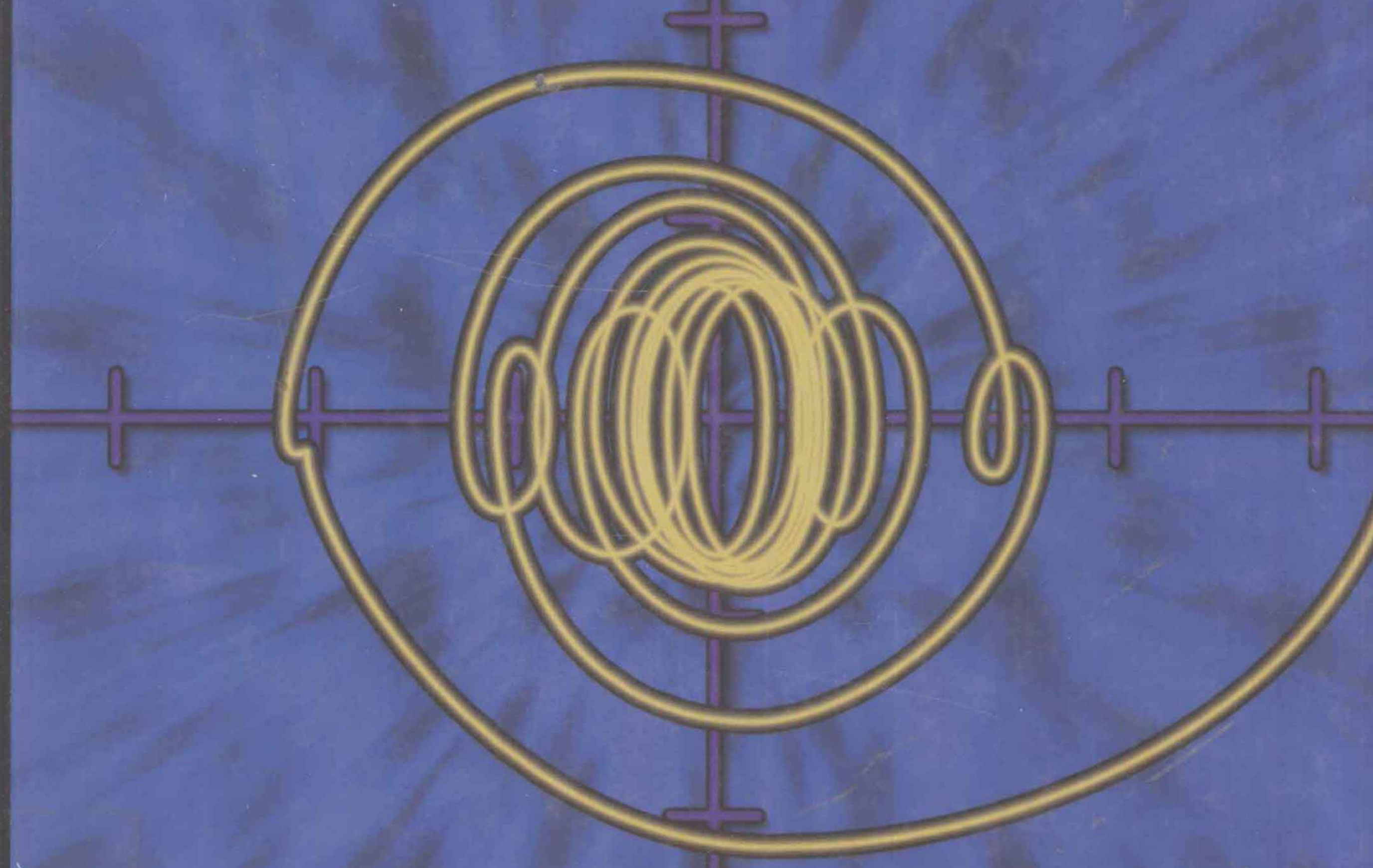
ELEMENTARY DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

SIXTH EDITION



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RICHARD C. DIPRIMA

SIXTH EDITION

Elementary Differential Equations and Boundary Value Problems

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To Elsa and Maureen

To Siobhan, James, Richard, Jr., Carolyn, and Ann

And to the next generation:
Charles, Aidan, Stephanie, Veronica, and Deirdre

The Authors

William E. Boyce received his B.A. degree in Mathematics from Rhodes College, and his M.S. and Ph.D. degrees in Mathematics from Carnegie-Mellon University. He is a member of the American Mathematical Society, the Mathematical Association of America, and the Society of Industrial and Applied Mathematics. In 1991 he received the William H. Wiley Distinguished Faculty Award given by Rensselaer. He is currently the Edward P. Hamilton Distinguished Professor of Science Education (Department of Mathematical Sciences) at Rensselaer. He is the author of numerous technical papers in boundary value problems and random differential equations and their applications. He is the author of several textbooks including two differential equations texts, and is the coauthor (with M.H. Holmes, J.G. Ecker, and W.L. Siegmann) of a text on using Maple to explore Calculus. He is also coauthor (with R.L. Borrelli and C.S. Coleman) of *Differential Equations Laboratory Workbook* (Wiley 1992), which received the EDUCOM Best Mathematics Curricular Innovation Award in 1993. Professor Boyce is extremely active in curriculum innovation and reform, including the development of materials that most effectively use technology to teach and learn. He is the Principal Investigator of the NSF–supported project "Mathematics and its Applications in Engineering and Science: Building the Links," is a member of the NSF–sponsored CODEE (Consortium for Ordinary Differential Equations Experiments), and the Initiator of the "Computers in Calculus" project at Rensselaer, partially supported by the NSF. Among the consequences of this project is the pioneering use of studio classes in basic mathematics courses.

Richard C. DiPrima (deceased) received his B.S., M.S., and Ph.D. degrees in Mathematics from Carnegie-Mellon University. He joined the faculty of Rensselaer Polytechnic Institute after holding research positions at MIT, Harvard, and Hughes Aircraft. He held the Eliza Ricketts Foundation Professorship of Mathematics at Rensselaer, was a fellow of the American Society of Mechanical Engineers, the American Academy of Mechanics, and the American Physical Society. He was also a member of the American Mathematical Society, the Mathematical Association of America, and the Society of Industrial and Applied Mathematics. He served as the Chairman of the Department of Mathematical Sciences at Rensselaer, as President of the Society of Industrial and Applied Mathematics, and as Chairman of the Executive Committee of the Applied Mechanics Division of ASME. In 1980, he was the recipient of the William H. Wiley Distinguished Faculty Award given by Rensselaer. He received Fulbright fellowships in 1964–65 and 1983 and a Guggenheim fellowship in 1982–83. He was the author of numerous technical papers in hydrodynamic stability and lubrication theory and two texts on differential equations and boundary value problems. Professor DiPrima died on September 10, 1984.

PREFACE

This edition, like its predecessors, is written from the viewpoint of the applied mathematician, whose interest in differential equations may be highly theoretical, intensely practical, or somewhere in between. We have sought to combine a sound and accurate (but not abstract) exposition of the elementary theory of differential equations with considerable material on methods of solution, analysis, and approximation that have proved useful in a wide variety of applications.

The book is written primarily for undergraduate students of mathematics, science, or engineering, who typically take a course on differential equations during their first or second year of study. The main prerequisite for reading the book is a working knowledge of calculus, gained from a normal two- or three-semester course sequence or its equivalent.

A Changing Learning Environment

The environment in which instructors teach, and students learn, differential equations has changed enormously in the past few years and continues to evolve at a rapid pace. Computing equipment of some kind, whether a graphing calculator, a notebook computer, or a desktop workstation is available to most students of differential equations. This equipment makes it relatively easy to execute extended numerical calculations, to generate graphical displays of a very high quality, and, in many cases, to carry out complex symbolic manipulations.

The fact that so many students now have these capabilities enables instructors, if they wish, to modify very substantially their presentation of the subject and their expectations of student performance. Not surprisingly, instructors have widely varying opinions as to how a course on differential equations should be taught under these circumstances.

One option is to focus somewhat less attention on the manipulative details of finding solutions, and correspondingly more attention on the conclusions that can be drawn from them. Consequently, at many colleges and universities courses in differential equations are rapidly becoming much more visual, much more quantitative, and much less formula-centered than in the past.

Mathematical Modeling

The main reason for solving many differential equations is to try to learn something about an underlying physical process that the equation is believed to model. It is basic to the importance of differential equations that even the simplest equations correspond to useful physical models, for example, exponential growth and decay, spring-mass systems, or electrical circuits. Gaining an understanding of a complex natural process is usually accomplished by combining or building upon simpler and more basic models. Thus a thorough knowledge of these models, the equations that describe them, and their solutions, is the first and indispensable step toward the solution of more complex and realistic problems.

More difficult problems often require the use of a variety of tools, both analytical and numerical. We believe strongly that pencil and paper methods must be combined with effective use of a computer. Quantitative results and graphs, often produced by a computer, serve to illustrate and clarify conclusions that may be obscured by complex analytical expressions. On the other hand, the implementation of an efficient numerical procedure typically rests on a good deal of preliminary analysis – to determine the qualitative features of the solution as a guide to computation, to investigate limiting or special cases, or to discover which ranges of the variables or parameters may require or merit special attention.

Thus, a student should come to realize that analysis and computation must frequently be combined, and that results (however they were obtained) are often most easily understood if presented in graphical form.

A Flexible Approach

From a student's point of view, the problems that are assigned as homework and those that appear on examinations drive the course. We believe that the most outstanding feature of this book is the number, and above all the variety and range, of the problems that it contains. There are far more problems than any instructor can use in any given course, and this provides instructors with a multitude of possible choices in tailoring their course to meet their own goals and the needs of their students.

The new learning environment, based on the emerging role of computing technology in instruction, calls for a new kind of flexibility in a textbook. For instance, many more or less routine problems, such as those requesting the solution of a first or second order initial value problem, are now easy to solve by a computer algebra system. This revision includes quite a few such problems, just as the earlier editions did. We do not state in these problems how they should be solved, because we believe that it is up to each instructor to specify whether their students should solve such problems by hand, with computer assistance, both ways, or perhaps not at all. Also, there are many problems that call for a graph of the solution. Instructors have the option of specifying whether they want an accurate computer-generated plot or a hand-drawn sketch, or perhaps both.

We have also added a great many problems, as well as some examples in the text, that call for conclusions to be drawn about the solution. Sometimes this takes the form of asking for the value of the independent variable at which the solution has a certain property. Other problems ask for the effect of variations in a parameter or for the determination of a critical value of a parameter at which the solution experiences a substantial change. Such problems are typical of those that arise in the applications of differential equations, and, depending on the goals of the course, an instructor has the option of assigning few or many of these problems.

To be widely useful a textbook must be adaptable to a variety of instructional strategies. This implies that instructors should have maximum flexibility to choose both the particular topics that they wish to cover and also the order in which they want to cover them. We provide this flexibility by making sure that, so far as possible, individual chapters are independent of each other. Thus, after the basic parts of the first three chapters are completed (roughly Sections 1.1, 2.1 through 2.3, and 3.1 through 3.6) the selection of additional topics, and the order and depth in which they are covered, is at the discretion of the instructor. For example, while there is a good deal of material on applications of various kinds, especially in Chapters 2, 3, 9, and 10, most of this material appears in separate sections, so that an instructor can easily choose which applications to include and which to omit. Alternatively, an instructor who wishes to emphasize the details of numerical algorithms for approximating solutions of differential equations (the subject matter of Chapter 8) can use this chapter immediately after, or in conjunction with, the material in Chapter 2 on first order equations. Or, an instructor who wishes to emphasize a systems approach to differential equations can take up Chapter 7 (Linear Systems) and perhaps even Chapter 9 (Nonlinear Autonomous Systems) immediately after Chapter 2. Or, while we present the basic theory of linear equations first in the context of a single second order equation (Chapter 3), many instructors have combined this material with the corresponding treatment of higher order equations (Chapter 4) or of linear systems (Chapter 7). Many other choices and combinations are also possible and have been used effectively with earlier editions of this book.

Although we note repeatedly that computers are extremely useful for investigating differential equations and their solutions, and although many of the problems are best approached with computational assistance, the book is adaptable to courses having various levels of computer involvement, ranging from little or none to intensive. The text is independent of any particular hardware platform or software package. For courses having a strong computer component, however, students may need supplements on the locally available computing platforms and software.

Two software packages that are widely used in differential equations courses are *Maple* and *Mathematica*. The books *Differential Equations with Maple* and *Differential Equations with Mathematica*, by K. R. Coombes, B. R. Hunt, R. L. Lipsman, J. E. Osborn, and G. J. Stuck, all at the University of Maryland, are available with this book for those who make use of these packages.

There is also a solutions manual, by C. W. Haines of Rochester Institute of Technology, that contains detailed solutions to many of the problems in the book.

Major Changes in the Sixth Edition

Readers who are familiar with earlier editions will find that this one has the same general organizational structure. In addition to a host of minor improvements, there are the following major revisions.

1. *The chapter on numerical methods has been considerably revised.* The backward Euler method is introduced in Section 8.1, foreshadowing a more extended discussion of higher order backward differentiation formulas in Section 8.5. There is also a more detailed description of multistep methods in general, and more discussion of factors to consider when choosing a method. Adaptive methods are introduced in Section 8.3, and illustrated with the Euler and improved Euler formulas. Section 8.6 contains an expanded discussion of error control and stability, including an example illustrating how the reduction of step size affects truncation and roundoff errors.

2. *There are nearly 300 new problems*, many of which assume the availability of computing technology.

3. *Many new (and revised) problems and examples investigate the manner in which a solution depends on one or more parameters.* These problems and examples support the idea that often it is more important to understand how a solution depends on a parameter than to obtain the solution for some particular value of the parameter. Consequently, bifurcation points and other critical parameter values, at which a solution experiences significant change, are explored frequently.

4. *The material on the method of Frobenius,* where the indicial equation has equal roots or roots differing by an integer, has been consolidated into a single section (Section 5.7).

5. *Some new examples, and quite a few new problems, have been added to the chapter on Laplace transforms.* Again, quantitative conclusions, graphs, and an investigation of parameter dependence are often called for in these problems.

6. *There is a greater emphasis on visualization.* All figures have been redrawn, there are more examples whose solutions are presented graphically, and more problems that ask students to generate graphs, or to draw conclusions from them.

7. *The more quantitative and geometrical point of view is particularly evident in Chapter 10,* which deals with partial differential equations and Fourier series. Many more problems ask for quantitative conclusions to be drawn from a solution, and many more graphs of solutions are requested.

As the subject matter of differential equations continues to grow, as new technologies become commonplace, as old areas of application are expanded, and as new ones appear on the horizon, the content and viewpoint of courses and their textbooks must also evolve. This is the spirit we have sought to express in this book.

William E. Boyce
Troy, New York
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For most of the past several years I have collaborated in teaching differential equations with my good friend and colleague at Rensselaer, Bill Siegmann. My ideas on the effective use of computer technology in instruction have been significantly influenced by many stimulating conversations with him.

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W. E. B.

CONTENTS

C H A P T E R 1

Introduction

In this brief chapter we try to give perspective to your study of differential equations. First, we indicate several ways of classifying equations, in order to provide organizational structure for the remainder of the book. Later, we outline some of the major figures and trends in the historical development of the subject. The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. Nevertheless, it remains a dynamic field of inquiry today, with many interesting open questions.

1.1 Classification of Differential Equations

Many important and significant problems in engineering, the physical sciences, and the social sciences, when formulated in mathematical terms, require the determination of a function satisfying an equation containing one or more derivatives of the unknown function. Such equations are called **differential equations**. Perhaps the most familiar example is Newton's law $F = ma$. If $u(t)$ is the position at time t of a particle of mass m acted on by a force F, then we obtain

$$m\frac{d^2u}{dt^2} = F\left[t, u, \frac{du}{dt}\right], \tag{1}$$

where the force F may be a function of t, u, and the velocity du/dt. To determine the motion of a particle subject to a given force F it is necessary to find a function u satisfying the differential equation (1).

The main purpose of this book is to discuss some of the properties of solutions of differential equations, and to describe some of the methods that have proved effective in finding solutions, or in some cases approximating them. To provide a framework for our presentation we first mention several useful ways of classifying differential equations.

Ordinary and Partial Differential Equations. One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation**. In the second case the derivatives are partial derivatives, and the equation is called a **partial differential equation**.

Two examples of ordinary differential equations, in addition to Eq. (1), are

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{2}$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, inductance L, and impressed voltage $E(t)$; and the equation governing the decay with time of an amount $R(t)$ of a radioactive substance, such as radium,

$$\frac{dR(t)}{dt} = -kR(t), \tag{3}$$

where k is a known constant. Typical examples of partial differential equations are the potential, or Laplace's, equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \tag{4}$$

the diffusion or heat conduction equation

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \tag{5}$$

and the wave equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{6}$$

Here α^2 and a^2 are certain constants. The potential equation, the diffusion equation, and the wave equation arise in a variety of problems in the fields of electricity and magnetism, elasticity, and fluid mechanics. Each is typical of distinct physical phenomena (note the names), and each is representative of a large class of partial differential equations.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions, then a system of equations is required. For example,