UNIVERSITY

MATHEMATICAL
TEXTS

PARTIAL DIFFERENTIATION

BY

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OLIVER AND BOYD
EDINBURGH AND LONDON
NEW YORK: INTERSCIENCE PUBLISHERS, INC.
1954

First Published 1951

Second Edition 1954

PRINTED IN HOLLAND BY
KONINKLIJKE VERENIGDE DRUKKERIJEN HOITSEMA N.V., GRONINGEN
FOR OLIVER AND BOYD LTD., EDINBURGH

PREFACE

This book is based on a set of lectures delivered at Glasgow University. I wish to thank Professor T. M. MacRobert and Dr. T. S. Graham whose lecture notes were of great help to me in preparing these lectures. My thanks are also due to Dr. Elizabeth A. McHarg and to Dr. Daniel Martin for their valuable suggestions and for the care with which they have corrected the proofs.

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Glasgow.

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CHAPTER I

PARTIAL DERIVATIVES

§ 1. Functions of Two Variables

The subject of this book, partial differentiation, is concerned with the rates of change of functions of more than one variable. It involves extensions of the ideas and methods of the differential calculus as applied to functions of a single variable, and it will be assumed that the reader is familiar with these ideas and methods.

A simple example of a function of the two independent variables x and y is that of the function z defined to be $2x^2 + 3y^2$. The value of z depends on the values given independently to the two variables x and y, and to obtain a value of z, x and y must simultaneously be given values. Geometrically, these pairs of values of x and y can be represented as points in the (x, y) plane and so a function z of the two variables x and y is defined at points in the (x, y) plane, just as a function of the single variable x is defined at points on the x-axis. Functions of two variables x and y may be defined in regions of the (x, y) plane, just as functions of the single variable x are defined in intervals of the x-axis. Thus the function $\sqrt{(1-x^2-y^2)}$ has real values only inside and on the circle whose centre is the origin of coordinates and whose radius is 1.

When z is a function of the independent variables x and y, it is defined for pairs of values (x, y), and for each such pair there exists an ordered set of three numbers (x, y, z), viz., the pair x, y and the corresponding value of z. Each of these sets represents a point in (x, y, z) space and the values of the function correspond to the aggregate of these points in space. In the case of the function $z = 2x^2 + 3y^2$ this aggregate consists of the points of the paraboloid with this equation. In general the functions dealt with are defined over regions of the (x, y) plane and are represented by surfaces in (x, y, z) space, and it will be assumed throughout that the reader is familiar with the elements of the theory of analytical geometry of three dimensions. *

The discussion of the rate of change of a function of two variables is essentially a more complicated matter than that of the rate of change of a single variable, since the variables vary independently. The simplest way to approach the problem is to find the rate of change of the function with respect to one of the variables while the other is kept constant. For example, if $z = f(x, y) = 2x^2y + 3xy^2$, the rate of change of z with respect to x may be found, where y is held constant. To do this z is differentiated with respect to x, treating y as a constant, and the expression $4xy + 3y^2$ is obtained. This quantity is called the **partial derivative** of z at (x, y) with respect to x and is denoted by $\partial z/\partial x$ or f_x. The process of differentiating a function with respect to one variable, while treating the other variables on which the function depends as constants, is called **partial differentiation.** If, in the above example, x is kept constant and z is differentiated with respect to y, the expression $2x^2 + 6xy$, the partial derivative at (x, y) of z with respect to y, is obtained. This derivative, as above, is written $\partial z/\partial y$ or f_y.

In the case of a function $y = f(x)$ of the single variable x, the derivative dy/dx is represented geometrically by the gradient of the tangent at the point (x, y) on the curve with equation $y = f(x)$, and from this can be deduced a geometrical interpretation of the above-defined partial derivatives. The value of the partial derivative of the function $z = f(x, y)$ with respect to x, when $x = x_0$,

* See W. H. McCrea, *Analytical Geometry of Three Dimensions.*

$y = y_0$, is the value, when $x = x_0$, of the function of x, $f_x(x, y_0)$. Now $z = f(x, y_0)$, $y = y_0$ is the equation of the intersection of the surface $z = f(x, y)$ by the plane $y = y_0$, i.e., it is the equation of a curve on the surface $z = f(x, y)$ and the value of the partial derivative $f_x(x_0, y_0)$ is therefore the tangent of the angle which the tangent to this curve at the point (x_0, y_0) makes with a line in the plane $y = y_0$ parallel to the x-axis. Similarly $\partial z/\partial y$ for $x = x_0$, $y = y_0$ is the value of the derivative of the function of y, $f(x_0, y)$, when $y = y_0$, and hence is the tangent of the angle which the tangent at the point (x_0, y_0) on the curve of intersection of the surface $z = f(x, y)$ by the plane $x = x_0$, makes with a line parallel to the y-axis in this plane.

Since the partial derivative with respect to x of the function $f(x, y)$ when $x = x_0$, $y = y_0$ is the derivative of the function of x, $f(x, y_0)$, when $x = x_0$, it is defined as the limit

$$\lim_{h\to 0} \{f(x_0 + h, y_0) - f(x_0, y_0)\}/h,$$

if this limit exists. Similarly the partial derivative of $f(x, y)$ when $x = x_0$, $y = y_0$, with respect to y is the limit

$$\lim_{k\to 0} \{f(x_0, y_0 + k) - f(x_0, y_0)\}/k,$$

if it exists.

These two limits are examples of **simple limits** of a function of two variables and the general definition of such limits will now be given. If in the function $f(x, y)$, y is given the constant value y_0 and the function is considered as the function $f(x, y_0)$ of the single variable x, it may happen that the limit of this function of x

$$\lim_{x\to x_0} f(x, y_0)$$

exists. Such a limit is called a simple limit of a function of two variables. As y is given different values the limit,

$\lim_{x \to x_0} f(x, y)$, will assume values, when they exist, corresponding to the values of y, i.e., the limit is a function of y, say

$$\lim_{x \to x_0} f(x, y) = \varphi(y).$$

Similarly the limit

$$\lim_{y \to y_0} f(x, y) = \psi(x)$$

is a function of x.

In obtaining a simple limit one of the two variables is held constant, but it will now be shown that a limit of a function of two variables can be defined in which both variables vary simultaneously. The function $f(x, y)$ is said to have the **double limit** $\mathbf{L}$ as x tends to x_0 and y tends to y_0, if, given any positive number ε, a non-zero number η can be found such that $|f(x, y) - \mathrm{L}| < \varepsilon$ for those values (x, y) for which $f(x, y)$ is defined and for which $|x - x_0| < \eta$ and $|y - y_0| < \eta$, This double limit is written

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = \mathrm{L}.$$

It is easy to show that an alternative form of the definition of a double limit can be obtained by replacing the condition $|x - x_0| < \eta$, $|y - y_0| < \eta$ by the condition $\sqrt{\{(x - x_0)^2 + (y - y_0)^2\}} < \eta$. For it is clear that if the first of these conditions is satisfied, the second is satisfied, and if, on the other hand, the second condition is satisfied, then $|f(x, y) - \mathrm{L}| < \varepsilon$ for those values (x, y) for which $f(x, y)$ is defined and for which $|x - x_0| < \eta/\sqrt{2}$ and $|y - y_0| < \eta/\sqrt{2}$, so that the first condition is satisfied using $\eta/\sqrt{2}$ as the η of the condition. Geometrically, the first form of the definition means that, given ε, there can be found a square of side 2η, with (x_0, y_0) as its centre-point and with sides parallel to the coordinate axes, such that $|f(x, y) - \mathrm{L}| < \varepsilon$ at all points inside the square, while the second form means that, given ε, a circle can be found, with centre (x_0, y_0) and radius η, such that $|f(x, y) - L| < \varepsilon$ at all points inside the circle.

Ex. 1. Show that the double limit of the function $(x^3+y^3)/(x^2+y^2)$ for (x, y) tending to $(0, 0)$ is zero.

Ex. 2. Show that the double limit of the function $xy/(x^2 + y^2)$ for (x, y) tending to $(0, 0)$ does not exist.

Corresponding to the ideas of simple and double limits of a function of two variables there can be introduced the ideas of **simple continuity,** i.e., continuity with respect to one of the two variables by itself, and of **double continuity,** i.e., continuity with respect to the pair of variables. The function $f(x, y)$ is said to be **continuous with respect to x** at (x_0, y_0) if

$$\lim_{x \to x_0} f(x, y_0) = f(x_0, y_0),$$

and it is said to be **continuous with respect to y** at (x_0, y_0) if

$$\lim_{y \to y_0} f(x_0, y) = f(x_0, y_0).$$

The function $f(x, y)$ is said to be **continuous with respect to the pair of variables (x, y)** at (x_0, y_0) if the double limit of $f(x, y)$ as (x, y) tends to (x_0, y_0) is $f(x_0, y_0)$, i.e., given any positive number ε, a non-zero number η can be found such that

$$|f(x, y) - f(x_0, y_0)| < \varepsilon \text{ if } \sqrt{\{(x - x_0)^2 + (y - y_0)^2\}} < \eta.$$

If in this inequality those (x, y) pairs are taken for which $y = y_0$, the condition is

$$|f(x, y_0) - f(x_0, y_0)| < \varepsilon \text{ if } |x - x_0| < \eta,$$

so that $f(x, y)$ is continuous with respect to x at (x_0, y_0). Similarly it can be shown that $f(x, y)$ is continuous with respect to y at (x_0, y_0), and thus if $f(x, y)$ is continuous with respect to the pair of variables (x, y) at (x_0, y_0) it is continuous there with respect to the variables separately.

If a function of two variables is continuous at a point with respect to the variables separately, it does not follow that it is continuous there with respect to the pair of variables, as is seen by considering the function $f(x, y)$, defined as $xy/(x^2 + y^2)$ when x and y are not both zero, and as zero at $(0, 0)$. This function is continuous at $(0, 0)$ with respect to the variables separately, but since it possesses no double limit as (x, y) tends to $(0, 0)$ it is not continuous at $(0, 0)$ with respect to the pair of variables.

A function $f(x, y)$ is said to be continuous with respect to the pair of variables (x, y) in a region R of the (x, y) plane, if it is continuous at every point of the region; i.e.,

(i) $f(x, y)$ has a definite value at every point of the region,

(ii) corresponding to any point (x_0, y_0) of the region, given any positive number ε, a non-zero number η can be found such that $|f(x, y) - f(x_0, y_0)| < \varepsilon$, if $|x - x_0| < \eta$, $|y - y_0| < \eta$ and (x, y) belongs to the region. The value of η depends on ε and, in general, on the particular point (x_0, y_0). If an η can be found which serves for every (x_0, y_0) of the region, the function is said to be **uniformly continuous in the region.** It can be proved * that a function which is continuous with respect to the pair of variables in a closed region, i.e., a region which contains its boundaries, is uniformly continuous in that region.

§ 2. Functions of Three or More Variables

It is easy to extend the ideas of the last paragraph to functions of more than two variables. A function such as

* See, for example, Hobson, *The Theory of Functions of a Real Variable* (Second Edition), Vol. I, p. 274.

$f(x, y, z) = x^2y + xyz + xz^2 + y^2z$ is a function of the three independent variables x, y, z. A function of three variables is defined at points in three-dimensional space and it will be represented by a "surface" in four-dimensional space. Partial derivatives of functions of three variables are defined as in the case of functions of two variables. The partial derivative of $f(x, y, z)$ with respect to x at the point (x, y, z), written as above $\partial f/\partial x$ or f_x, is the rate of change of f at this point with respect to x, where y and z are kept constant. Similarly $\partial f/\partial y$ or f_y is the rate of change of f with respect to y, where x and z are kept constant, and $\partial f/\partial z$ or f_z is the rate of change of f with respect to z, where x and y are kept constant. In the above example, where $f = x^2y + xyz + xz^2 + y^2z$, straightforward differentiation gives

$$f_x = 2xy + yz + z^2, \quad f_y = x^2 + xz + 2yz, \quad f_z = xy + 2xz + y^2.$$

The partial derivatives of $f(x, y, z)$ at the point (x_0, y_0, z_0) are given by the following limits, if they exist:

$$f_x(x_0, y_0, z_0) = \lim_{h\to 0} \{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)\}/h,$$

$$f_y(x_0, y_0, z_0) = \lim_{k\to 0} \{f(x_0, y_0 + k, z_0) - f(x_0, y_0, z_0)\}/k,$$

$$f_z(x_0, y_0, z_0) = \lim_{l\to 0} \{f(x_0, y_0, z_0 + l) - f(x_0, y_0, z_0)\}/l.$$

Continuity of the function $f(x, y, z)$ with respect to the variables (x, y, z) is defined analogously to the continuity of $f(x, y)$ with respect to the pair (x, y). The function $f(x, y, z)$ is said to be continuous with respect to the variables (x, y, z) at (x_0, y_0, z_0) if

(i) $f(x, y, z)$ has a definite value $f(x_0, y_0, z_0)$ at (x_0, y_0, z_0),

(ii) given any positive number ε, a non-zero number η can be found such that

$$| f(x, y, z) - f(x_0, y_0, z_0) | < \varepsilon \text{ if } | x - x_0 | < \eta,$$
$$| y - y_0 | < \eta, \ | z - z_0 | < \eta.$$

Just as in the case of a function of two variables a function of three variables can possess continuity with respect to the variables separately and also in this case the function may possess continuity with respect to any pair of the three variables.

It is clear that partial derivatives may be defined for a function of any number of variables. For example the partial derivative of the function $f(x_1, x_2, \ldots, x_n)$ with respect to the variable x_1 at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ is defined to be the limit

$$\lim_{h_1 \to 0} \{f(x_1^0 + h_1, x_2^0, \ldots, x_n^0) - f(x_1^0, x_2^0, \ldots, x_n^0)\}/h_1.$$

In differentiating such a function partially with respect to one particular variable, all the other variables are treated as constants. The definition of continuity is easily extended from that of a function of three variables to that of a function of any number of variables.

§ 3. Higher Partial Derivatives

In the case of a function of a single variable x, $y = f(x)$, the second derivative d^2y/dx^2 or $f''(x)$ is defined as the rate of change of dy/dx or $f'(x)$ with respect to x, the third derivative d^3y/dx^3 or $f'''(x)$ as the rate of change of d^2y/dx^2 with respect to x, and so on. Extending this idea to $f(x, y)$, a function of two variables, $\partial^2 f/\partial x^2$ or f_{xx} is the rate of change at (x, y) with respect to x of the function $\partial f/\partial x$ or f_x, y being held constant, and $\partial^2 f/\partial y^2$ or f_{yy} is the rate of change at (x, y) with respect to y of the function $\partial f/\partial y$ or f_y, x being held constant. For example, if

$$f(x, y) = 4x^3y^2 - 5xy^4 + 2x^2y^2 + 3x^2 + 4xy,$$
$$f_x = 12x^2y^2 - 5y^4 + 4xy^2 + 6x + 4y, \quad f_{xx} = 24xy^2 + 4y^2 + 6,$$
$$f_y = 8x^3y - 20xy^3 + 4x^2y + 4x, \quad f_{yy} = 8x^3 - 60xy^2 + 4x^2.$$

The partial derivatives f_{xx}, f_{yy} are said to be of the **second order.** The value of the partial derivative f_{xx}

at (x_0, y_0) is given by the limit

$$\lim_{h \to 0} \{f_x(x_0 + h, y_0) - f_x(x_0, y_0\ \}/h$$

and similarly f_{yy} at (x_0, y_0) is given by

$$\lim_{k \to 0} \{f_y(x_0, y_0 + k) - f_y(x_0, y_0)\}/k.$$

If these limits do not exist, the derivatives f_{xx} and f_{yy} do not exist at (x_0, y_0).

Besides f_{xx} and f_{yy} there may be defined two more partial derivatives of the second order of the function $f(x, y)$, viz., the rate of change of the function f_x with respect to y, when x is held constant, a derivative which is denoted by $\partial^2 f/\partial y \partial x$ or f_{yx}, and the rate of change of the function f_y with respect to x, when y is held constant, a derivative which is denoted by $\partial^2 f/\partial x \partial y$ or f_{xy}. Hence the value of f_{yx} at the point (x_0, y_0) is given by the limit

$$\lim_{k \to 0} \{f_x(x_0, y_0 + k) - f_x(x_0, y_0)\}/k,$$

and the value of f_{xy} at the same point by

$$\lim_{h \to 0} \{f_y(x_0 + h, y_0) - f_y(x_0, y_0)\}/h.$$

In the above example

$$f_{yx} = 24x^2y - 20y^3 + 8xy + 4, \quad f_{xy} = 24x^2y - 20y^3 + 8xy + 4,$$

the remarkable thing being that f_{yx} and f_{xy} are exactly the same functions, and it can easily be verified by differentiating other functions that in general this is the case. It will be shown later in this paragraph that when certain continuity conditions are satisfied f_{yx} is always identical with f_{xy}. In other words the order in which differentiation with respect to x and with respect to y takes place is irrelevant. This property is known as the **commutative** property of partial differentiation and throughout this book it will be assumed, unless otherwise stated, that the partial derivatives of the functions dealt with possess the commutative property. Derivatives in which the function is differentiated with respect to more than one of the variables are called **mixed** derivatives.

Mixed derivatives of orders higher than the second are formed in a similar fashion to those of the second order. For example, the partial derivative $\partial^3 f/\partial x^2 \partial y$ is obtained by

(i) differentiating with respect to y,
(ii) then differentiating with respect to x,
(iii) then differentiating with respect to x again.

From the above mentioned commutative property,

$$f_{xxy} = f_{xyx} = f_{yxx},$$

for

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$
$$= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^3 f}{\partial y \partial x^2}.$$

For any mixed derivative a similar proof shows that the differentiation may be performed in any order.

Higher derivatives of functions of more than two variables are defined exactly as in the case of functions of two variables and, as before, the differentiation may be performed in any order.

The following are straightforward examples on partial differentiation involving derivatives of higher order than the first.

1. If $u = \log (x^2 + y^2)$, show that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.

2. If $u = \tan^{-1} \{xy/\sqrt{(1 + x^2 + y^2)}\}$, show that
$$u_{xy} = (1 + x^2 + y^2)^{-\frac{3}{2}},\ u_{xxyy} = 15xy/(1 + x^2 + y^2)^{\frac{7}{2}}.$$

3. If $a^2x^2 + b^2y^2 - c^2z^2 = 0$, show that
$$z_{xx}z_{yy} = (z_{xy})^2.$$

4. If $u = \log (x^2 + y^2 + z^2)$, show that
$$xu_{yz} = yu_{zx} = zu_{xy}.$$

The Commutative Property of Partial Differentiation

A set of conditions will now be given, sufficient to