after Proposition 2.3.

Let X be a real linear space; C a non-empty subset of X. C is called a *cone* (with vertex 0) if for all x, y in C and scalars $\lambda \geq 0$,

$$\text{(a)} \quad \lambda x + y \in C$$

and

$$\text{(b)} \quad x \text{ and } -x \in C \text{ imply } x = 0 .$$

For a given cone C in X, define a relation "$\leq$" on X by: for x, y in X, $x \leq y$ provided $y - x \in C$. We then have that $(X, \leq)$ is a *partially ordered vector space*. That is, "$\leq$" is a partial order on X so that for all x, y, z in X and scalars $\lambda \geq 0$, if $x \leq y$ then

$$x + z \leq y + z \quad \text{and} \quad \lambda x \leq \lambda y$$

and evidently $C = \{x \in X : x \geq 0\}$. (Conversely every partially ordered vector space has its order uniquely determined by the cone C of its non-negative elements). C is called a *lattice cone* provided C is a cone so that if $\leq$ is the corresponding order relation on X and $Z = C - C$, then $(Z, \leq)$ is a *vector-lattice*; that is, every pair x, y of elements of Z has a least upper bound in Z, denoted $x \vee y$ (equivalently, a greatest lower bound in Z, denoted $x \wedge y$).

Finally, a non-empty convex subset K of X is said to be the *base* of a cone C provided for every non-zero element y of C there exist unique k in K and $\lambda > 0$ with $y = \lambda k$; we call C the cone *generated* by K. We note the following important, simple result.

Proposition 2.1. *Let K be a non-empty convex subset of X. Then the following are equivalent.*

1. *K is the base of a cone.*
2. *$0 \notin \text{Aff}(K)$.*
3. *There is a hyperplane H in X with $K \subset H$ and $0 \notin H$.*
4. *There is a linear functional p on X with $p(k) = 1$ for all $k \in K$.*

(We denote the *linear span* of K by span K; $\text{Aff}(K)$ denotes the *smallest affine subspace* containing K, that is, the smallest set Y so that $K \subset Y$ and if $y_1 \neq y_2$ in Y, then $L \subset Y$ where $L = \{ty_1 + (1-t)y_2 : t \in \mathbb{R}\}$ is the *line* joining y_1 and y_2. It is easily seen that if $k_0 \in K$, then $\text{Aff}(K) = [\text{span}(K - k_0)] + k_0$.)

We say that K is in *algebraically general position* in X provided one (and hence any) of the conditions of 2.1 hold. When this occurs, we call $C = \bigcup_{\lambda \geq 0} \lambda K$, the cone *generated* by

K and the corresponding order relation $\leq$ the *order* induced by K; for short, we also simply say that "$(X, \leq)$ is ordered by K". It is worth pointing out that if $(X, \leq)$ is ordered by K and p satisfies 4 of Proposition 2.1, p is *strictly positive* on X. That is, for $x \in X$ with $x > 0$, $p(x) > 0$. (Conversely an ordered vector space has a base for its cone of positive elements provided it admits a strictly positive linear functional.)

The following fundamental proposition summarizes the connection between affine properties of convex sets and order properties of the cones they induce. (For convex sets K_1 and K_2, a map $\alpha : K_1 \to K_2$ is called an *affine equivalence* if α is one-one onto with $\alpha(\lambda x + (1-\lambda)y) = \lambda\alpha(x) + (1-\lambda)\alpha(y)$ for all $x, y \in K$, and λ with $0 \leq \lambda \leq 1$. For partially ordered vector spaces X_1 and X_2, $T : X_1 \to X_2$ is called an *order isomorphism* provided T is linear, one-one onto, and for all x, y in X_1, $x \leq y$ if and only if $Tx \leq Ty$.)

Proposition 2.2. *Let K_1, K_2 be convex subsets of real linear spaces X_1, X_2 and assume K_i is in algebraically general position in X_i with $X_i = \operatorname{span} K_i$ and $(X_i, \leq)$ ordered by K_i for $i = 1, 2$. Then X_1 and X_2 are order isomorphic if K_1 and K_2 are affinely equivalent; precisely, given $\alpha : K_1 \to K_2$ an affine equivalence, there exists a unique order isomorphism $T : X_1 \to X_2$ with $T \mid K_1 = \alpha$.*

Now recall the fundamental definition given in the introduction.

A convex non-empty subset K of X is said to be a simplex provided $K \times \{1\}$ is the base of a lattice-cone in $X \times \mathbb{R}$.

Evidently $K \times \{1\}$ is in algebraically general position in $X \times \mathbb{R}$; it follows immediately from the preceding proposition that K is a simplex if and only if for any (resp. some) K' in algebraically general position in a real linear space X', with K' affinely equivalent to K, X' is a vector lattice, where $(X', \leq)$ is ordered by K' with $X' = \operatorname{span} K'$. As noted in the introduction, the uniqueness theorem (Theorem 2)and the Choquet-Kendall characterization theorem yield that if X if finite-dimensional, K is a simplex if and only if K is a *classical simplex*, *i.e.*, the convex hull of a finite affinely independent set.

We next wish to review some standard vector-lattice concepts (cf. [**25**]). For A a non-empty subset of a partially ordered vector space X, we say that $\sup A$ exists provided there is a (necessarily unique) element x of X which is a least upper bound of A; we then let $\sup A$ denote this element.

Proposition 2.3. *Let $(X, \leq)$ a vector-lattice, x, y, z elements of X, and $\lambda \geq 0$ be given.*

(a) $(x+z) \vee (y+z) = (x \vee y) + z$ *and* $(x+z) \wedge (y+z) = (x \wedge y) + z$;
$\lambda(x \vee y) = \lambda x \vee \lambda y$ *and* $\lambda(x \wedge y) = \lambda x \wedge \lambda y$.

(b) *Setting* $x^+ = x \vee 0$ *and* $x^- = -x \vee 0$, *then* $x = x^+ - x^-$ *and* $x^+ \wedge x^- = 0$. *Moreover if* $x = y - z$ *and* $y \wedge z = 0$, *then* $y = x^+$ *and* $z = x^-$.

(c) *Let* A *be a non-empty subset of* X *such that* $\sup A$ *exists, and set* $x \wedge A = \{x \wedge a : a \in A\}$. *Then* $\sup(x \wedge A)$ *exists and* $\sup(x \wedge A) = x \wedge \sup A$.

(d) *Let* $x, y, z \geq 0$. *Then* $x \wedge (y + z) \leq x \wedge y + x \wedge z$.

The proof of the "if" part of Theorem 2 follows easily from standard results and the above considerations. We first require the following notion. Let K be a convex subset of a linear topological space X. *We say that* K *is in general position provided there exists a* $p \in X^*$ *with* $p(k) = 1$ *for all* $k \in K$. (As usual, X^* denotes the set of continuous linear functionals on X.) Evidently if K is in general position, K is in algebraically general position and $0 \notin \overline{\text{Aff}\, K}$; if K spans X or if X is locally convex, then conversely $0 \notin \overline{\text{Aff}\, K}$ implies K is in general position. Finally, $K \times \{1\}$ is obviously in general position in $X \times \mathbb{R}$.

Now assume that K is a compact metrizable convex subset of a locally convex space X. We may assume without loss of generality that K *is in general position in* X *with* $\operatorname{span} K = X$. Let B be a non-empty Borel subset of K, let $\mathcal{M}_+(B)$ denote the set of all finite non-negative Borel measures on K supported on B, $\mathcal{M}(B)$, the span of $\mathcal{M}_+(B)$, and set $\mathcal{P}(B) = \mathcal{P}(K) \cap \mathcal{M}_+(B)$. We then have that $\mathcal{P}(B)$ *is a base for* $\mathcal{M}_+(B)$ *and* $\mathcal{M}_+(B)$ *is a lattice cone for* $\mathcal{M}(B)$; *moreover* $\mathcal{M}(B)$ *is a sub-lattice of* $\mathcal{M}(K)$. (We shall not prove this important, standard result. We note that if $\mu, \nu \in \mathcal{M}(K)$ and $\lambda = |\mu| + |\nu|$ (with $|\mu|, |\nu|$ the total variation of μ and ν respectively), then $d(\mu \wedge \nu) = f\, d\lambda$ and $d(\mu \vee \nu) = g d\lambda$ where

$$f = \min\left\{\frac{d\mu}{d\lambda}, \frac{d\nu}{d\lambda}\right\} \quad , \quad g = \max\left\{\frac{d\mu}{d\lambda}, \frac{d\nu}{d\lambda}\right\}$$

and $d\mu/d\lambda$ and $d\nu/d\lambda$ denote the Radon-Nikodým derivatives of μ and ν respectively, with respect to λ. If μ, ν are supported on B, so is λ and hence then $\mu \wedge \nu$ and $\mu \vee \nu$ belong to $\mathcal{M}(B)$.)

We now easily obtain the "if" part of the uniqueness theorem. Indeed, define the map $\alpha : \mathcal{P}(\text{Ext}\, K) \to K$ by $\alpha(\mu) = \int_K k\, d\mu(k)$ for all $\mu \in \mathcal{P}(\text{Ext}\, K)$. Theorem 1 shows that α is a well-defined surjection and of course α is an affine map. Hence if every point of K is uniquely represented by a member of $\mathcal{P}(\text{Ext}\, K)$, α is a bijection and so $(X, \leq)$ is order-isomorphic to $\mathcal{M}(\text{Ext}\, K)$ and thus a vector-lattice by the above considerations and Proposition 2.2.

We finally deal with the deep part of Theorem 2, namely the "only if" assertion. The proof we give is a variation of the arguments given in Choquet's original treatment [8], with some "geometric crystallization" of his discussion. Throughout the remainder of this section, we *let* K *be a fixed convex infinite subset of a linear space* X, *with* K *in algebraically general* *position in X and* $\operatorname{span} K = X$*; $\leq$ denotes the order relation on X induced by K. We also let p be the unique linear functional on X with $p \mid K \equiv 1$.*

We first require some basic definitions. Non-negative elements x and y of X are said to be *orthogonal* if whenever u is a non-negative element of X with $u \leq x$ and $u \leq y$, then $u = 0$; we use the notation $x \perp y$ to denote that x and y are orthogonal. We note that if $(X, \leq)$ is a vector lattice, then x and y are orthogonal if and only if $x \wedge y = 0$. (Indeed, one direction is immediate; for the other, suppose x and y are orthogonal and u in X is such that $u \leq x$ and $u \leq y$. Then also $u^+ \leq x$ and $u^+ \leq y$, hence $u^+ = 0$. Since $u = u^+ - u^-$, we deduce that $u \leq 0$, hence 0 is the greatest lower bound of x and y, *i.e.*, $x \wedge y = 0$.) Finally, subsets A and B of non-negative elements of X are *orthogonal*, denoted $A \perp B$, provided a and b are orthogonal for all a in A and b in B.

A convex subset F of K is called a *face* of K provided F is *extremal*; that is, whenever x and y are in K and $\lambda x + (1-\lambda)y \in F$ for some $0 < \lambda < 1$, x and y belong to F. For x in K, F_x denotes the *smallest face of K containing* x; that is,

$$F_x = \bigcap \{\ F : F \text{ is a face of } K \text{ and } x \in F\ \}\ . \tag{13}$$

(It is easily seen that the intersection of an arbitrary family of faces is also a face; thus the right side of (13) is indeed the smallest face containing x.) The next elementary result (cf. [**13**]), gives a fundamental relationship between orthogonality and faces.

Proposition 2.4. *Let x and y be elements of K.*

(a) $F_x = \{k \in K :$ there is a $k' \in K$ and $0 < \lambda < 1$ with $x = \lambda k + (1-\lambda)k'\}$.

(b) x and y are orthogonal if and only if F_x and F_y are disjoint.

Remark. F_x is sometimes called the *face generated by* x. It is obvious that the "λ" in (a) may be chosen of the form $\lambda = \frac{1}{n}$ for n a positive integer; hence one obtains immediately from (a) that

$$F_x = \bigcap_{s>1} [sx - (s-1)K] \cap K = \bigcap_{n=1}^{\infty} [nx - (n-1)K] \cap K\ .$$

Proof of 2.4. To see (a), let provisionally G_x denote the right side of (a). Now if $k \in G_x$, there are k' in K and $0 < \lambda < 1$ with $x = \lambda k + (1-\lambda)k'$; it follows that if F is a face of K containing x, then $k \in F$, so $F_x \supset G_x$. Next, we observe that G_x is convex. To see this geometrically, for x_1, x_2 in X, set $\overline{x_1 x_2} = \{\lambda x_1 + (1-\lambda)x_2 : 0 \leq \lambda \leq 1\}$. *Say that $x \in X$ is interior to $\overline{x_1 x_2}$* if x is an interior point of $\overline{x_1 x_2}$ in its relative topology; *i.e.*, if $x = \beta x_1 + (1-\beta)x_2$ for some $0 < \beta < 1$. Now suppose then $k_1, k_2 \in G_x$ and k is interior to $\overline{k_1 k_2}$. Choose $k_1', k_2' \in K$ so that x is interior to $\overline{k_i k_i'}$ for $i = 1, 2$. A geometric picture now reveals that the line joining