DIFFERENTIAL EQUATIONS

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SECOND EDITION

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Addison-Wesley Publishing Company
Reading, Massachusetts ▪ Menlo Park, California ▪ New York
Don Mills, Ontario ▪ Wokingham, England ▪ Amsterdam ▪ Bonn
Sydney ▪ Singapore ▪ Tokyo ▪ Madrid ▪ Bogotá ▪ Santiago ▪ San Juan

Sponsoring Editor: **Thomas N. Taylor** ▪ Production Supervisor: **Marion E. Howe** ▪ Art Consultant: **Loretta Bailey** ▪ Copy Editor: **Lorraine Ferrier** ▪ Illustrator: **Textbook Art Associates** ▪ Manufacturing Supervisor: **Roy Logan** ▪ Cover Designer: **Marshall Henrichs** ▪ Text Designer: **Melinda Grosser**

Library of Congress Cataloging-in-Publication Data

Sanchez, David A.
Differential equations.

Includes index.
1. Differential equations. I. Allen, Richard C.
II. Kyner, Walter T. III. Title.
QA372.S17 1988 515.3′5 87-22949
ISBN 0-201-15407-2

Reprinted with corrections, June 1989

CDEFGHIJ-DO-89

PREFACE

This text is based on a course, successfully given at the University of New Mexico for several years, that is an introduction to differential equations for mathematics majors, engineering students, and majors in the physical sciences. This second edition has incorporated an additional chapter on partial differential equations and Fourier series. This material can be used in a more extensive course on differential equations.

The text differs from more traditional books in that numerical methods are used from the beginning and throughout as a tool to analyze the qualitative behavior of solutions as well as to approximate them. If one examines textbooks that have appeared subsequent to the first edition of this book, one sees that this approach has taken hold. This reflects in part the dramatic availability of personal computers to the undergraduate population.

Nevertheless we emphasize that the book is not intended as an introduction to numerical methods for ordinary differential equations. Our aim is to give the important topics and analytical tools needed to study ordinary and partial differential equations. A course taught with the book should be one with mathematical depth, but attractive to an audience with a large proportion of engineering and physical sciences students.

FEATURES

The following briefly describes the main features of the text.

- **Emphasis on applications.** Applications in areas of mechanics, circuit theory, astronomy, and biology are introduced throughout the text and in the problem sets.
- **Exercises.** Exercises are presented in order of difficulty and matched with the order of the material presented, and the worked examples.
- **Classroom presentation.** Every effort was made to match section lengths with the amount of material needed for a classroom lecture presentation.
- **Algorithmic format.** Upon completion of the discussion of a solution technique, an algorithm is presented to assist the students in using the technique. Examples following the algorithm show its use step-by-step.
- **Numerical techniques.** Numerical methods are introduced throughout the text, beginning with the most elementary ones. Students are asked to compare different algorithms, and to vary step sizes, with the intent of giving them a real appreciation for the use of numerical methods to study the qualitative behavior of solutions of ordinary differential equations.
- **Linear systems.** The emphasis is primarily on two- and three-dimensional systems in which the eigenvalues and eigenvectors can be easily found. The fundamental matrix is computed using both the eigenvector method and the more efficient Laplace transform.
- **Partial differential equations.** An elementary but careful presentation of Fourier series is given using the normal modes of vibration of a taut string as motivation. This is followed by the method of separation of variables used to construct solutions to the classical, second order equations of mathematical physics. A novel addition is a brief study of first order hyperbolic systems using the eigenvalue–eigenvector methods discussed previously; this section can be taught independently of the previous material.
- **Computer programs.** Each elementary numerical algorithm is presented in FORTRAN, but appendixes are included giving the algorithm in BASIC and PASCAL. Later in the text the program RKF45, which uses simultaneously a fourth and fifth order Runge–Kutta method to adjust step size, is introduced and some numerical examples are given. It is not necessary to understand the details of the code to be able to use it effectively.
- **Phase plane.** The phase plane is briefly introduced in the chapter on second order linear equations, as an aid to analyze the behavior of solutions, and later in the chapter on nonlinear systems.
- **Theoretical considerations.** Where such concepts as existence and uniqueness of solutions, direction fields, linear independence, and fundamental sets of solutions are introduced, students are given prior examples and exercises to help underpin their understanding of the concept.

- **Conservative systems.** The chapter on nonlinear systems includes a discussion of one degree of freedom conservative systems analyzed using energy methods. If a brief discussion of nonlinear systems is needed, which does not require phase plane analysis, this section can be taught independently.

ORGANIZATION

We have written a text that offers a great deal of flexibility in the choice of material to be covered. A basic course in ordinary differential equations for engineers and physical sciences majors would consist of Chapters 1, 2, 3 (Runge–Kutta methods only), 4, 5 and 7 (selected topics). Where Laplace transform methods are not required to be taught, Chapter 4 can be omitted, and Chapter 6 or further topics in nonlinear systems from Chapter 7 can be taught.

A course in linear systems, including both ordinary and partial differential equations, would consist of Chapters 1, 2, 4, 5, 6, and 9. A one-quarter separate course in partial differential equations, or as part of an engineering mathematics course, could be constructed from the material in Chapters 6 and 9.

The book contains more material than can be used in a one-semester course. But whether the course desired is a basic introductory course, or one emphasizing linear or nonlinear systems, or introducing partial differential equations, the authors believe the material presented in the text can be suitably arranged to meet any need.

CHANGES IN THE SECOND EDITION

We have made extensive changes in the second edition of *Differential Equations.* It should be emphasized that many of the changes came as a result of classroom experience, both here at New Mexico and at other institutions, as well as from reviewers' comments. We believe these changes will greatly improve the presentation of the material to the student reader as well as to the instructor. Some examples of some of the changes in the second edition are as follows:

- There is a 30% increase in the number of exercises, and exercises are ordered by level of difficulty and to match the material presented and the worked examples, which have also been increased.
- A new chapter (Chapter 9) on Fourier series and partial differential equations has been added. The final section of this chapter discusses first order hyperbolic systems and transmission line equations using eigenvalue–eigenvector methods—a first in introductory texts, to our knowledge.

- Much of the theoretical material has been introduced only after students have had some hands on experiences solving the relevant differential equations.
- Numerical methods based on the trapezoidal rule and extrapolation have been used in Chapter 1 to solve first order linear equations whose solutions cannot be obtained by quadrature.
- A discussion of higher order linear equations with application to beam problems has been added to Chapter 2.
- The discussion of linear independence and fundamental sets of solutions in Chapter 2 has been greatly simplified and better motivated, as has the discussion of the phase plane.
- The emphasis of the Heaviside formulas in the discussion of the Laplace transform (Chapter 4) has been reduced, and partial fractions expansions are used more.
- More emphasis has been given to the Problem/Solution format in worked examples.
- Material has been reordered and some sections split to make for better classroom presentation and reader absorption.
- An appendix has been added in which the elementary numerical techniques are written in BASIC.

ACKNOWLEDGMENTS

We would like to give our thanks to the following reviewers of this current edition for their valuable help and suggestions:

Christopher L. Cox, Clemson University
Charles K. Cook, University of South Carolina, Sumter
Zita Divis, Ohio State University
Juan A. Gatica, University of Iowa
Harry Hochstadt, Polytechnic University of New York
James A. Hummel, University of Maryland
Joseph Johnson, Rutgers University
Kenneth M. Larson, Brigham Young University
Gerald D. Ludden, Michigan State University
Walter J. Neath, California State University, Chico
Merle D. Roach, University of Alabama in Huntsville
John T. Scheick, Ohio State University
David R. Scribner, University of Maine at Farmington
Monty J. Strauss, Texas Tech University
William F. Trench, Trinity University
Fred Van Vleck, University of Kansas
Joseph. J. Wolcin, U. S. Coast Guard Academy

We would like to express our special appreciation to Steven G. Krantz of Washington University in St. Louis for his insightful assistance throughout the entire development of this edition.

The reviewers of the first edition gave valuable advice and suggestions. Our thanks to:

W.J. Kammerer, Georgia Institute of Technology
F.H. Mathis, Baylor University
R.E. Plant, University of California, Davis
L.F. Shampine, Sandia National Laboratories
R.E. Showalter, The University of Texas at Austin
A.P. Stone, University of New Mexico
D.D. Warner, Clemson University
K. Schmitt, University of Utah
T. Bowman, University of Florida

We wish to thank Gary Bauerschmidt, Scott Dumas, David Rutschman, and Ferenc Varadi for their technical assistance and Jackie Damrau for her expert typing. Finally, we thank the staff of Addison-Wesley for their support and advice during the writing of this text, and for their assistance in the technical preparation of this book.

Albuquerque, New Mexico

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1

FIRST ORDER DIFFERENTIAL EQUATIONS

1.1 GENERAL REMARKS

An *ordinary differential equation*[1] is an equation expressing a relationship among derivatives of an unknown function of a *single variable*. Such equations often result from the mathematical expression of scientific laws connecting physical quantities and their rates of change. For example, Newton's law of cooling states that the rate of change of temperature of a body is proportional to the difference in temperature between a cooling body and its surroundings. If $T(t)$ represents the temperature of the body at time t and S is the constant temperature of the surroundings, Newton's law leads to the differential equation

$$\frac{dT}{dt} = -k(T - S), \tag{1.1.1}$$

where k is a constant of proportionality. We know from experience that the temperature of the body will change monotonically until we cannot detect a difference between its temperature and that of its surroundings. But we shall need some mathematics to answer questions such as: How rapidly does $T(t)$ approach S? or How does $T(t)$ depend on the proportionality constant k and on the temperature at the time we started our observations?

1. It is not clear why "ordinary" ever became standard terminology in a subject that motivated the invention of the calculus, that has such a wide applicability to science and technology, and that contains so many fascinating ideas and methods.

EXAMPLE 1 Show that if $T(t) = S + Ce^{-kt}$ is substituted into (1.1.1) the expression becomes an identity in t. Write C in terms of S and the temperature of the body at $t = 0$.

SOLUTION We substitute $T(t)$ into (1.1.1) to obtain

$$\frac{d}{dt}T(t) = \frac{d}{dt}[S + Ce^{-kt}] = -kCe^{-kt} = -k[T(t) - S],$$

an identity in t.

If we set $t = 0$ in the formula for $T(t)$, then

$$T(0) = S + C,$$

or

$$C = S - T(0).$$ ■

The function $T(t) = S + (T(0) - S)e^{-kt}$ is said to satisfy the differential equation (1.1.1). We see that $T(t)$ approaches S exponentially as t increases and that the proportionality constant k appears in the argument of the exponential function. It is called a rate constant.

Techniques for constructing solutions to differential equations will be presented in subsequent sections, but we need only the tools of calculus to test if a function is or is not a solution to a differential equation. To do this we differentiate the function as many times as needed and substitute the function and its derivatives into the differential equation. This is illustrated in the following examples.

EXAMPLE 2 Show that $y(t) = 6 \sin 2t + 7 \cos 3t$ is a solution to the differential equation $y'' + 4y = -35 \cos 2t$.

SOLUTION From

$$y'' = -24 \sin 2t - 63 \cos 2t,$$
$$4y = 24 \sin 2t + 28 \cos 2t,$$

we have

$$y'' + 4y = -35 \cos 2t,$$

as required. ■

EXAMPLE 3 Show that $y(t) = 6 \sin t + 7 \cos 3t$ is not a solution to the differential equation $y'' + 4y = -5 \cos 2t$.

SOLUTION From

$$y'' = -6 \sin t - 63 \cos 3t,$$
$$4y = 24 \sin t + 28 \cos 3t,$$

we have

$$y'' + 4y = 18 \sin t - 35 \cos 3t \neq -5 \cos 2t$$

on any t-interval of nonzero length. This illustrates the fact that to be a solution a function must satisfy the differential equation in a nontrivial way. ■

Our goal in studying differential equations is to determine both the qualitative and quantitative properties of those functions that satisfy them; such functions are called *solutions.* This can sometimes be done by representing the solutions as elementary functions, e.g., polynomials or trigonometric functions. More often, geometric arguments are needed to determine the qualitative properties, and numerical methods are needed to determine the quantitative properties. Numerical techniques are especially important when an explicit representation of the solution cannot be found.

Three differential equations that arise quite often in applications and that will be discussed later in the book are:

1. $\dfrac{dy}{dt} = k(t)F(y)$, the growth equation;
2. $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{C} = E(t)$, the LCR oscillator equation; and
3. $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{l}\sin\theta = 0$, the pendulum equation.

In all three equations t is called the *independent variable,* and the unknown functions y, Q, and θ, whose derivatives explicitly appear, are called the *dependent variables.* A solution is a function $y(t)$, $Q(t)$, or $\theta(t)$ that satisfies the equation on an open t-interval. Hence the solutions y, Q, and θ are functions of t but, in contrast with the explicit notation $k(t)$ and $E(t)$, this functional dependence is to be understood from the context.

Our study is made easier by grouping together those differential equations that have a significant common property. The most important property is that of *order.* The *order of a differential equation* is the order of the highest derivative of the dependent variable which appears in the equation. For example, the oscillator and pendulum equations are second order, while the growth equation is first order. In the remainder of this chapter, first order differential equations will be studied, and in later chapters higher order equations will be discussed.

A *partial differential equation* is a relationship involving an unknown function of *at least two* variables and one or more of its derivatives. For instance, a partial differential equation that describes the temperature $u(x, t)$ in a thin heated wire as a function of position x and time t is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{or} \qquad u_t = ku_{xx},$$

where k is a constant dependent on the physical properties of the wire. An introduction to some of the partial differential equations of mathematical physics is given in Chapter 9.

EXERCISES 1.1

In Exercises 1–6 determine which of the given equations are ordinary differential equations and which are partial differential equations. Identify the dependent and independent variables.

1. $y'' + 3y' + 2y = \sin t$ **2.** $\left(\frac{dy}{dt}\right)^2 + 4y = t^2$ **3.** $u_{xx} - 9u_{tt} = 3\sin x\cos t$

4. $\frac{\partial u}{\partial x} + t^2\frac{\partial u}{\partial t} = 0$ **5.** $\frac{d^3x}{dt^3} - 4\frac{dx}{dt} + 7x = 4e^{-t}$ **6.** $\varphi'' + (\cos t)\varphi = 0$

In Exercises 7–12 determine the order of each of the given differential equations.

7. $y' + 4y = 0$ **8.** $\frac{d^2y}{dt^2} + 4y^2 = 7$ **9.** $y'' + \epsilon y'(y^2 - 1) + 4y = 3\sin t$

10. $(p(t)y')' + q(t)y = 0$ **11.** $y'y'' + 4yy' = 0$ **12.** $(y')^2 + y^2 = 4$

In Exercises 13–20 verify by direct substitution that the given function $y(t)$ is a solution.

13. $\frac{dy}{dt} = 10(y - t), \quad y(t) = t + 0.1$

14. $\frac{dy}{dt} = 4ty^2, \quad y(t) = \frac{1}{(1 - 2t^2)}$

15. $\frac{dy}{dt} = -2y + e^t, \quad y(t) = Ce^{-2t} + \frac{1}{3}e^t,$ C any constant

16. $\frac{dy}{dt} = \frac{t(3 - 2y)}{(t^2 - 1)}, \quad y(t) = \frac{1}{(t^2 - 1)} + \frac{3}{2}$