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SECOND EDITION

# DIFFERENTIAL EQUATIONS

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# PREFACE

This text is based on a course, successfully given at the University of New Mexico for several years, that is an introduction to differential equations for mathematics majors, engineering students, and majors in the physical sciences. This second edition has incorporated an additional chapter on partial differential equations and Fourier series. This material can be used in a more extensive course on differential equations.

The text differs from more traditional books in that numerical methods are used from the beginning and throughout as a tool to analyze the qualitative behavior of solutions as well as to approximate them. If one examines textbooks that have appeared subsequent to the first edition of this book, one sees that this approach has taken hold. This reflects in part the dramatic availability of personal computers to the undergraduate population.

Nevertheless we emphasize that the book is not intended as an introduction to numerical methods for ordinary differential equations. Our aim is to give the important topics and analytical tools needed to study ordinary and partial differential equations. A course taught with the book should be one with mathematical depth, but attractive to an audience with a large proportion of engineering and physical sciences students.

# **FEATURES**

The following briefly describes the main features of the text.

- Emphasis on applications. Applications in areas of mechanics, circuit theory, astronomy, and biology are introduced throughout the text and in the problem sets.
- Exercises. Exercises are presented in order of difficulty and matched with the order of the material presented, and the worked examples.
- **Classroom presentation.** Every effort was made to match section lengths with the amount of material needed for a classroom lecture presentation.
- Algorithmic format. Upon completion of the discussion of a solution technique, an algorithm is presented to assist the students in using the technique. Examples following the algorithm show its use step-by-step.
- Numerical techniques. Numerical methods are introduced throughout the text, beginning with the most elementary ones. Students are asked to compare different algorithms, and to vary step sizes, with the intent of giving them a real appreciation for the use of numerical methods to study the qualitative behavior of solutions of ordinary differential equations.
- **Linear systems.** The emphasis is primarily on two- and three-dimensional systems in which the eigenvalues and eigenvectors can be easily found. The fundamental matrix is computed using both the eigenvector method and the more efficient Laplace transform.
- Partial differential equations. An elementary but careful presentation of Fourier series is given using the normal modes of vibration of a taut string as motivation. This is followed by the method of separation of variables used to construct solutions to the classical, second order equations of mathematical physics. A novel addition is a brief study of first order hyperbolic systems using the eigenvalue–eigenvector methods discussed previously; this section can be taught independently of the previous material.
- Computer programs. Each elementary numerical algorithm is presented in FORTRAN, but appendixes are included giving the algorithm in BASIC and PASCAL. Later in the text the program RKF45, which uses simultaneously a fourth and fifth order Runge–Kutta method to adjust step size, is introduced and some numerical examples are given. It is not necessary to understand the details of the code to be able to use it effectively.
- **Phase plane.** The phase plane is briefly introduced in the chapter on second order linear equations, as an aid to analyze the behavior of solutions, and later in the chapter on nonlinear systems.
- Theoretical considerations. Where such concepts as existence and uniqueness of solutions, direction fields, linear independence, and fundamental sets of solutions are introduced, students are given prior examples and exercises to help underpin their understanding of the concept.

■ Conservative systems. The chapter on nonlinear systems includes a discussion of one degree of freedom conservative systems analyzed using energy methods. If a brief discussion of nonlinear systems is needed, which does not require phase plane analysis, this section can be taught independently.

# ORGANIZATION

We have written a text that offers a great deal of flexibility in the choice of material to be covered. A basic course in ordinary differential equations for engineers and physical sciences majors would consist of Chapters 1, 2, 3 (Runge–Kutta methods only), 4, 5 and 7 (selected topics). Where Laplace transform methods are not required to be taught, Chapter 4 can be omitted, and Chapter 6 or further topics in nonlinear systems from Chapter 7 can be taught.

A course in linear systems, including both ordinary and partial differential equations, would consist of Chapters 1, 2, 4, 5, 6, and 9. A one-quarter separate course in partial differential equations, or as part of an engineering mathematics course, could be constructed from the material in Chapters 6 and 9.

The book contains more material than can be used in a one-semester course. But whether the course desired is a basic introductory course, or one emphasizing linear or nonlinear systems, or introducing partial differential equations, the authors believe the material presented in the text can be suitably arranged to meet any need.

# CHANGES IN THE SECOND EDITION

We have made extensive changes in the second edition of *Differential Equations*. It should be emphasized that many of the changes came as a result of classroom experience, both here at New Mexico and at other institutions, as well as from reviewers' comments. We believe these changes will greatly improve the presentation of the material to the student reader as well as to the instructor. Some examples of some of the changes in the second edition are as follows:

- There is a 30% increase in the number of exercises, and exercises are ordered by level of difficulty and to match the material presented and the worked examples, which have also been increased.
- A new chapter (Chapter 9) on Fourier series and partial differential equations has been added. The final section of this chapter discusses first order hyperbolic systems and transmission line equations using eigenvalue—eigenvector methods—a first in introductory texts, to our knowledge.

- Much of the theoretical material has been introduced only after students have had some hands on experiences solving the relevant differential equations.
- Numerical methods based on the trapezoidal rule and extrapolation have been used in Chapter 1 to solve first order linear equations whose solutions cannot be obtained by quadrature.
- A discussion of higher order linear equations with application to beam problems has been added to Chapter 2.
- The discussion of linear independence and fundamental sets of solutions in Chapter 2 has been greatly simplified and better motivated, as has the discussion of the phase plane.
- The emphasis of the Heaviside formulas in the discussion of the Laplace transform (Chapter 4) has been reduced, and partial fractions expansions are used more.
- More emphasis has been given to the Problem/Solution format in worked examples.
- Material has been reordered and some sections split to make for better classroom presentation and reader absorption.
- An appendix has been added in which the elementary numerical techniques are written in BASIC.

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Albuquerque, New Mexico

D.A.S.

R.C.A. W.T.K.

# **CONTENTS**

FIDST	ORDER	DIFFERENTIAL	FOUATIONS	-
LIU21	UNDER	DIFFERENTIAL	EQUATIONS	

	_
1.1	General Remarks 1
1.2	Linear First Order Differential Equations 5
1.3	Inhomogeneous Linear First Order Differential Equations 17
1.4	The Method of Undetermined Coefficients and Some Numerical Integration Rules 31
1.5	Separable Equations 42
1.6	Direction Fields and Existence and Uniqueness Theorems 58
1.7	Some Elementary Numerical Methods 67
1.8	Implicit Solutions of First Order Equations 81
1.9	A Mathematical Model of an Electric Circuit Problem 97
1.10	A Model of Population Growth 104
1.11	A Simple Mixing Problem and Feedback Control 112
	Summary 121
	Miscellaneous Exercises 121
	References 125

LINEAR SECOND ORDER

	DIFFERENTIAL EQUATIONS 127
2.1	Introduction 127
2.2	Examples from Mechanics and Circuit Theory 127
2.3	General Theory—Introduction 136
2.4	Constant Coefficient Homogeneous Equations 140
2.5	Constant Coefficient Homogeneous Equations with Complex Characteristic Roots 150
2.6	Theory of Homogeneous Second Order Linear Differential Equations 157
2.7	Inhomogeneous Equations—The Method of Undetermined Coefficients 167
2.8	Inhomogeneous Equations—The Variation of Parameters Method 173
2.9	Unforced Oscillations of Electrical and Mechanical Systems 181
2.10	Forced Oscillations of Electrical and Mechanical Systems 204
2.11	Elementary Numerical Methods for Second Order Equations 215
2.12	Higher Order Equations 224
	Summary 236
	Miscellaneous Exercises 237
	References 240

# 3

# **ELEMENTARY NUMERICAL METHODS** 243

3.1	Introduction	243

- 3.2 A General One-Step Method 243
- 3.3 Taylor Series Methods 246
- 3.4 Runge–Kutta Methods 253

Summary 265

Miscellaneous Exercises 266

References 270



# THE LAPLACE TRANSFORM 271

- 4.1 Introduction 271
- 4.2 The Laplace Transform of  $e^{at}$  272

4.3 4.4 4.5 4.6 4.7 4.8 4.9	An Application to First Order Equations 275 Further Properties and Transform Formulas 279 Solving Constant Coefficient Linear Equations 287 Partial Fractions and the Heaviside Formulas 296 The Convolution Integral: Weighting Functions 307 The Unit Step Function: Transforms of Nonsmooth Functions 315 The Unit Impulse Function: Transfer Functions 328 Summary 334 Miscellaneous Exercises 334 References 337
5.1 5.2 5.3 5.4 5.5	LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS 339  Introduction 339  Vectors and Matrices 344  Homogeneous Linear Systems with Constant Coefficients 361  The Fundamental Matrix 382  The Inhomogeneous Linear System—Variation of Parameters 397  Summary 413  Miscellaneous Exercises 414  References 422
_6	NONCONSTANT COEFFICIENT SECOND ORDER LINEAR EQUATIONS AND SERIES SOLUTIONS 423
6.1	Introduction 423
6.2	Series Solutions—Part 1 425
6.3	Series Solutions—Part 2 433
6.4	The Method of Reduction of Order and the Logarithmic Case of a Regular Singular Point 440
6.5	Some Special Functions and Tools of the Trade 459

Summary 466

References 468

Miscellaneous Exercises 466

Introduction 587

Fourier Series 599

The Wave Equation 588

Normal Modes of Vibration 590

9.1

9.2

9.3

9.4

9.5

7.1	An Introductory Example—The Pendulum 472
7.2	Phase Paths, Equilibrium States, and Almost Linear Systems 479
7.3	The Phase Plane and Stability 487
7.4	Almost Linear Systems 503
7.5	Energy Methods for Systems with One Degree of Freedom 513
7.6	Mathematical Models of Two Populations 532
	Summary 549
	Miscellaneous Exercises 550
	References 556
8	MORE ON NUMERICAL METHODS 559
0	MURE UN NUMERICAL METHODS 559
8.1	Introduction 559
8.2	Errors, Local and Global 559
8.3	Estimating Local Errors 563
8.4	A Step Size Strategy 566
8.5	The Subroutine RKF45 567
8.6	Some Examples 572
0.0	Summary 581
	Miscellaneous Exercises 581
	References 584
	References 501
Λ	
9	FOURIER SERIES AND PARTIAL DIFFERENTIAL EQUATIONS 587

Odd, Even, and Periodic Extensions, Fourier Sine and Cosine Series 612

NONLINEAR DIFFERENTIAL EQUATIONS 471

9.7	The Heat Equation 656
9.8	Laplace's Equation 668
9.9	Transmission Line Equations and Systems of Partial Differential Equations 677
	Summary 702
	Miscellaneous Exercises 702
	References 709
	Appendix 1 The Fundamental Local Existence and Uniqueness Theorem 711
	Appendix 2 PASCAL Programs 729
	Appendix 3 BASIC Programs 737
	Appendix 4 Listing of Subroutine RKF45 743
	Appendix 5 Power Series, Complex Numbers, and Euler's Formula 753
	Answers to Selected Exercises A-1
	Index I-1

The Wave Equation and the Separation-of-Variables Method 631

9.6

# FIRST ORDER DIFFERENTIAL EQUATIONS

# ${f l.1}$ GENERAL REMARKS

An ordinary differential equation<sup>1</sup> is an equation expressing a relationship among derivatives of an unknown function of a single variable. Such equations often result from the mathematical expression of scientific laws connecting physical quantities and their rates of change. For example, Newton's law of cooling states that the rate of change of temperature of a body is proportional to the difference in temperature between a cooling body and its surroundings. If T(t) represents the temperature of the body at time t and S is the constant temperature of the surroundings, Newton's law leads to the differential equation

$$\frac{dT}{dt} = -k(T - S),\tag{1.1.1}$$

where k is a constant of proportionality. We know from experience that the temperature of the body will change monotonically until we cannot detect a difference between its temperature and that of its surroundings. But we shall need some mathematics to answer questions such as: How rapidly does T(t) approach S? or How does T(t) depend on the proportionality constant k and on the temperature at the time we started our observations?

<sup>1.</sup> It is not clear why "ordinary" ever became standard terminology in a subject that motivated the invention of the calculus, that has such a wide applicability to science and technology, and that contains so many fascinating ideas and methods.

**EXAMPLE 1** Show that if  $T(t) = S + Ce^{-kt}$  is substituted into (1.1.1) the expression becomes an identity in t. Write C in terms of S and the temperature of the body at t = 0.

**SOLUTION** We substitute T(t) into (1.1.1) to obtain

$$\frac{d}{dt} T(t) = \frac{d}{dt} [S + Ce^{-kt}] = -kCe^{-kt} = -k[T(t) - S],$$

an identity in t.

If we set t = 0 in the formula for T(t), then

$$T(0) = S + C,$$

or

$$C = S - T(0).$$

The function  $T(t) = S + (T(0) - S)e^{-kt}$  is said to satisfy the differential equation (1.1.1). We see that T(t) approaches S exponentially as t increases and that the proportionality constant k appears in the argument of the exponential function. It is called a rate constant.

Techniques for constructing solutions to differential equations will be presented in subsequent sections, but we need only the tools of calculus to test if a function is or is not a solution to a differential equation. To do this we differentiate the function as many times as needed and substitute the function and its derivatives into the differential equation. This is illustrated in the following examples.

**EXAMPLE 2** Show that  $y(t) = 6 \sin 2t + 7 \cos 3t$  is a solution to the differential equation  $y'' + 4y = -35 \cos 2t$ .

### **SOLUTION** From

$$y'' = -24 \sin 2t - 63 \cos 2t,$$
  
$$4y = 24 \sin 2t + 28 \cos 2t,$$

we have

$$y'' + 4y = -35\cos 2t,$$

as required.

**EXAMPLE 3** Show that  $y(t) = 6 \sin t + 7 \cos 3t$  is not a solution to the differential equation  $y'' + 4y = -5 \cos 2t$ .

# **SOLUTION** From

$$y'' = -6 \sin t - 63 \cos 3t,$$
  
$$4y = 24 \sin t + 28 \cos 3t,$$

we have

$$y'' + 4y = 18 \sin t - 35 \cos 3t \neq -5 \cos 2t$$

on any t-interval of nonzero length. This illustrates the fact that to be a solution a function must satisfy the differential equation in a nontrivial way.

Our goal in studying differential equations is to determine both the qualitative and quantitative properties of those functions that satisfy them; such functions are called solutions. This can sometimes be done by representing the solutions as elementary functions, e.g., polynomials or trigonometric functions. More often, geometric arguments are needed to determine the qualitative properties, and numerical methods are needed to determine the quantitative properties. Numerical techniques are especially important when an explicit representation of the solution cannot be found.

Three differential equations that arise quite often in applications and that will be discussed later in the book are:

1. 
$$\frac{dy}{dt} = k(t)F(y)$$
, the growth equation;

2. 
$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t)$$
, the LCR oscillator equation; and

3. 
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$
, the pendulum equation.

In all three equations t is called the *independent variable*, and the unknown functions y, Q, and  $\theta$ , whose derivatives explicitly appear, are called the dependent variables. A solution is a function y(t), Q(t), or  $\theta(t)$  that satisfies the equation on an open tinterval. Hence the solutions y, Q, and  $\theta$  are functions of t but, in contrast with the explicit notation k(t) and E(t), this functional dependence is to be understood from the context.

Our study is made easier by grouping together those differential equations that have a significant common property. The most important property is that of order. The order of a differential equation is the order of the highest derivative of the dependent variable which appears in the equation. For example, the oscillator and pendulum equations are second order, while the growth equation is first order. In the remainder of this chapter, first order differential equations will be studied, and in later chapters higher order equations will be discussed.

# First Order Differential Equations

A partial differential equation is a relationship involving an unknown function of at least two variables and one or more of its derivatives. For instance, a partial differential equation that describes the temperature u(x, t) in a thin heated wire as a function of position x and time t is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_t = k u_{xx},$$

where k is a constant dependent on the physical properties of the wire. An introduction to some of the partial differential equations of mathematical physics is given in Chapter 9.

### **EXERCISES**

In Exercises 1-6 determine which of the given equations are ordinary differential equations and which are partial differential equations. Identify the dependent and independent variables.

1. 
$$y'' + 3y' + 2y = \sin t$$
 2.  $\left(\frac{dy}{dt}\right)^2 + 4y = t^2$ 

$$2. \left(\frac{dy}{dt}\right)^2 + 4y = t^2$$

3. 
$$u_{xx} - 9u_u = 3 \sin x \cos t$$

4. 
$$\frac{\partial u}{\partial x} + t^2 \frac{\partial u}{\partial t} = 0$$

5. 
$$\frac{d^3x}{dt^3} - 4\frac{dx}{dt} + 7x = 4e^{-t}$$
 6.  $\varphi'' + (\cos t)\varphi = 0$ 

$$6. \varphi'' + (\cos t)\varphi = 0$$

In Exercises 7–12 determine the order of each of the given differential equations.

7. 
$$y' + 4y = 0$$

$$8. \ \frac{d^2y}{dt^2} + 4y^2 = 7$$

8. 
$$\frac{d^2y}{dt^2} + 4y^2 = 7$$
 9.  $y'' + \epsilon y'(y^2 - 1) + 4y = 3 \sin t$ 

**10.** 
$$(p(t)y')' + q(t)y = 0$$

11. 
$$y'y'' + 4yy' = 0$$
 12.  $(y')^2 + y^2 = 4$ 

12. 
$$(y')^2 + y^2 = 4$$

In Exercises 13-20 verify by direct substitution that the given function y(t) is a solution.

**13.** 
$$\frac{dy}{dt} = 10(y - t), \quad y(t) = t + 0.1$$

**14.** 
$$\frac{dy}{dt} = 4ty^2$$
,  $y(t) = \frac{1}{(1-2t^2)}$ 

15. 
$$\frac{dy}{dt} = -2y + e^t$$
,  $y(t) = Ce^{-2t} + \frac{1}{3}e^t$ , C any constant

**16.** 
$$\frac{dy}{dt} = \frac{t(3-2y)}{(t^2-1)}, \quad y(t) = \frac{1}{(t^2-1)} + \frac{3}{2}$$