Lecture Notes in Mathematics

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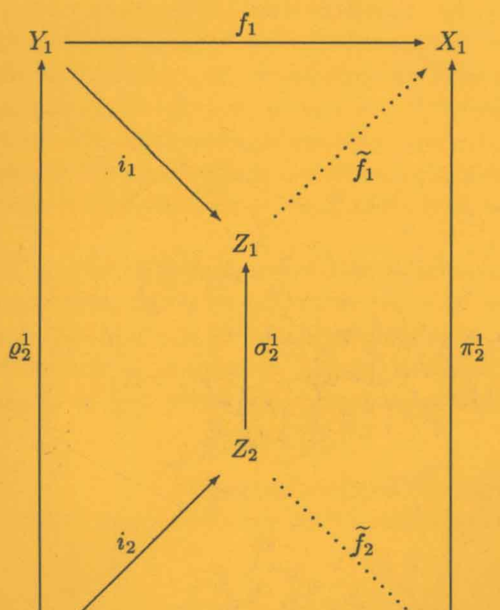
Jochen Wengenroth

Derived Functors in Functional Analysis



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Contents

1

Introduction

In the last years, the part of functional analysis which contributes to the solution of analytical problems using various techniques from the theory of locally convex spaces gained a lot of strength from new developments in topics which are related to category theory and homological algebra. In particular, progress about the derived projective limit functor (which measures the obstacle against the construction of a global solution of a problem from local solutions) and the splitting theory for Fréchet and more general spaces (which is concerned with the existence of solution operators) allowed new applications for instance to problems about partial differential or convolution operators.

The connection between homological algebra and the theory of locally convex spaces had been established by V.P. Palamodov [50] in 1969. He pointed out that a number of classical themes from functional analysis can be viewed as exactness problems in appropriate categories and thus can be investigated with the aid of derived functors. After developing suitable variants of tools from category theory he constructed the derivatives of a fairly wide class of functors and proved concrete representations, characterizations and relations for several functors acting on the category of locally convex spaces, like the completion, duality or Hom-functors. A major role in these investigations was played by the projective limit functor assigning to a countable projective limit of locally convex spaces its projective limit. A very detailed study of this functor was given by Palamodov in [49].

Starting in the eighties, D. Vogt reinvented and further developed large parts of these results in [62] (which never had been published) and [61, 63, 64, 65] with a strong emphasis on the functional analytic aspects and avoiding most of the homological tools. He thus paved the way to many new applications of functional analytic techniques. Since then, the results (in particular about the projective limit functor) have been improved to such an extent that they now constitute a powerful tool for solving analytical problems.

The aims of this treatise are to present these tools in a closed form, and on the other hand to contribute to the solution of problems which were left open in Palamodov's work [50, §12]. We try to balance between the homological viewpoint, which often illuminates functional analytic results, and techniques from the theory of locally convex spaces, which are easier accessible for the typical reader we have in mind. Therefore we assume a good familiarity with functional analysis as presented e.g. in the books of Bonet and Pérez-Carreras [51], Jarchow [36], Köthe [39], or Meise and Vogt [45]. Except for some examples we will not need anything beyond these text books. On the other hand, no knowledge about homological algebra is presumed. Chapter 2 reviews the definitions and results (including some ideas for the proofs) that will be used in the sequel. This is only a small portion of the material presented and needed in Palamodov's work. Readers who are interested in the relation of topological vector spaces to more sophisticated concepts of category theorey may consult the articles [52, 53] of F. Prosmans.

The key notions in chapter 2 are that of short exact sequences in suitable categories (for instance, in the category of locally convex spaces

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an exact sequence if f is a topological embedding onto the kernel of g which is a quotient map) and the notion of an additive functor which transforms an object X into an object $F(X)$ and a morphism $f : X \to Y$ into a morphism $F(X) \longrightarrow F(Y)$. The derived functors are used to measure the lack of exactness of the complex

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0.$$

If the values $F(X)$ are abelian groups or even vector spaces then exactness of the sequence means that $F(f)$ is injective, its image is the kernel of $F(g)$, and $F(g)$ is surjective. For example, if E is a fixed locally convex space and F assigns to every locally convex space X the vector space $\mathrm{Hom}(E, X)$ of continuous linear maps and to $f : X \to Y$ the map $T \mapsto f \circ T$, then the exactness of the sequence above means that each operator $T : E \to Z = Y/X$ has a lifting $\tilde{T} : E \to Y$.

If the functor F has reasonable properties, one can construct derived functors F^k such that every exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is transformed into an exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F^1(X) \longrightarrow F^1(Y) \longrightarrow \ldots$$

Then $F^1(X) = 0$ means that

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0$$

is always exact.

Chapter 3 develops the theory of the countable projective limit functor starting in 3.1 with a "naive" definition of the category of projective spectra where the objects $\mathscr{X} = (X_n, \varrho^n_{n+1})$ consist of linear spaces X_n and linear spectral maps ϱ^n_{n+1}, and the morphisms $f = (f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{N}}$ consist of linear maps commuting with the spectral maps. This definition differs from the one given by Palamodov but has the advantage of being very simple. The functor Proj (which is also denoted by $\varprojlim$ in the literature) then assigns to a spectrum $\mathscr{X}$ its projective limit

$$X = \operatorname{Proj}\mathscr{X} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho^n_{n+1}(x_{n+1}) = x_n \right\}$$

and to a morphism f the linear map $\operatorname{Proj}(f) : (x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}$. If we consider the "steps" X_n as the local parts of X and we are concerned with the problem whether a given map $f^* : X \to Y$ is surjective, we can try to solve the problem locally which requires to find a morphism f with surjective components $f_n : X_n \to Y_n$ such that $f^* = \operatorname{Proj}(f)$, and then we can hope to conclude the surjectivity of f^* which requires knowledge about $\operatorname{Proj}^1\mathscr{K}$ where $\mathscr{K}$ is the spectrum consisting of the kernels $\ker f_n$.

After presenting the homological features of this functor and comparing its applicability with Palamodov's original definition, we give in section 3.2 a variety of characterizations and sufficient conditions for $\operatorname{Proj}^1\mathscr{X} = 0$. The unifying theme of all these results is the Mittag-Leffler procedure: one seeks for corrections in the kernels of the local solutions which force the corrected solutions to converge to a global solution. If the steps of the spectrum are Fréchet spaces this idea leads to a characterization of $\operatorname{Proj}^1\mathscr{X} = 0$ due to Palamodov. We present three proofs of this which stress different aspects and suggest variations in several directions. One of the proofs reduces the result to the classical Schauder lemma which is a version of the open mapping theorem. It is this proof which easily generalizes to a theorem of Palamodov and Retakh [50, 54] about $\operatorname{Proj}^1\mathscr{X} = 0$ for spectra consisting of (LB)-spaces and clarifies the role of the two conditions appearing in that theorem: the first is the continuity and the second is the density required for the Mittag-Leffler procedure. Knowing this, it is very surprising that in many cases the theorem remains true without the first assumption. The argument behind is again a version of the Schauder lemma (even a very simple one). This trick tastes a bit like lifting oneself by the own bootstraps, but in our case it works. After discussing this circle of results with an emphasis on spectra consisting of (LS)-spaces, we consider in section 3.3 topological consequences (like barrelledness conditions and quasinormability) for a projective limit if some representing spectrum satisfies $\operatorname{Proj}^1\mathscr{X} = 0$, and we solve one of Palamodov's questions about Proj considered as a functor with locally convex spaces as values: the algebraic property $\operatorname{Proj}^1\mathscr{X} = 0$ does not imply topological exactness in general, but it does indeed under an additional assumption which is satisfied in all situations which appear in analysis.

Section 3.4 contains some applications of the results obtained in 3.2 and 3.3. We start with some very classical situations like the Mittag-Leffler theorem or the surjectivity of $\overline{\partial}$ on $\mathscr{C}^\infty(\Omega)$ for open set $\Omega \subseteq \mathbb{C}$. The techniques based on the projective limit functor nicely separate the two aspects of the standard proofs into a local and a global part. We also give a proof of Hörmander's characterization of surjective partial differential operators on $\mathscr{D}'(\Omega)$ and finally explain results of Braun, Meise, Langenbruch, and Vogt about partial differential operators on spaces of ultradifferentiable functions.

Encouraged by the results of chapter 3 and the simple observation that every complete locally convex space is the limit of a projective spectrum of Banach spaces (which is countable only for Fréchet spaces), we investigate in chapter 4 the homological behaviour of arbitrary projective limits. In a different context, this functor has been investigated e.g. by C.U. Jensen [37]. In section 4.1 the algebraic properties are developed similarly as in 3.1 for the countable case, and we present Mitchell's [47] generalization of the almost trivial fact that $\mathrm{Proj}^k \mathscr{X} = 0$ for $k \geq 2$ and countable spectra: if $\mathscr{X}$ consists of at most $\aleph_n$ objects (in our case linear spaces) then $\mathrm{Proj}^k \mathscr{X} = 0$ for $k \geq n+2$.

Before we consider spectra of locally convex spaces, we insert a short section about the completion functor with a result of Palamodov and a variant due to D. Wigner [72] who observed a relation between the completion functor and the derivatives of the projective limit functor which is presented in 4.3. Besides this, we prove a generalization of Palamodov's theorem about reduced spectra $\mathscr{X}$ of Fréchet spaces in the spirit of Mitchell's result mentioned above: if $\mathscr{X}$ consists of at most $\aleph_n$ spaces then $\mathrm{Proj}^k \mathscr{X} = 0$ holds for $k \geq n+1$. This seems to be the best possible result: using ideas of Schmerbeck [55], we show that under the continuum hypothesis (in view of the result above this set-theoretic assumptions appears naturally) the canonical representing spectrum of the space φ of finite sequences endowed with the strongest locally convex topology satisfies $\mathrm{Proj}^k \mathscr{X} = 0$ for $k \geq 2$ but $\mathrm{Proj}^1 \mathscr{X} \neq 0$. The same holds for all complete separable (DF)-spaces satisfying the "dual density condition" of Bierstedt and Bonet [6] (this is the only place where we use arguments of [51] which do not belong to the standard material presented in books about locally convex spaces). These negative results lead to a negative answer to another of Palamodov's questions. The essence of chapter 4 is that the first derived projective limit functor for uncountable spectra hardly vanishes (we know essentially only one non-trivial example given in 4.1) and that this theory is much less suitable for functional analytic applications than in the countable case.

In chapter 5 the derivatives $\mathrm{Ext}^k(E,\cdot)$ of the functors $\mathrm{Hom}(E,\cdot)$ are introduced, and we explain the connection to lifting, extension, and splitting properties (it is this last property which is used to find solution operators in applications). We show that for a Fréchet space X there is a close relation between $\mathrm{Ext}^k(E,X)$ and $\mathrm{Proj}^k \mathscr{Y}$ for a suitable spectrum $\mathscr{Y}$ and use this to give a simplified proof of the fact that $\mathrm{Ext}^k(E,X) = 0$ for all $k \geq 1$ whenever E is a complete (DF)-space and X is a Fréchet space and one of them is nuclear (this may serve as a guide for the case of two Fréchet spaces considered in 5.2). The rest of section 5.1 is devoted to a conjecture of Palamodov that under the same assumptions for E and X also $\mathrm{Ext}^k(X, E) = 0$ holds. The only Fréchet space X for which we can provide some information is $X = \omega = \mathbb{K}^{\mathbb{N}}$. For this case, we could show jointly with L. Frerick that $\mathrm{Ext}^1(\omega, E) = 0$ for "most" (DF)-spaces. On the other hand, the negative results of chapter 4 eventually lead to $\mathrm{Ext}^2(\omega, \varphi) \neq 0$ at least under the continuum hypothesis.

In 5.2 we present Vogt's [63] arguments which led to a fairly complete characterization of $\mathrm{Ext}^1(E, F)$ for pairs of Fréchet spaces in [29]. We deduce from the splitting theorem the most important results about the structure of nuclear Fréchet spaces (which are due to Vogt [59] and Vogt and Wagner [67]) to compare these with results in 5.3 about splitting in the category of (PLS)-spaces (in particular, spaces of distributions). We first present very recent results of P. Domański and Vogt [24, 25] about the structure of complemented subspaces of $\mathscr{D}'$ (with only minor modifications of their proof, but having the aesthetical advantage of staying in the category of (PLS)-spaces) and deduce from this an improvement of their result about $\mathrm{Ext}^1_{PLS}(E, F) = 0$ which shows that $\mathscr{D}'$ plays exactly the same role for splitting in the category of (PLS)-spaces as s does for nuclear Fréchet spaces. This has immediate applications for the splitting of distributional complexes.

In the sixth chapter about inductive limits we explain the relation to the projective limit functor which gives several characterizations of acyclic (LF)-spaces. We provide a very short proof for the completeness of these spaces and show that for (LF)-spaces acyclicity is equivalent to many regularity conditions of the inductive limit. Because of the close connection to projective spectra of (LB)-spaces and in view of existing literature about inductive limits (in particular the book of Bonet and Pérez-Carreras [51]) this discussion is rather short. The rest of the chapter is devoted to questions of Palamodov whether inductive limits of complete locally convex spaces are always complete and regular. We provide positive answers under a very weak extra assumption.

The final chapter is devoted to the duality functor assigning to a locally convex space its strong dual and to a continuous linear map the transposed operator. For an exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of locally convex spaces neither f^t nor g^t need be open onto its range.

This "lack of openness" is measured by the derived functors $D^+(X)$ and $D^1(X)$, respectively. We derive this characterization from the homological definitions and provide a quite simple proof of a result due to Palamodov [50], Merzon [46], and Bonet and S. Dierolf [8] characterizing the quasinormable Fréchet spaces by $D^1(X) = 0$ and a lifting property for bounded sets, where again the Schauder lemma plays the main role. Moreover, we show that beyond the class of Fréchet spaces quasinormability is not sufficient for vanishing of D^+ nor of D^1 (we suspect that these answers to further questions of Palamodov were probably known to many people for quite a while). We finish with a surprisingly general positive result about the (topological) exactness of

$$0 \longrightarrow Z'_\beta \xrightarrow{g^t} Y'_\beta \xrightarrow{f^t} X'_\beta \longrightarrow 0,$$

where the strict Mackey condition (which is dual to quasinormability) enters the game, and apply this to projective limits of (LB)-spaces.

As we said above, a good portion of this treatise (in particular chapter 4 and partly 5.1, 6, and 7) is motivated by the list of unsolved problems in Palamodov's work. These parts are probably much less important for applications than other parts. But one should keep in mind that the efforts for searching counterexamples led to several positive results which allow applications.

In this work we touch various fields of the theory of locally convex spaces which have quite a long tradition. It would have been expedient or even necessary to explain the background of many results with much more care. I refrained from really trying to do so because this would have changed the character of this work and because there are many people who are much better qualified for this.