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Edited by A. Dold, B. Eckmann and F. Takens

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Computational Methods and Function Theory

Proceedings, Valparaíso 1989

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held in Valparaíso, Chile, March 13–18, 1989

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Preface

This volume contains the proceedings of the international conference on 'Computational Methods and Function Theory', held at the Universidad Técnica Federico Santa María, Valparaíso, Chile, March 13-18, 1989.

That conference had two goals. The first one was to bring together mathematicians representing two somewhat distant areas of research to strengthen the desirable scientific cooperation between their respective disciplines. The second goal was to have this conference in a country where mathematics as a field of research is developing and scientific contacts with foreign experts are very neccessary. It seems that the conference was successful in both regards. Besides, for many of the non-Chilean participants this was the first visit to South-America and these days left them with valuable personal impressions about the regional problems, an experience which may lead to active support and cooperation in the future.

About 40 half- and one-hour lectures were presented during the conference. They are listed on the last pages of this volume. Of course, not all of them led to a contribution for these proceedings since many have been published elsewhere. However, the papers in this volume are fairly representative for the areas covered.

To hold such a conference, in a place somewhat distant from the international mathematical centers, obviously requires strong support from funding agencies, and it is the organizer's pleasure to acknowledge those contributions at this point. The local organization was made possible through generous grants from the Fundación Andes, Chile, and from our host, the Universidad Técnica Federico Santa María. In addition, foreign participants were supported by a special grant of the National Science Foundation (NSF), USA, and by other national agencies such as the Deutsche Forschungsgemeinschaft (DFG), FRG, the German Academic Exchange Service (DAAD), FRG, the British Council, UK, etc.

We also wish to thank the Universidad Técnica Federico Santa María for the hospitality on its marvellous campus overlooking the beautiful Bay of Valparaíso, and the many people who did help us with the organization. Especially, we wish to thank Ruth Ruscheweyh, who assisted the organizers during the conference and the hot phase of its preparation, and also was responsible for the typesetting (in LaTeX) of the papers in this volume. Finally, we should like to thank Springer-Verlag for accepting these proceedings for its Lecture Notes series.

For the editors:
Stephan Ruscheweyh

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Open Problems and Conjectures in Complex Analysis

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Introduction

This article surveys some of the open problems and conjectures in complex analysis that the author has been interested in and worked on over the last several years. They include problems on polynomials, geometric function theory, and special functions with a frequent mixture of the three. The problems that will be discussed and the author's collaborators associated with each problem are as follows:

1. Polynomials with nonnegative coefficients

We first discuss a series of conjectures which have as one of their sources the work of Rigler, Trimble and Varga in [66]. In [66] these authors considered two earlier papers by Beauzamy and Enflo [23] and Beauzamy [22], which are connected with polynomials and the classical Jensen inequality. To describe their results, let

$$p(z) = \sum_{j=0}^{m} a_j z^j = \sum_{j=0}^{\infty} a_j z^j, \quad \text{where } a_j = 0,\ j > m,$$

be a complex polynomial ($\not\equiv 0$), let d be a number in the interval $(0, 1)$, and let k be a nonnegative integer. Then (cf [22], [23]) p is said to have concentration d of degree at most k if

$$\sum_{j=0}^{k} |a_j| \geq d \sum_{j=0}^{\infty} |a_j|. \tag{1}$$

Beauzamy and Enflo showed that there exists a constant $\hat{C}_{d,k}$, depending only on d and k, such that for any polynomial p satisfying (1), it is true that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta - \log \left(\sum_{j=0}^{\infty} |a_j| \right) \geq \hat{C}_{d,k}. \tag{2}$$

In the case of $k = 0$ in (2) the inequality is equivalent to the Jensen inequality [23],

$$\frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \geq \log |a_0|.$$

Rigler, etc., in [66] considered the extension of this inequality from the class of polynomials to the class of H^∞ (cf. Duren [36]) functions. For $f \in H^\infty$ the functional

$$J(f) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta - \log \left(\sum_{j=0}^{\infty} |a_j| \right)$$

can be well-defined and is finite. They let

$$C_{d,k} = \inf\{J(f) : f \in H^\infty \text{ and } f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0) \text{ satisfies (1)}\}. \tag{3}$$

For a (fixed) $d \in (0, 1)$ and a (fixed) nonnegative integer k, it was shown that there exists an unique positive integer n (dependent on d and k) such that

$$\frac{1}{2^n} \sum_{j=0}^{k} \binom{n}{j} \leq d < \frac{1}{2^{n-1}} \sum_{j=0}^{k} \binom{n-1}{j}.$$

For this n, set

$$\rho = \frac{\binom{n-1}{k}}{\sum_{j=0}^{k}\binom{n-1}{j} - d2^{n-1}} - 1.$$

With these definitions the following conjecture was made in [66].

Conjecture 1. *Let $C_{d,k}$ be defined by (3). Then*

$$C_{d,k} = \log\left(\frac{\rho}{(\rho+1)2^{n-1}}\right). \tag{4}$$

In [66] Conjecture 1 was verified for $k = 0$ and for the subclass of Hurwitz polynomials, i.e., those polynomials with real coefficients and having all their zeros in the left half-plane. In order to verify the conjecture for the entire class an interim step was suggested. This step was one of the motivations for the following problem which was solved recently by this author and others in [10]. Let p be a real polynomial with nonnegative coefficients. Can a conjugate pair of zeros be factored from p so that the resulting polynomial still has nonnegative coefficients? We gave an answer to one proposed choice for factoring out a pair of zeros. Fairly straightforward arguments show that if the degree of the polynomial is less than 6 then a conjugate pair of zeros of *greatest real part* can be factored out and the resulting polynomial will still have non-negative coefficients. However, the example

$$p(z) = 140 + 20z + z^2 + 1000z^3 + 950z^4 + 5z^5 + 20z^6$$

shows that the statement is not true for arbitrary polynomials with non-negative coefficients. A large amount of computer data had suggested the following:

Conjecture 2. *The nonnegativeness of the coefficients of a real polynomial is preserved upon factoring out a conjugate pair of zeros of smallest positive argument in absolute value.*

Interestingly this last conjecture also arose quite independently in the work of Brian Conrey in analytic number theory in his work on one of Polya's conjectures. Conrey announced Conjecture 2 at the annual West Coast Number Theory Conference in December 1987. The conjecture was communicated to this author by the number theorist Ron Evans. Indeed Evans, using a large amount of computer evidence, has generated a closely related conjecture which we include.

Conjecture 3. *If a polynomial of degree $2n$ has zeros*

$$e^{i(t+a_k)} \text{ and } e^{-i(t+a_k)}, \quad k = 1, 2, \cdots n,$$

where the a_k lie between 0 and π, then all the coefficients are nondecreasing functions of t for small $t > 0$ provided the coefficients are all nonnegative for $t = 0$.

A special case of Conjecture 3 where the zeros on the upper semicircle are equally spaced would be of special interest. Although Conjecture 2 was verified in [10] the techniques do not appear applicable to Conjecture 3.

2. The center divided difference of polynomials

Another series of polynomial problems was generated in classical number theory by the work of Evans and Stolarsky in [37]. Given a polynomial p and a real number λ define $\delta_\lambda(p)$, the center divided difference of p, by

$$\delta_\lambda(p) = \begin{cases} \dfrac{p(x+\lambda) - p(z-\lambda)}{2\lambda}, & \lambda \neq 0, \\ p'(z), & \lambda = 0. \end{cases}$$

We did a study of the behavior of the $\delta_\lambda(p)$ as a function of λ in [11]. A number of classical results of Walsh and Obrechkoff and of Kuipers [50] give some information about the zeros of $\delta_\lambda(p)$ as a function of λ. Let $W[p]$ equal the width of the smallest vertical strip containing the zeros of p. It follows from the classical work that

$$W[\delta_\lambda(p)] \leq W[p]$$

and that the diameter of the zero set of $\delta_\lambda(p)$ approaches ∞ as $|\lambda|$ approaches ∞. The Gauss-Lucas theorem shows that

$$W[p'] \leq W[p].$$

It was shown in [11] that

$$W[\delta_\lambda(p)] \leq W[p'] \tag{5}$$

and the conditions on p when equality holds in (5) are given. We were also able to prove that

$$W[\delta_\lambda(p)] = O(1/\lambda) \quad \text{as } |\lambda| \to \infty.$$

The numerical work done by the number theorists had suggested,

Conjecture 4. *$W[\delta_\lambda(p)]$ monotonically decreases to zero as $|\lambda| \to \infty$.*

In that direction it was shown in [11] that

$$W[\delta_{2\lambda}(p)] \leq W[\delta_\lambda(p)] \tag{6}$$

for all positive λ and conditions for equality in (6) were found. In addition, if the zero set of p is symmetric about a vertical line then

$$W[\delta_\lambda(p)] = 0 \text{ for all } \lambda \geq W[p']. \tag{7}$$

However, an example was given of a polynomial p_ε, that contradicts Conjecture 4 at least for some λ. The polynomial p_ε has its zero set symmetric about the imaginary axis and has the property that for small ε, $W[\delta_1(p_\varepsilon)] = 0$ and $W[\delta_\lambda(P_\varepsilon)] = 0$ for $\lambda \geq \sqrt{1+2\varepsilon} = W[p'_\varepsilon]$ while $W[\delta_\lambda(p_\varepsilon)] > 0$ for

$$1 < \lambda < \sqrt{1+2\varepsilon}.$$

Thus conjecture 4 needs to be modified to read

Conjecture 5. *$W[\delta_\lambda(p)]$ monotonically decreases to zero for $\lambda > W[p']$.*

The original question that motivated the number theorist's interest in this problem was the determination of the zeros of $\delta_\lambda(p_N)$ where

$$p_N(z) = \prod_{k=-N}^{N} (z-k).$$

Also occuring in their work were the iterates, $\delta^{(n)}$ of δ defined inductively by

$$\delta_\lambda^{(n)}(p_N) = \delta_\lambda[\delta_\lambda^{(n-1)}(p_N)]$$

with

$$\delta_\lambda^{(1)}(p_N) = \delta_\lambda(p_N).$$

The numerical work had suggested

Conjecture 6. *All nonreal zeros of $\delta_\lambda^{(n)}(p_N)$ are purely imaginary for all λ and all n.*

Conjecture 6 has been verified in [11] for $n = 1$. Indeed, an interesting problem, with other ramifications in number theory, see Stolarsky [71], would be to characterize those polynomials for which $\delta_\lambda^{(n)}$ has only real and pure imaginary roots.

3. Digital filters and zeros of interpolating polynomials

Some interesting problems arise when classical complex analysis techniques are applied to digital filter theory.

Polynomials to be used in interpolation of digital signals are called interpolating polynomials. These polynomials may require modification to assure convergence of their reciprocals on the unit circle. Such modifications provide the opportunity to apply classical analysis theory as was done by the author, Ford, and Wang in [12].

A real function, g, defined for all values of the real independent variable time, t, is called a signal. A digital signal, γ, is a real sequence, $\{\gamma_m : -\infty < m < \infty\}$, consisting of equally spaced values or samples, $\gamma_m = g(m\Delta t)$, from the signal, g, with a time increment or sample interval, Δt. Thus, the independent variable for digital signals such as γ is sample time, $m\Delta t$, or simply sample number, m.

The signal, g, is studied in terms of its classical Fourier transform, G, as a function of real frequency, ω. The digital analog of the Fourier transform consists of the study of a sequence such as γ in terms of its Z-transform, which is defined to be the power series, Γ, having γ_m as the coefficient of z^m. Frequency's digital analog comes from evaluation of Z-transforms such as Γ on the unit circle with the negative of the θ in $z = e^{i\theta}$ referred to as frequency. If the coefficients in Γ are used without any actual evaluation of $\Gamma(z)$ or g is used without computation of G, such use is said to be in the time domain. But if $\Gamma(z)$ is used with evaluation for some z of unit modulus or G is used, such use is said to be in the frequency domain.

Signals are based on even functions in a number of applications. This restricts digital signals to self-inversive cases meaning that $\Gamma(z) = \Gamma(z^{-1})$ for $z \neq 0$. Equivalently, γ is a symmetric sequence meaning that $\gamma_m = \gamma_{-m}$ for all m.

A second signal, f, with Fourier transform, F, poses as a filter of the signal, g, if the convolution integral, $g * f$, of g and f is considered. Of course, the Fourier transform of $g * f$ is the product of the Fourier transforms, G of g and F of f. The discrete analogy consists of the product of Z-transforms, Γ and Φ, where the latter refers to the power series with the sample, $\Phi_m = f(m\Delta t)$, taken from the filter, f, as the coefficient of z^m.

Reduction of certain frequencies is a fundamental aim in the application of a filter, f, to a function, g. This can involve the definition of f by the requirement that $F(\omega)$ be a constant, c, for $|\omega| < \omega_0$ but zero otherwise. If so, c can be chosen so that

$$f(t) = \text{ sinc } \omega_0 t, \tag{8}$$

where sinc is defined by

$$\text{sinc } x = \frac{\sin x}{x}. \tag{9}$$

These equations illustrate that the definition of a real signal is determined from the specifications of its Fourier transform. Similarly, digital signals are often defined by the specification of Z-transforms.

The Fourier transform, F, of the f in (8) is referred to as a frequency window since it has compact support in frequency. Application of such a window to a signal, g, is known as a frequency windowing. These problems concern discrete time windowing. This consists of the scaled truncation of an infinite sequence such as γ to obtain a finite sequence of the form $\{c_m\gamma_m : -L < m < L\}$ wherein the finite sequence, $\{c_m : -L < m < L\}$, is referred to as a time window.

Suppose a given digital signal, $\{b_k : -\infty < k < \infty\}$, is such that b_k is understood to correspond to the time, $kN\Delta t$, with the sample interval, $N\Delta t$, where N is a natural number such that $N > 1$. If this digital signal is to be compared with digital signals based on the smaller sample interval, Δt, the given digital signal must be interpolated to the smaller sample interval, Δt. For example, insertion of $N - 1$ zeros between every b_k and b_{k+1}, followed by multiplication of the Z-transform of the result by the interpolating series, P_N, defined by

$$P_N(z) = 1 + \sum_{m=1}^{\infty} (z^m + z^{-m}) \text{ sinc } \frac{m\pi}{N}, \tag{10}$$

leads to

$$A(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \left(\sum_{j=-\infty}^{\infty} b_j z^{jN} \right) P_N(z). \tag{11}$$

Since the coefficient of z^{kN}, a_{kN}, in A comes from products of b_j and $\text{sinc}(m\pi/N)$ such that $kN = jN \pm m$, it follows that $m \equiv 0 \ (\text{mod} N)$, $\text{sinc}(m\pi/N)$=0 for nonzero m, and $a_{kN} = b_k$. Thus, A is an interpolation of the given B with coefficients, b_j.

A major goal is to study possible alternatives to the interpolation used in (10) in terms of truncation of the interpolating series in (11). In practice one truncates P to obtain the interpolating polynomial, $P_{N,L}$ defined by

$$P_{N,L}(z) = z^{L-1}\left(1 + \sum_{m=1}^{L-1}(z^m + z^{-m}) \operatorname{sinc} \frac{m\pi}{N}\right), \tag{12}$$

where $N > 1$.

To assure stability and accuracy of evaluation it is important that alternative P's have *no* zeros on the unit circle. It is shown in [12] that all of the zeros of $P_{N,L}$ are of unit modulus when $L \leq N$ and examples are given showing that when $L > N+1$ almost any combination of zeros inside, on, and outside the unit circle can occur. A number of classical results are then combined to give sharp conditions on real sequences $\{c_m : 1 \leq m \leq \infty\}$ so that the function $P^*_{N,L}$ defined by

$$P^*_{N,L}(z) = z^{L-1}\left[1 + \sum_{m=1}^{L-1}(z^m + z^{-m})c_m \operatorname{sinc} \frac{m\pi}{N}\right] \tag{13}$$

has no zero of unit modulus. In particular, in order to define a useful test to determine if a specific sequence of numbers will work for the c_m's in (13) the following theorem was proved in [12].

Theorem 1. *If a real sequence, $\{b_m : 0 \leq m < L, b_0 = 1\}$ is such that*

$$\begin{vmatrix} 1 & b_1 & \cdots & & b_{k-1} & b_k \\ b_1 & 1 & b_1 & \cdots & & b_{k-1} \\ \vdots & & & & & \\ b_{k-1} & \cdots & & b_1 & 1 & b_1 \\ b_k & b_{k-1} & \cdots & & b_1 & 1 \end{vmatrix} \geq 0$$

for $0 < k < L$, let

$$c_m = b_m\left(1 - \frac{2\log L}{L}\right)^m$$

define the coefficients in (13). Then $P^*_{N,L}$ has no zero of unit modulus.

A number of the standard "windows" that occur in the engineering literature are then shown to be just special cases of those defined in Theorem 1, including the very generalized Hamming window and the Hanning window. (see Rabniner and Gold's book, *Theory and Application of Digital Signal Processing.*)

The distribution of zeros and the orthogonality property of the sinc functions determine the interpolating properties in (11) and enables the classical results to be applied. Thus one can ask, can the sinc functions be replaced by more general orthogonal functions, e.g., Jacobi polynomials, to create a more general setting in which many more applications can be found? Discussions with several engineers have suggested this.

4. Omitted values problems

We now discuss a number of open problems in geometric function theory. Let

$$\Delta_r = \{z : |z| < r\}, \text{ with } \Delta_1 = \Delta.$$

Let S denote the class of univalent functions f in Δ normalized by $f(0) = 0$ and $f'(0) = 1$. The problem of omitted values was first posed by Goodman [38] in 1949, restated by MacGregor [57] in his survey article in 1972, then reposed in a more general setting by Brannan [5] in 1977. It also appears in Bernardi's survey article [24] and has appeared in several open problem sets since then including [27],[40] and [60].

For a function f in S, let $A(f)$ denote the Lebesgue measure of the set $\Delta \backslash f(\Delta)$ and let $L(f, r)$ denote the Lebesgue measure of the set $\{\Delta \backslash f(\Delta)\} \bigcap \{w : |w| = r\}$ for some fixed $r, 0 < r < 1$. Two explicit problems posed by Goodman and by Brannan were to determine

$$A = \sup_{F \varepsilon S} A(f), \tag{14}$$

and

$$L(r) = \sup_{F \varepsilon S} L(f, r). \tag{15}$$

Goodman [38] showed that $.22\pi < A < .50\pi$. The lower bound which he obtained was generated by a domain of the type shown in Figure 1.

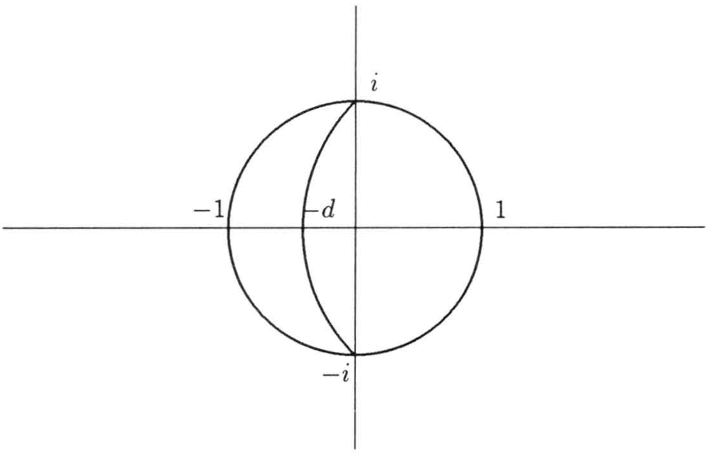

Figure 1

Later, Goodman and Reich [39] gave an improved upper bound of $.38\pi$ for A. Using variational methods developed by the author in [6] and some deep results of Alt and Caffarelli [4] in partial differential equations for free boundary problems, a geometric description for an extremal function for A was given by the author in [9] and by Lewis in [54]. This can be described as follows: There is an f_0 in S with $A = A(f_0)$ such that $f_0(\Delta)$ is circularly symmetric with respect to the positive real axis, i.e., it has the property that for $0 < r < 1$,

$$\frac{\partial}{\partial\theta}|f_0(re^{i\theta})| \text{ and } \frac{\partial}{\partial\theta}|f_0(re^{-i\theta})| \leq 0, \text{ for } 0 < \theta < \pi$$

(cf. Hayman [44]). Moreover the boundary of $f_0(\Delta)$ consists of the negative real axis up to -1, an arc γ of the unit circle that is symmetric about -1 and an arc λ lying in Δ, except for its endpoints. The arc λ is symmetric about the reals, connects the endpoints of γ and has monotonically decreasing modulus in the closure of the upper half disc. These results follow by standard symmetrization methods. Much deeper methods are needed to show (as in [9] and in [54]) that f_0 has a piecewise analytic extension to λ with f_0' continuous on $f_0^{-1}(\lambda)$ and $|f_0'(f_0^{-1}(w))| \equiv c < 1$ for all $w \in \lambda \bigcap \{\Delta \backslash (-1,1)\}$. Using these properties of f_0 it was shown by the author and Pearce in [19] that by "rounding the corners" in certain gearlike domains a close approximation to the extremal function could be obtained. This gives the best known lower bound of

$$.24\pi < A.$$

The upper bound is conceptually harder since it requires an estimate on the omitted area of each function in S. Indeed, it appears difficult to use the geometric description of f_0 to calculate A directly. However, an indirect proof was used by the author and Lewis in [17] to obtain the best known upper bound of

$$A < .31\pi.$$

Open problem. *Show that f_0 is unique and determine A explicitly.*

For the class S^* of functions in S whose images are starlike with respect to the origin, the problem of determining the corresponding

$$A^* = \sup_{f \in S^*} A(f)$$

has been completely solved by Lewis in [54]. The extremal function $f_1 \in S^*$ defined by

$$A^* = A(f_1) \cong .235\pi$$

is unique (up to rotation). The boundary of $f_1(\Delta)$ has two radial rays projecting into Δ with their end points connected by an arc λ_1 that is symmetric about the reals and has $|f_1'(\zeta)| \equiv c_1$ for all $\zeta \in f_1^{-1}(\lambda_1)$.

The problem of determining $L(r)$ in (15) was solved by Jenkins in [47] where he proved that for a fixed $r, 1/4 < r < 1$,

$$L(r) = 2r \arccos(8\sqrt{r} - 8r - 1).$$

The extremal domain in this case is the circular symmetric domain (unique up to rotation) having as its boundary the negative reals up to $-r$ and a single arc of $\{w : |w| = r\}$ symmetric about the point $-r$.

The corresponding problem for starlike functions of determining $L^*(r) = \sup_{f\in S^*} L(f,r)$ was solved by Lewandowski in [53] and by J. Stankiewicz in [70]. The extremal domain in that case is the circularly symmetric domain (unique up to rotation) having as its boundary two radial rays and the single arc of $\{w : |w| = r\}$ connecting their endpoints. An explicit formula for the mapping function in this case was first given by Suffridge in [72].

For the class S^c of functions in S whose images are convex domains the corresponding problem of determining

$$A^c(r) = \sup_{f\in S^c} A(f,r) \tag{16}$$

and

$$L^c(r) = \sup_{f\in S^c} L(r,v). \tag{17}$$

where $A(f,r)$ denotes the Lebesgue measure of $\Delta_r/f(\Delta)$, presents some interesting difficulties. One particular difficulty is that the basic tool of circular symmetrization used in the solution to each of the previous determinations is no longer useful. The example of starting with the convex domain bounded by a square shows that convexity is not always preserved under circular symmetrization. However, Steiner symmetrization (cf. Hayman [44]) can still be used in certain cases such as sectors. Another difficulty is the introduction of distinctly different extremal domains for different ranges of r. Since every function in S^c covers a disk of radius 1/2 (cf. Duren [36]) r needs only to be considered in the interval $(1/2, 1)$. Waniurski has obtained some partial results in [74]. He defined r_1 and r_2 to be the unique solutions to certain transcendental equations where $r_1 \approx .594$ and $r_2 \approx .673$. If $F_{\pi/2}$ is the map of Δ onto the half plane $\{w : \operatorname{Re} w > -1/2\}$ and F_α maps Δ onto the sector

$$\left\{w : \left|\arg\left(w + \frac{\pi}{4\alpha}\right)\right| < \alpha\right\}$$

whose vertex, $v = -\pi/4\alpha$, is located inside the disk, then

$$\begin{aligned} A^c(r) &= A(F_{\pi/2}, r) \text{ for } 1/2 < r < r_1, \\ L^c(r) &= L(F_{\pi/2}, r) \text{ for } 1/2 < r < r_1, \end{aligned}$$

and

$$L^c(r) = L(F_\alpha, r) \text{ for } r_1 < r < r_2.$$

This author had announced in his survey talk on open problems in complex analysis at the 1985 *Symposium on the Occasion of the Proof of the Bieberbach Conjecture* the following conjecture:

Conjecture 7. *The extremal domains in determining $A^c(r)$ and $L^c(r)$ will be half-planes, symmetric sectors and domains bounded by singles arcs of $|w| = r$ along with tangent lines to the endpoints of these arcs, the different domains depending on different ranges of r in $(1/2, 1)$.*

This conjecture was also made independently by Waniurski at the end of his paper [74] in 1987.

Another conjecture that was announced at the *Symposium on the Proof of the Bieberbach Conjecture* arose out of this author and Pearce's work on the omitted values problem. A significant part of characterizing the extremal domains for $A^c(r)$ and $L^c(r)$ in (16) and (17) via the variational method developed in [6] would be the verification of the following:

Conjecture 8. *If* $f \in S^c$ *then*

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{f'(re^{i\theta})}\right| d\theta \le \sup_{z\in\Delta}\left|\frac{z}{f(z)}\right|. \tag{18}$$

Using standard integral means notation this is equivalent to showing that the smallest c such that

$$\mathcal{M}_1\left[1/f'\right] \le c\mathcal{M}_\infty\left[z/f(z)\right] \tag{19}$$

holds is $c = 1$. Well known results (cf. Duren [36], pp. 214) on integral means show that the smallest c for all functions in S is two, while unpublished results of the author and Pearce show that the smallest c for the class of functions starlike of order $1/2$ [cf Goodman [40]] (a slightly larger class than S^c) is $c = 4/\pi$. It was also shown that equality holds in (18) for all domains bounded by regular polygons and it was conjectured that equality holds for those convex domains bounded by single arcs of $\{w : |w| = r\}$ and tangent lines at the endpoints of these arcs. Verification of Conjecture 8 would give an interesting geometric inequality. Let a convex curve Γ have length L and have its minimum distance from the origin be denoted by d. An application of the isoperimetric inequality along with the conjecture would imply

$$\sqrt{\frac{2d\pi}{L}} \le \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|f'(e^{i\theta})|}} \le \frac{L}{2d\pi}. \tag{20}$$

We note that the normalization for the functions f in S^c would force the first and last terms in inequality (20) to go to one as d goes to one.

Determining explicit values for $A^c(r)$ and $L^c(r)$ would involve computing the map that takes Δ onto the convex domains bounded by an arc of $\{w : |w| = r\}$ along with the two tangent lines at the endpoints of this arc. The function defining this map involves the quotient of two hypergeometric functions (cf. Nehari, [62]). In particular an extensive verification shows that the function g as shown in Figure 2

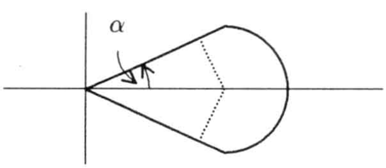

Figure 2