Stochastic Methods in Reliability Theory

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IN
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PREFACE

The primary objective of this book is to present the analysis of repairable systems in the area of reliability theory under a unified framework of stochastic processes. While there are several books in the area of reliability theory to meet the needs of researchers and practitioners, this book meets the existing gap in the literature to equip and provide researchers with various applications of stochastic processes in reliability analysis.

An attempt has been made here to provide a systematic treatment of repairable systems starting with elementary situations to fairly advanced models. A conscious choice has been made in the presentation in unfolding and building the main concepts in the analysis of redundant repairable systems rather than an attempt to present all the reported research in this area. Clearly I have been guided by my interest and what I believe as the major developments and contributions in this area. It is hoped that the presentation of the analysis of systems by the unified theme of the underlying stochastic processes, rather than a presentation of an analysis of a string of reliability models would be useful. While no claim on completeness of reported research is made, it is ensured by and large the major developments in this area are discussed.

A sound working knowledge of probability as discussed at the level of Feller Volume 1 is assumed for the readers. The book is organised in ten chapters. The first two chapters provide the background information for this work. Chapters 3 and 4 deal with Markovian models. The next module is on the applications of renewal theory and its extensions. This is followed by a special chapter on Monto Carlo simulation. Yet another chapter discusses the relevance and interconnections of reliability system with other areas like inventory and queueing models. The final chapter provides a survey of the methods used in the book in the context of a special system.

The book can be used as a resource material for a graduate course on reliability theory with special emphasis on stochastic process. The material in the book can also be used as a sound supporting material for graduate courses in applied stochastic process. It is hoped that the material presented here would enable the reliability researchers and practitioners to obtain a complete picture on the state of art, and hence would serve as a meaningful basis for further research.

Much of the basic material discussed in this work was taught to me while I was a graduate student at the Department of Mathematics, Indian Institute of Technology, Madras, by Professors R. Subramanian and S.K. Srinivasan. I remember with gratitude the support and kindness shown by my teachers during my study at IIT, Madras. Needless to emphasis that any shortcomings in this work are entirely due to me and my understanding. It was Professor G. Sankaranarayanan, who introduced the fascinating subject of stochastic processes to us at an early stage in our academic life at Annamalai University, Chidambaram. I recollect with appreciation the support of Professor G. Sankaranarayanan during my study at Annamalai University.

Part of this work was lectured at the Department of Mathematics and Statistics of Cochin University of Science and Technology and at the Department of Statistics, University of Pune, under the UGC visiting fellowship scheme. I thank the support provided by Professors T. Thiruvikraman, and A. Krishnamurthy of CUSAT and that of Professor B.K. Kale and A.D. Dharmadhikari of Pune. Special thanks are due to Dr N.R. Sheth, Director, Indian Institute of Management, Ahmedabad, for extending all the help in completing this work.

6 August 1989 **N. Ravichandran**
Ahmedabad.

CONTENTS

Chapter 1

PRELIMINARIES

1.1 INTRODUCTION

The aim of this book is to present in a concise form the modelling concepts of redundant repairable systems. Our concern in this monograph is to review and explain the relevance and use of various kinds of stochastic processes in the analysis of redundant repairable systems.

Analysis of repairable systems has been the concern of many industrial engineers, applied probabilists and statisticians. Consequently, several directions of research has been reported on this topic. While the statisticians investigate the problems from the view-point of estimating the life-time characteristics of these systems, industrial engineers are concerned with the methods and means of improving the system design and performance within acceptable costs. The applied probabilists are concerned with the evaluation of the operating characteristics of these systems by using appropriate stochastic models.

The models proposed in the literature for the analysis of such systems vary depending on the context and background of individuals. Techniques like Markov Process, Semi-Markov Process, Renewal Theory, Regenerative Stochastic Process and Stochastic Point Process have been extensively used.

This chapter introduces the main concepts in reliability theory in a generalized setting. Briefly, the contents of the chapter are : the concept of a redundant system, specific examples of systems, definition of performance measures, and review of some appropriate distributions in reliability studies.

1.2 THE CONCEPT OF A SYSTEM

We define a system as an entity consisting of several units with specified interaction between them. The definition of units is arbitrary and remains context dependent. A system is said to be a redundant system if it has more than the necessary units for its proper functioning. This means the system has some spares or back-up units. A redundant system is completely described by specifying the number of units in the system, the conditions under which the system is said to be operating, the status of the system corresponding to the failure of the system, the repairable/non-repairable nature of the units, the lifetime durations of the units in operation, and their repair time durations (if they are repairable), the interaction of spares with the operating units, the repair policy in terms of priority, the number of repair channels, and maintenance schedules if any.

1.3 A GENERAL SYSTEM DESCRIPTION

1. The system consists of n ($\geqslant 1$) units. The system requires k ($\leqslant n$) units for its successful operation.
2. Initially k units are operative, and $(n - k)$ units are kept as standbys.
3. There is a repair facility with r ($\geqslant 1$) repair-channels. The repair policy is first in, first out (FIFO).
4. The lifetime and repair time durations of the units are independent random variables with known probability distribution functions.
5. The lifetime of the standby unit is a random variable with known (usually negative exponential) distribution.
6. On failure, an operating unit is moved to the repair facility instantaneously where it is repaired according to the 'first in, first out' queue discipline.
7. The standby units are also taken to the repair facility on failure to follow the same course of action as the online operating units.
8. When the number of operable units is less than k, the system is said to be non-operable or in the degraded state of operation with reduced output.
9. Once repaired, a unit joins the pool of spares if there are sufficient number of operable units to render the system operative. Otherwise, the newly repaired unit is switched online.
10. Repair is assumed to be perfect. All random variables are assumed to be mutually independent and identically distributed whenever it is meaningful.

EXAMPLES

1. Consider a cinema hall in a state where power supply is irregular. In order to ensure uninterrupted supply of power, apart from the regular source of supply, a generator is kept as standby. The generator is switched on as and when the main supply is disrupted and is switched off as and when the main supply is resumed. The main source of supply and generator together constitute the system. The system functions as long as one of the units functions without failure.
2. For a person interested in a local sports meet, the television is the main communication link. It enables him to watch the event. In case of power shortage or TV transmission problems, a transistor radio may be used as a backup.
3. Consider a communication network with all possible links from city **A** to city **B**. If a specific path is busy at a given time, any other path can be used to communicate a message from city **A** to city **B**. Here,

we have an example of multiple (path) unit redundant system. The system functions as long as even one path in the network connecting the two cities is operational.

4. Consider a computing centre, where there is a large main frame computer with several terminals and many small personal computers (PCs). For a person who arrives at the computing centre, with a specific task to be fulfilled any one of the terminals or the PCs could serve the purpose. The collection of PCs and the main frame terminals serve as a multiple unit redundant system. The system functions as long as one of the (suitable) units is available for the specific task on hand.

1.4 ADDITIONAL FEATURES

There are several additional characteristics of a system which can be individually and independently added to the characteristics of a general system defined in section 1.3 to model different classes of systems, some of which we describe here. Several other types and examples can be traced in the literature.

(a) PRIORITY SYSTEMS

There are systems in which some units are given preference for operation as well as for repair. Such a system consists of two classes of units. Units of one set are called priority units (p-units) and the other as ordinary units (o-units). The following additional statements characterize priority systems.

The priority unit is never kept as a standby. When the priority unit fails, it is repaired immediately by pre-empting the repair of the o-unit if need be. The pre-empted repair of the o-unit is resumed on the completion of repair of the p-unit. There are several rules by which the remaining repair time of the o-unit can be described. We discuss only two of them, namely, (a) Pre-emptive repeat (under this rule, the remaining repair time of the o-unit is equivalent to the original repair time). This means the amount of work done earlier by the repair crew on the unit has no impact, and (b) Pre-emptive resume (under this rule, the remaining repair time of the o-unit is continued from where it was left due to the arrival of the p-unit). The repaired priority unit normally replaces the online operating o-unit, if any, and the operable o-unit is switched off as standby.

(b) SYSTEM WITH IMPERFECT SWITCHOVER

In this case the assumption (6) in the general system description is modified to incorporate the non-instantaneous switchover times. This means, it requires a random duration of time to switch a unit from operating place to the repair facility and vice-versa. It is further assumed that these random variables are independent of all other random variables in the model and their distributions are known.

(c) PARALLEL SYSTEMS

These are systems in which no unit is kept as a standby. All the operable units are switched operative. The operating conditions are specified in terms of the number of operable units at any time t. All the other assumptions/descriptions remain.

(d) SYSTEMS WITH PREVENTIVE MAINTENANCE

These are systems in which the operating units are sent for preventive maintenance according to an age specific probability distribution. When the age of an operating unit is in $(x, x + \Delta x)$, it is sent for Preventive Maintenance (PM) with probability $a(x)\,\Delta x$, provided this action of scheduling the preventive maintenance of the unit does not result in a system failure. Otherwise, the preventive maintenance is either deferred until one of the units under repair becomes available or the preventive maintenance is skipped. The concept of PM is useful only when the failure rate of a unit is an increasing function of (age) x and the expected duration of maintenance is not larger than that of repair time duration.

(e) INTERMITTENTLY USED SYSTEMS

In all the previous models there is a basic assumption that the system is required continuously. However there are situations where the systems' functions are not required continuously. Such systems are known as intermittently used/needed systems. These systems are needed and not needed during alternative periods (of time) which are governed by a pair of (a sequence of mutually independent pairwise identical) random variables.

The features added in this section can be independently combined to the basic models described in section 1.3, to meet the requirements of a situation which requires specific investigation. Some possible models incorporating various aspects are given below:

1. A parallel system with imperfect switchover,
2. An intermittently used warm standby with preventive maintenance,
3. A multiple unit system with non-instantaneous switch-over times, and
4. A multiple unit parallel system with single repair facility.

1.5 PERFORMANCE MEASURES

In the preceding section we have defined several types of redundant systems. Now, we concentrate and explain some of the performance measures of these systems which are of interest from the system's design viewpoint and analysis. We confine ourselves to those measures that are proposed and discussed in the literature. It is convenient to begin with the following definition.

Let $Z(t) = 1$, if the system is operable at time t, and $Z(t) = 0$, when it is not operable. The two-valued stochastic process $\{Z(t), t \geqslant 0\}$ can be used effectively to define the performance measures.

We also assume throughout the discussion of this section that the value of $Z(0)$ is specified and all the measures we define are conditional on this initial state of $Z(0)$.

1. *Reliability* $R(t)$: This is a classical measure defined as the probability that the system functions well in the interval $(0, t]$ and is, symbolically, defined as $R(t) = Pr\,\{Z(u) = 1, 0 \leqslant u \leqslant t \mid Z(0) = 1\}$. Equivalently, let X be a random variable representing the duration of first system failure starting with an initial operable condition at $t = 0$. Then, the reliability $R(t)$ of the system is equal to $Pr\,\{X > t\}$.

2. *Mean Time to System Failure* (*MTSF*): Often, one is interested in a summarized measure of the random variable X, representing the life time of the system, namely, its first moment. This summarized measure is aptly described as the Mean Time to System Failure (MTSF), as it corresponds to the average duration between successive system failures. By using standard manipulations, we get

$$E[X] = \int_0^{\infty} xfx\,(x)\,dx = \int_0^{\infty} R(u)\,du.$$

3. *Availability* $A(t)$: This measure is defined as the probability that the system is operational at time t.
Hence,

$$A(t) = Pr\{Z(t) = 1 \mid Z(0) = 1\}.$$

We note the following differences between the measures reliability and availability. The reliability is an interval function while the availability is a point function describing the behaviour of the system at a specified epoch. Secondly, the reliability function precludes the failure of the system during the interval under consideration, while availability function does not impose any such restriction on the behaviour of the system.

Associated with the measure $A(t)$ is another measure $\bar{A}(t) = 1 - A(t)$, probability that the system is unavailable at time t. Also, $A(t)$, the availability measure of a system is a time dependent function and as t increases for a class of systems which are well behaved, we may expect this probability to converge to a specific limit β, called stationary availability of the system.

$$\therefore \qquad \beta = \lim_{t \to \infty} A(t)$$

4. *Interval Reliability* $R(t, \tau)$: We next define another performance measure which combines the reliability and availability measure. This measure is known as the interval reliability. It is the probability that the system functions well during the interval $(t, t + \tau)$, where t and τ are quite arbitrary. We use the symbol $R(t, \tau)$ to denote this measure. In terms of $Z(t)$ process

$$R(t, \tau) = Pr\{Z(u) = 1, \qquad t \leqslant u \leqslant t + \tau \mid Z(0) = 1\}$$

Notice that $R(t, 0) = A(t)$ and $R(0, \tau) = R(\tau)$. Hence the interval reliability function is the prediction function corresponding to the successful performance of the system for an arbitrary duration (τ) beginning at an arbitrary epoch t.

An associated asymptotic measure corresponding to $R(t, \tau)$, called stationary interval reliability denoted as $R(\tau)$, is obtained by using the relation

$$R(\tau) = \lim_{t \to \infty} R(t, \tau)$$

Finally, we introduce the multidimensional generalizations of the availability and interval reliability functions.

5. *Joint Availability* $A(t_1, t_2)$: It is the joint probability that the system is operable at time t_1 and at time t_2, given the initial event at $t = 0$. Symbolically,

$$A(t_1, t_2) = Pr\{Z(t_1) = 1 \quad \text{and} \quad Z(t_2) = 1 \mid Z(0) = 1\}$$

6. *Joint Interval Reliability*: $R(t_1, x_1; t_2, x_2)$: This is the joint probability that the system is operable in the disjoint intervals $(t_1, t_1 + x_1)$ and $(t_2, t_2 + x_2)$ given the initial event at $t = 0$. Symbolically,

$$R(t_1, x_1; t_2, x_2) = Pr\{Z(u) = 1,\ t_i \leqslant u \leqslant t_i + x_i,\quad i = 1, 2 \mid Z(0) = 1\}$$

The multinominal versions of Joint availability and Joint interval reliability are similarly defined. We conclude this section with a few comments. While the measures reliability, meantime to system failure, availability, stationary availability, interval reliability are of interest from the view-point of system design and analysis, the measures joint availability and joint interval reliability are of academic interest.

To illustrate the usefulness of these measures consider the example of a computer center. From the systems manager view-point, who is responsible for the maintenance and utilization of the computing facility, the reliability and MTSF measures are of importance and are of his immediate concern. From the users view-point the measure of availability of the system at a specified time and the same in the long run are relevant. From the system analyst's view, who enters large programs, the effective functioning of the system from a specified epoch for a specified duration, namely, the interval reliability is relevant.

Historically, the measures reliability, availability and MTSF are treated in Barlow and Proschan (1965). The concept of interval reliability can be found in the book by Barlow and Proschan (1965) and the first applied paper seems to be due to Nakagawa (1978). Further developments of this measure in the context of redundant systems can be traced in Ravichandran (1979). The concept of joint availability is due to Baxter (1982).

1.6 CHOICE OF DISTRIBUTIONS IN RELIABILITY MODELLING

The previous two sections dealt with the concept of a system and some performance measures associated with the behaviour of these systems. In any true modelling one needs to specify or identify the lifetime and repair time durations of the units. In this section we review some properties of probability distribution functions which are useful in identifying the appropriate distribution of the random variables representing the duration of life and repair times for a given situation.

Analysing a system without specifying any functional form for the distribution of the random variable representing life time and repair durations would be most ideal. Unfortunately, for such a very general set-up (in terms of the random variables involved) not much progress is possible in obtaining the relevant performance measures of the system. Hence, a compromise is necessary in terms of specifying the functional form of the distribution. Of course, care must be taken to ensure that this specification of the functional form of the distribution does not detach the model from reality. Hence, the choice of these distributions should be made with a combined consideration of mathematical tractability and the ability to represent acceptable physical properties of the equipment/units in the system.

In this section, we briefly review some of the distributions with reference to their mathematical tractability. Further, we also discuss two important physical measures (of units), useful in selecting the appropriate distributions, viz., memory and hazard rate.

1.7 MATHEMATICAL TRACTABILITY

As observed earlier, it will be ideal not to assume any functional forms for the distributions of random variables involved in reliability modelling. However, even for simpler systems this situation makes the analysis difficult, if not impossible. More specifically the process $\{X(t), t \geqslant 0\}$, $X(t)$ representing a suitable state of the system (for example the number of operable units at time t) turns out to be a process which is difficult to analyse either for transient durations and/or for stationary case.

On the other hand, one can make a very simple and somewhat unrealistic assumption of negative exponential life and/or repair time durations, which renders the process $\{X(t), t \geqslant 0\}$ as (simply) Markovian about which relatively more information is known.

It is perhaps worthwhile to know how we can accommodate more general functional forms of the life and repair time distributions in the analysis, without much damaging the Markovian nature of the system state process $\{X(t), t \geqslant 0\}$. It is known that for Erlangian and generalized Erlangian (an Erlangian with different stage parameters) distributions of life times/repair times, the process $\{X(t), t \geqslant 0\}$ is Markovian. This is achieved by enlarging the state space of $X(t)$ suitably.

Use of Erlangian and generalized Erlangian has a (direct) requirement on the data, namely, the coefficient of variation of the data should be < 1 (see Cox, 1962) which is perhaps an undesirable effect. This restriction can be relaxed by the consideration of what are known as Coxian distributions

(distributions whose Laplace transform is a rational function) of which an important sub-class is the convex combinations of exponential distributions.

Moreover, recently a wide class of distributions known as the phase type distributions which correspond to first passage time to a recurrent state in an irreducible aperiodic Markov chain were introduced by Neuts (1975). When these distributions are used to represent the life/repair time duration of units, the system results in an $\{X(t), t \geqslant 0\}$ process which is Markovian and tractable. Thus, it seems completely feasible to model a large class of systems without really damaging the Markovian nature of the $\{X(t), t \geqslant 0\}$ process. This aspect shall be treated extensively in the later chapters of this monograph.

Generally, using distributions other than those mentioned above, will distort the Markovian nature of the $\{X(t), t \geqslant 0\}$ process. Nevertheless considerable progress is possible for a wider class of systems as will be demonstrated in the later chapters.

1.8 PHYSICAL PROPERTIES

(a) HAZARD RATE

Let X be a random variable, denoting the life time/repair time of an equipment. Let $f(x)$ be the probability density function (pdf) of the random variable and $F(x)$ its cumulative distribution function (Cdf). Consider the function,

$$h(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr\{x < X \leqslant x + \Delta \mid X > x\}$$

$$= \frac{f(x)}{1 - F(x)} = \frac{f(x)}{\bar{F}(x)}, \qquad \bar{F}(x) \neq 0$$

$h(x)\Delta x$ represents the probability that the random variable has value in $(x, x + \Delta)$ given the information that the value of X is greater than x: in physical terms this means the probability of a failure in $(x, x + \Delta x)$ given the information that the equipment survived up to x or the chances of service completion in $(x, x + \Delta x)$ given the information that the service duration exceeded the value x. The function $h(x)$ is called the hazard rate of the random variable X.

Distributions can be classified according as $h(x)$ is an increasing/decreasing or constant function of x, which has the direct physical meaning that more the chronological age of the equipment more/less (equal) the chances of failure of an equipment. Table 1.1 gives specific examples of distribution functions with different types of hazard rates.

We close this sub-section by remarking that specifying the hazard function uniquely specifies the survival function of a random variable by means of the following relation:

$$Pr(X > t) = \bar{F}(t) = 1 - F(t) = \exp\left\{-\left[\int_0^t h(u)\, du\right]\right\}$$

Table 1.1

Examples of Distributions with different types of hazard rates

Name of the distribution	Functional form of the pdf	Expression for $h(x)$	Remarks on the hazard rate
Negative exponential	$\lambda e^{-\lambda x} > 0$	λ	Constant
Gamma	$\frac{1}{\beta_\alpha \Gamma\alpha} e^{-x/\beta} x^{\alpha-1}$ $\alpha, \beta, x > 0$	$\frac{x^{\alpha-1} x^{-x/\beta}}{\beta^\alpha\left[\Gamma(\alpha) - \Gamma\left(\alpha, \frac{x}{\beta}\right)\right]}$ $\Gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt$	$\alpha > 1$ IFR $\alpha = 1$ Constant $\alpha < 1$ DFR
Weibull	$\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]$ $\alpha, \beta > 0,\ 0 < x < \infty$	$\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}$	$\beta > 1$ IFR $\beta = 1$ Constant $\beta < 1$ DFR
Raleigh	$\beta = 2$	$\frac{2}{\alpha^2} x$	Linear function
Normal	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ $\sigma > 0,\ t < \infty,$	$\frac{\varphi\left(\frac{x-\mu}{\sigma}\right)}{\left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]\sigma}$ $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ $\Phi(z) = \int_{-\infty}^{Z} \varphi(u)\, du$	IFR
Hyper exponential	$\frac{cd}{d-c}[e^{-cx} - e^{-dx}]$ $x > 0\quad d > c > 0$	$\frac{cd[e^{-ct} - e^{-dt}]}{de^{-ct} - ce^{-dt}}$	IFR

(b) MEMORY

Another associated physical characteristic in modelling life time of an equipment is what is known as memory. Several distributions can be classified by using this property.

For any t, define $r(t)$, the remaining expected life time of a unit as:

$$r(t) = E[X - t \mid X > t]$$

and further define the virtual age of a unit at time t as

$$v(t) = r(0) - r(t)$$

The measure $v(t_2) - v(t_1)$ for any $t_1 < t_2$ gives the amount of damage caused to the equipment due to the chronological age equivalent to $t_2 - t_1$. Hence, this measure can be thought of as the 'memory' of the interval of length $t_2 - t_1$ of the distribution representing the life time of the unit. Using this we define the interval memory as

$$m(t_1, t_2) = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

The distribution is said to have positive/perfect/negative memory according as $m(t_1, t_2)$ is < 1, $= 1$ and > 1.

Physically, a distribution with positive memory means that the mean residual life time of the equipment decreases faster than the actual age of the equipment. In the case of perfect memory the reduction in the mean residual life time duration is exactly equal to the difference in the chronological time, and in the case of negative memory the change in the mean residual time is less fast than the actual age. Table 1.2 gives examples of some distribution with different kinds of memory. Given the statistical evidence about the life and repair time of an equipment, it is possible to use the physical properties, memory and hazard rate to choose an appropriate distribution to represent the life/repair time durations. This added with the notion of mathematical tractability will provide a meaningful choice.

Table 1.2

Examples of Distributions with Different Types of Memory

Name of the distribution	Functional form of the pdf	Remarks on the memory property
1. Negative exponential	$\lambda e^{-\lambda x}$	No memory
2. Unifrom in (a, b)	$\frac{1}{b-a}$; $b > a$, $a < x < b$	Perfect memory in $(0, a)$; Positive memory in (a, b)
3. Log normal	$\frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2} \log (t-\mu)^2\right]$ $t > 0, \sigma > 0$ $-\infty < \mu < \infty$	Positive memory in $(0, \alpha)$ and negative memory in (α, ∞) α—function of parameters
4. Geometric	pq^t*	No memory
5. Binomial	$\binom{n}{t} p^t q^{n-t}$*	Positive memory

*$p + q = 1$; $0 \leqslant p \leqslant 1$

The concept of hazard rate is available in the literature for long time (Barlow and Proschan, 1965). The notion of using memory as a property to classify distributions is due to Muth (1980).

We close this chapter by establishing a relation between the mean residual time and hazard function.

By definition

$$h(t) = \frac{f(t)}{\bar{F}(t)} = \frac{-\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)}$$

$$r(t) = E[X - t \mid X > t]$$

where $\bar{F}(t) = Pr\{X > t\}$