Lecture Notes in Mathematics

S. Albeverio
W. Schachermayer
M. Talagrand

Lectures on Probability Theory and Statistics

1816

Ecole d'Eté de Probabilités
de Saint-Flour XXX – 2000

Editor: P. Bernard

Springer

SEMMES

S. Albeverio W. Schachermayer M. Talagrand

Lectures on Probability Theory and Statistics

Ecole d'Eté de Probabilités de Saint-Flour XXX - 2000

Editor: Pierre Bernard

Springer

Authors

Sergio Albeverio
Institute for Applied Mathematics,
Probability Theory and Statistics
University of Bonn
Wegelerstr. 6
53115 Bonn, Germany

e-mail: albeverio@uni-bonn.de

Walter Schachermayer
Department of Financial and
Actuarial Mathematics
Vienna University of Technology
Wiedner Hauptstraße 8-10/105
1040 Vienna, Austria

e-mail: wschach@fam.tuwien.ac.at

Michel Talagrand
Equipe d'Analyse
Université Paris VI
4 Place Jussieu
75230 Paris Cedex 05
France

e-mail: mit@ccr.jussieu.fr

Editor

Pierre Bernard
Laboratoire de Mathématiques Appliquées
UMR CNRS 6620, Université Blaise Pascal
Clermont-Ferrand, 63177 Aubière Cedex
France

e-mail: pierre.bernard@math.univ-bpclermont.fr

Cover: Blaise Pascal (1623-1662)

Cataloging-in-Publication Data applied for
Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at http://dnb.ddb.de

Mathematics Subject Classification (2000):
60-01, 60-06, 60G05, 60G60, 60J35, 60J45, 60J60, 70-01, 81-06, 81T08, 82-01, 82B44, 82D30, 90-01, 90A09

ISSN 0075-8434 Lecture Notes in Mathematics
ISSN 0721-5363 Ecole d'Eté des Probabilités de St. Flour
ISBN 3-540-40335-3 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York a member of BertelsmannSpringer
Science + Business Media GmbH

http://www.springer.de

Typesetting: Camera-ready TEX output by the authors

SPIN: 10931677 41/3142/du - 543210 - Printed on acid-free paper

INTRODUCTION

This volume contains lectures given at the Saint-Flour Summer School of Probability Theory during the period August 17th - September 3d, 2000. This school was Summer School 2000 of the European Mathematical Society.

We thank the authors for all the hard work they accomplished. Their lectures are a work of reference in their domain.

The School brought together 90 participants, 39 of whom gave a lecture concerning their research work.

At the end of this volume you will find the list of participants and their papers.

Thanks. We thank the European Math Society, the European Commission DG12, Blaise Pascal University, the CNRS, the UNESCO, the city of Saint-Flour, the department of Cantal, the Region of Auvergne for their helps and sponsoring.

Finally, to facilitate research concerning previous schools we give here the number of the volume of "Lecture Notes" where they can be found:

Lecture Notes in Mathematics

1971 : n° 307 – 1973 : n° 390 – 1974 : n° 480 – 1975 : n° 539 –
1976 : n° 598 – 1977 : n° 678 – 1978 : n° 774 – 1979 : n° 876 –
1980 : n° 929 – 1981 : n° 976 – 1982 : n° 1097 – 1983 : n° 117 –
1984 : n° 1180 – 1985–1986 et 1987 : n° 1362 – 1988 : n° 1427 –
1989 : n° 1464 – 1990 : n° 1527 – 1991 : n° 1541 – 1992 : n° 1581 –
1993 : n° 1608 – 1994 : n° 1648 – 1995 : n° 1690 – 1996 : n° 1665 –
1997 : n° 1717 – 1998 : n° 1738 – 1999 : n° 1781 – 2000 : n° 1816

Lecture Notes in Statistics

1986 : n° 50

Table of Contents

Part I Sergio Albeverio: Theory of Dirichlet forms and applications

Part II Walter Schachermayer: Introduction to the Mathematics of Financial Markets

Part III Michel Talagrand: Mean field models for spin glasses: a first course

Part I

Sergio Albeverio: Theory of Dirichlet forms and applications

Table of Contents

Summary. The theory of Dirichlet forms, Markov semigroups and associated processes on finite and infinite dimensional spaces is reviewed in an unified way. Applications are given including stochastic (partial) differential equations, stochastic dynamics of lattice or continuous classical and quantum systems, quantum fields and the geometry of loop spaces.

0 Introduction

The theory of Dirichlet forms is situated in a vast interdisciplinary area which includes analysis, probability theory and geometry.

Historically its roots are in the interplay between ideas of analysis (calculus of variations, boundary value problems, potential theory) and probability theory (Brownian motion, stochastic processes, martingale theory).

First, let us shortly mention the connection between the "phenomenon" of Brownian motion, and the probability and analysis which goes with it. As well known the phenomenon of Brownian motion has been described by a botanist, R. Brown (1827), as well as by a statistician, in connection with astronomical observations, T.N. Thiele (1870), by an economist, L. Bachelier (1900), (cf. [455]), and by physicists, A. Einstein (1905) and M. Smoluchowski (1906), before N. Wiener gave a precise mathematical framework for its description (1921-1923), inventing the prototype of interesting probability measures on infinite dimensional spaces (Wiener measure). See, e.g., [394] for the fascinating history of the discovery of Brownian motion (see also [241], [16] for subsequent developments).

This went parallel to the development of infinite dimensional analysis (calculus of variation, differential calculus in infinite dimensions, functional analysis, Lebesgue, Fréchet, Gâteaux, P. Lévy...) and of potential theory.

Although some intimate connections between the heat equation and Brownian motion were already implicit in the work of Bachelier, Einstein and Smoluchowski, it was only in the 30's (Kolmogorov, Schrödinger) and the 40's that the strong connection between analytic problems of potential theory and fine properties of Brownian motion (and more generally stochastic processes) became clear, by the work of Kakutani. The connection between analysis and probability (involving the use of Wiener measure to solve certain analytic problems) as further developed in the late 40's and the 50's, together with the application of methods of semigroup theory in the study of partial differential equations (Cameron, Doob, Dynkin, Feller, Hille, Hunt, Martin, ...).
The theory of stochastic differential equations has its origins already in work by P. Langevin (1911), N. Bernstein (30's), I. Gikhman and K. Ito (in the 40's), but further great developments were achieved in connection with the above mentioned advances in analysis, on one hand, and martingale theory, on the other hand.
By this the well known relations between Markov semigroups, their generators and Markov processes were developed, see, e.g. [162], [160], [207], [208], [209], [276], [463].
This theory is largely concerned with processes with "relatively nice characteristics" and with "finite dimensional state space" E (in fact locally compact state spaces are usually assumed). From many areas, however, there is a demand of extending the theory in two directions:

1) "more general characteristics", e.g. allowing for singular terms in the generators
2) infinite dimensional (and nonlinear) state spaces.

As far as 1) is concerned let us mention the needs of handling Schrödinger operators and associated processes in the case of non smooth potentials, see [70].
As far as 2) is concerned let us mention the theory of partial differential equations with stochastic terms (e.g. "noises"), see, e.g. [201], [28], [37], [38], [129], [127] the description of processes arising in quantum field theory (work by Friedrichs, Gelfand, Gross, Minlos, Nelson, Segal...) or in statistical mechanics, see, e.g. [16], [15], [344], [242]. Other areas which require infinite dimensional processes are the study of variational problems (e.g. Dirichlet problem in infinite dimensions) [278], the study of certain infinite dimensional stochastic equations of biology, e.g. [474], the representation theory of infinite dimensional groups, e.g. [68], the study of loop groups, e.g. [30], [12], the study of the development of interest rates in mathematical finance, e.g. [416], [337], [502].
The theory of Dirichlet forms is an appropriate tool for these extensions. In fact it is central for it to work with reference measures μ which are neither necessarily "flat" nor smooth and in replacing the Markov semigroups on continuous functions of the "classical theory" by Markov semigroups on

$L^2(\mu)$-spaces (thus making extensive use of "Hilbert space methods" [211]). The theory of Dirichlet forms was first developed by Feller in the 1-dimensional case, then extended to the locally compact case with symmetric generators by Beurling and Deny (1958-1959), Silverstein (1974), Ancona (1976), Fukushima (1971-1980) and others (see, e.g., [244], [258]).(Extensions to non symmetric generators were given by J. Elliott, S. Carrillo-Menendez (1975), Y. Lejan (1977-1982), a.a., see, e.g. [367]).
The case of infinite dimensional state spaces has been investigated by S. Albeverio and R. Høegh-Krohn (1975-1977), who were stimulated by previous analytic work by L. Gross (1974) and used the framework of rigged Hilbert spaces (along similar lines is also the work of P. Paclet (1978)). These studies were successively considerably extended by Yu. Kondratiev (1982-1987), S. Kusuoka (1984), E. Dynkin (1982), S.Albeverio and M.Röckner (1989-1991), N. Bouleau and F. Hirsch (1986-1991), see [39], [147], [278], [367], [230], [172], [465], [234], [235], [236], [237], [238], [239], [256].
An important tool to unify the finite and infinite dimensional theory was provided by a theory developed in 1991, by S. Albeverio, Z.M. Ma and M. Röckner, by which the analytic property of quasi regularity for Dirichlet forms has been shown in "maximal generality" to be equivalent with nice properties of the corresponding processes.
The main aim of these lectures is to present some of the basic tools to understand the theory of Dirichlet forms, including the forefront of the present research. Some parts of the theory are developed in more details, some are only sketched, but we made an effort to provide suitable references for further study.
The references should also be understood as suggestions in the latter sense, in particular, with a few exceptions, whenever a review paper or book is available we would quote it rather than an original reference. We apologize for this "distortion", which corresponds to an attempt of keeping the reference list into some reasonable bounds - we hope however the references we give will also help the interested reader to reconstruct historical developments.
For the same reason, all references of the form "see [X]" should be understood as "see [X] and references therein".

1 Functional analytic background: semigroups, generators, resolvents

1.1 Semigroups, Generators

The natural setting used in these lectures is the one of normed linear spaces B over the closed algebraic field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Some of the results are however depending on the additional structure of completeness, therefore we shall assume most of the time that B is a Banach space.
We are interested in describing operators like the Laplacian Δ and the associated semigroup (heat semigroup), and vast generalizations of them.
Let $L \equiv (L, D(L))$ be a linear operator on a normed space B over $\mathbb{K}$, defined on a linear subset $D(L)$ of B, the definition domain of L.
We say that two such operators $L_i, i = 1, 2$ are equal if $D(L_1) = D(L_2)$ and $L_1 u = L_2 u, \forall u \in D(L_1)$.
L is said to be bounded if $\exists C \geq 0$ s.t. $\|Lu\| \leq C\|u\|, \forall u \in D(L) = B$.
We then have, setting $\|L\| \equiv \sup_{u \in B, \|u\| \leq 1} \|Lu\| \in [0, +\infty]$

$$L \text{ bounded } \Leftrightarrow \|L\| < +\infty.$$

L is said to be continuous at 0 ($\in D(L)$!) if $u_n \to 0, u_n \in D(L)$ implies $Lu_n \to 0, n \to \infty$.
L is said to be continuous if $u_n \to u, u_n \in D(L)$ implies $u \in D(L)$ and $Lu_n \to Lu, n \to \infty$.
One easily shows

$$L \text{ bounded } \Leftrightarrow L \text{ continuous at } 0 \Leftrightarrow L \text{ continuous.}$$

We define $L = \alpha_1 L_1 + \alpha_2 L_2, \alpha_i \in \mathbb{K}, i = 1, 2$, by
$D(L) = D(L_1) \cap D(L_2), Lu = \alpha_1 L_1 u + \alpha_2 L_2 u, \forall u \in D(L)$.
Moreover we define for L_1, L_2
$L_1 L_2 u \equiv L_1(L_2 u), \forall u \in D(L_1 L_2) \equiv L_1 D(L_2) \equiv \{u \in B | L_2 u \in D(L_1)\}$

Definition 1. *A linear bounded operator A on a normed linear space B is a contraction if $\|A\| \leq 1$. A family $T = (T_t)_{t \geq 0}$ of linear bounded operators on B is said to be a strongly continuous semigroup or C_0-semigroup if*

i) $T_0 = 1$ *(the identity on B)*
ii) $\lim_{t \downarrow 0} T_t u = u, \forall u \in B$ *(strong continuity)*
iii) $(T_t)_{t \geq 0}$ *is a semigroup i.e.*
$T_t T_s = T_s T_t = T_{s+t}, \forall t, s > 0.$
$(T_t)_{t \geq 0}$ is said to be a C_0-semigroup of contractions or a

C_0-contraction semigroup if, in addition,
iv) T_t *is a contraction for all $t \geq 0$.*

Exercise 1. Show that i),ii),iv) imply that $t \to T_t u$ is continuous, for all $t \geq 0, \forall u \in B$.

Definition 2. *Let $T \equiv (T_t)_{t\geq 0}$ be a C_0-contraction semigroup on B. The linear operator L is said to be generator of T if:*

i) $D(L) \equiv \left\{ u \in B | \lim\limits_{t\downarrow 0} \frac{1}{t}(T_t u - u) \text{ exists in } B \right\}$
ii) $Lu = \lim\limits_{t\downarrow 0} \frac{1}{t}(T_t u - u) \, \forall u \in D(L)$

Exercise 2. Show that the "strong derivative" $\frac{d}{dt}T_t u \equiv \lim\limits_{h\downarrow 0} \frac{(T_{t+h}-T_t)u}{h}$ exists in B, for all $u \in D(L)$ and $\frac{d}{dt}T_t u = LT_t u = T_t L u \, \forall t \geq 0, \forall u \in D(L)$.
In particular $Lu = \frac{d}{dt}T_t u|_{t=0}, \forall u \in D(L)$.

It is easy to convince oneself that even simple operators like the Laplacian Δ are not bounded, e.g. in $B = L^2(\mathbb{R}^d)$. For this reason it is useful to introduce the concept of a closed operator.

Definition 3. *A linear operator L in B is called closed if $u_n \in D(L)$, $u_n \to u$ as $n \to \infty$, Lu_n convergent as $n \to \infty$, in B, imply that $u \in D(L)$, and $Lu_n \to Lu$.*

Exercise 3. Show that L closed $\Leftrightarrow$ $G(L)$ closed in $B \times B$, where $G(L) \equiv \{\{u, Lu\}, u \in D(L)\}$ is the graph of L.

Proposition 1. *Let $T = (T_t)_{t\geq 0}$ be a C_0-contraction semigroup on a Banach space B, with generator L. Then $T_t u = u + \int_0^t T_s L u \, ds, u \in D(L)$ where the integral on the r.h.s is to be understood in the natural sense of strong integrals on Banach spaces (Bochner integral [1]).*

Proof. This follows immediately from Exercise 2, via integration. □

Proposition 2. *The generator L of a C_0-contraction semigroup $T = (T_t)_{t\geq 0}$ on a Banach space is a closed operator.*

Proof. This easily follows from Proposition 1, the strong continuity (Exercise 1), the fact that for $u_n \to u$, Lu_n convergent to v, $\|T_s L u_n\| \leq \|Lu_n\| \leq C$, for some $C \geq 0$, independent of n, as Lu_n converges, and dominated convergence. □

Proposition 3. *The generator L of a C_0-contraction semigroup $T = (T_t)_{t\geq 0}$ on a Banach space is densely defined.*

---

[1] See, e.g. [506], p.132

Proof. One easily shows that for any $u \in B$, with $v_t \equiv \int_0^t T_s u ds$:

$$\frac{1}{r}[v_{t+r} - v_t] = \frac{1}{r}[T_r v_t - v_t] \to T_t u - u, \text{ as } r \downarrow 0$$

hence $v_t \in D(L)$.
On the other hand
$\frac{v_t}{t} \to u, t \downarrow 0$, yielding an approximation of an arbitrary $u \in B$ by elements $\frac{v_t}{t}$ in $D(L)$. □

Corollary 1. *If* $T = (T_t)_{t\geq 0}, S = (S_t)_{t\geq 0}$ *are two* C_0*-contraction semigroups on a Banach space with the same generator* L*, then* $T_t = S_t \quad \forall t \geq 0$.

Proof. From Exercise 2 we have easily $\frac{d}{ds}T_{t-s}S_s u = 0, \forall 0 \leq s \leq t, \forall u \in D(L)$ from which $T_t u = S_t u \forall u \in D(L)$ follows, hence $T_t = S_t$, these being bounded and $D(L)$ being dense. □

The above corollary implies that the usual notation $T_t = e^{tL}, t \geq 0$ for the semigroup with generator L is justified.
The question when a given densely defined linear operator L is the generator of a C_0-contraction semigroup is answered by the theory of Hille-Yosida. For this we recall some basic definitions.
If L is a linear injection (1-1 map), then L^{-1} is defined on $D(L^{-1}) = LD(L)$, by $L^{-1}u = v, u \in D(L^{-1})$, with v s.t. $Lv = u$.
For a linear operator L the resolvent set is defined by:
$\rho(L) \equiv \{\alpha \in \mathbb{K} | \alpha - L : D(L) \to B$ is an injection onto B i.e.
$D((\alpha - L)^{-1}) = B$. Moreover $(\alpha - L)^{-1}$ is bounded.$\}$

Exercise 4. Show that if $\rho(L) \neq 0$ then $\rho(L)$ is closed (use that $(\alpha - L)^{-1}$ for $\alpha \in \rho(L)$ is bounded).

The spectrum $\sigma(L)$ of L is by definition the complement in $\mathbb{K}$ of $\rho(L)$. For $\alpha \in \rho(L), G_\alpha \equiv (\alpha - L)^{-1}$ (which exists as a bounded operator on B) is called the resolvent of L at α.
$(G_\alpha)_{\alpha \in \rho(L)}$ is called the resolvent family associated to L.

Exercise 5. Show that $(G_\alpha)_{\alpha\in\rho(L)}$ satisfies the resolvent identity
$G_\alpha - G_\beta = (\beta - \alpha)G_\alpha G_\beta = (\beta - \alpha)G_\beta G_\alpha, \forall \alpha, \beta \in \rho(L)$.

Proposition 4. *Let* L *be the generator of a* C_0*-contraction semigroup on a Banach space. Then* $(0, \infty) \subset \rho(L)$ *and for any*
$Re\alpha > 0 : (\alpha - L)^{-1}u = G_\alpha u = \int_0^{+\infty} e^{-\alpha t} T_t u dt$
(where the integral is in Bochner's sense) and $\|G_\alpha\| \leq \frac{1}{Re\alpha}$.

Proof. Set $R_\alpha \equiv \int\limits_0^{+\infty} e^{-\alpha t} T_t dt$.
It is easily seen that $(\alpha - L)R_\alpha u = u, \forall u \in B, Re\alpha > 0$. Since L is closed for all $u \in D(L) : LR_\alpha u = R_\alpha Lu$, from which one deduces that $\alpha - L$ is injective for $Re\alpha > 0$ (in particular for $\alpha > 0$) and $R_\alpha = G_\alpha$. The bound in Proposition 4 then follows from the definition of R_α. □

Remark 1. G_α is the Laplace transform of T_t (in the sense given by Proposition 4).

Theorem 1. *(Hille-Yosida, for C_0-contraction semigroups): Let L be a linear operator in a Banach space B. The following are equivalent:*

i) L is the generator of a C_0-contraction semigroup $T = (T_t)_{t \geq 0}$ on B.
ii) L is densely defined and
α) $(0, \infty) \subset \rho(L)$
β) $\|\alpha(\alpha - L)^{-1}\| \leq 1 \quad \forall \alpha > 0$

Corollary 2. *If ii) is fullfilled then L is closed and uniquely determined.*

Proof. ii) implies i) by Theorem 1 and hence that L is closed by Proposition 2. The rest follows from Corollary 1. □

Proof. (of Theorem 1)
i) ⇒ ii): From i) we have L closed, densely defined (Propositions 2,3). That $(0, \infty) \subset \rho(L)$ and ii) holds follows from Proposition 4.
ii) ⇒ i): For details we refer to, e.g.[413]. In the proof the following Proposition is useful.

Proposition 5. *Let L satisfy the conditions ii) of Theorem 1. Set $G_\alpha = (\alpha - L)^{-1}, \alpha > 0$. Then*

i) $\alpha G_\alpha u \to u$ in B, as $\alpha \to +\infty$
ii) Define $L^{(\alpha)} \equiv -\alpha + \alpha^2 G_\alpha, \alpha > 0$ ("Yosida approximation of L"). Then $L^{(\alpha)}$ is bounded, $D(L^{(\alpha)}) = B, L^{(\alpha)}u \to Lu, \alpha \uparrow +\infty, u \in D(L)$, and $e^{tL^{(\alpha)}}u$ converges as $\alpha \uparrow +\infty$ for all $u \in D(L)$ to $\tilde{T}_t u$, where $\tilde{T}_t$ is a C_0-contraction semigroup, with generator L. Moreover $\tilde{T}_t$ coincides with the semigroup T_t generated by L mentioned in i).

Proof. For $u \in D(L)$ we have

$$\begin{aligned}
\|\alpha G_\alpha u - u\| &= \|\alpha(\alpha - L)^{-1}u - (\alpha - L)(\alpha - L)^{-1}u\| \\
&= \|L(\alpha - L)^{-1}u\| \\
&= \|(\alpha - L)^{-1}Lu\| \\
&\leq \frac{1}{\alpha}\|Lu\| \to 0, \alpha \uparrow +\infty
\end{aligned}$$

(where we used Proposition 4). But αG_α is a contraction by Proposition 4 and $D(L)$ is dense by assumption, hence $\alpha G_\alpha u \to u$ as $\alpha \uparrow +\infty$, for all $u \in B$.

From this it is easy to see that $\alpha G_\alpha Lu \to Lu, u \in D(L)$, as $\alpha \uparrow +\infty$, and thus $L^{(\alpha)}u = -\alpha u + \alpha^2 G_\alpha u = \alpha G_\alpha Lu \to Lu$ as $\alpha \uparrow +\infty$.
The rest follows by realizing that

$$e^{tL^{(\alpha)}} u = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^{(\alpha)^n} u = e^{\alpha t} e^{-\alpha^2 G_\alpha} u$$

Remark 2. Another useful "approximation formula" for T_t in terms of the resolvent is the following one:

$$T_t u = \lim_{n\to\infty} \left(\frac{n}{t}\right)^n \left(G_{\frac{n}{t}} u\right)^n, \forall u \in B$$

(see, e.g., [413], p. 33).

Remark 3. In the formulation of Hille-Yosida's theorem i) can be replaced by a statement involving the generator of a C_0-contraction resolvent family according to the following definition.

Definition 4. *A C_0-contraction resolvent family is a family $(G_\alpha)_{\alpha>0}$ such that*

$$\alpha G_\alpha u \to u, \alpha \uparrow +\infty, \|\alpha G_\alpha\| \leq 1, \alpha > 0$$

and the resolvent identity in Exercise 5 holds.
Hille-Yosida's theorem holds then with i) replaced by:

i') L is the generator of a C_0-contraction resolvent family $(G_\alpha)_{\alpha>0}$ in the sense that $G_\alpha = (\alpha - L)^{-1}$ on B. There is a one-to-one correspondence between C_0-contraction semigroups $(T_t)_{t\geq 0}$ and C_0-contraction resolvent families $(G_\alpha)_{\alpha>0}$ given by the Laplace-transform formula in Proposition 4 (and Remark 1) resp. Proposition 5 or Remark 2 after Proposition 5.

Hille-Yosida's characterization of generators L involves the resolvent G_α. A pure characterization of L, under some "direct restrictions" on L is given by the Lumer-Phillips theorem, for which we need a definition.

Definition 5. *The duality set $F(u)$ for any element u in a Banach space B is defined by*

$$F(u) \equiv \left\{u^* \in B^* | \langle u^*, u\rangle = \|u\|^2 = \|u^*\|^2\right\},$$

where B^* is the dual of B (the space of continuous linear functionals on B) and $\langle , \rangle$ is the dualization between B and B^*.
An operator L is dissipative on B if for any $u \in D(L)$ there exists some $u^* \in F(u)$ such that $Re\langle u^*, Lu\rangle \leq 0$.
($-L$ is then said to be accretive).