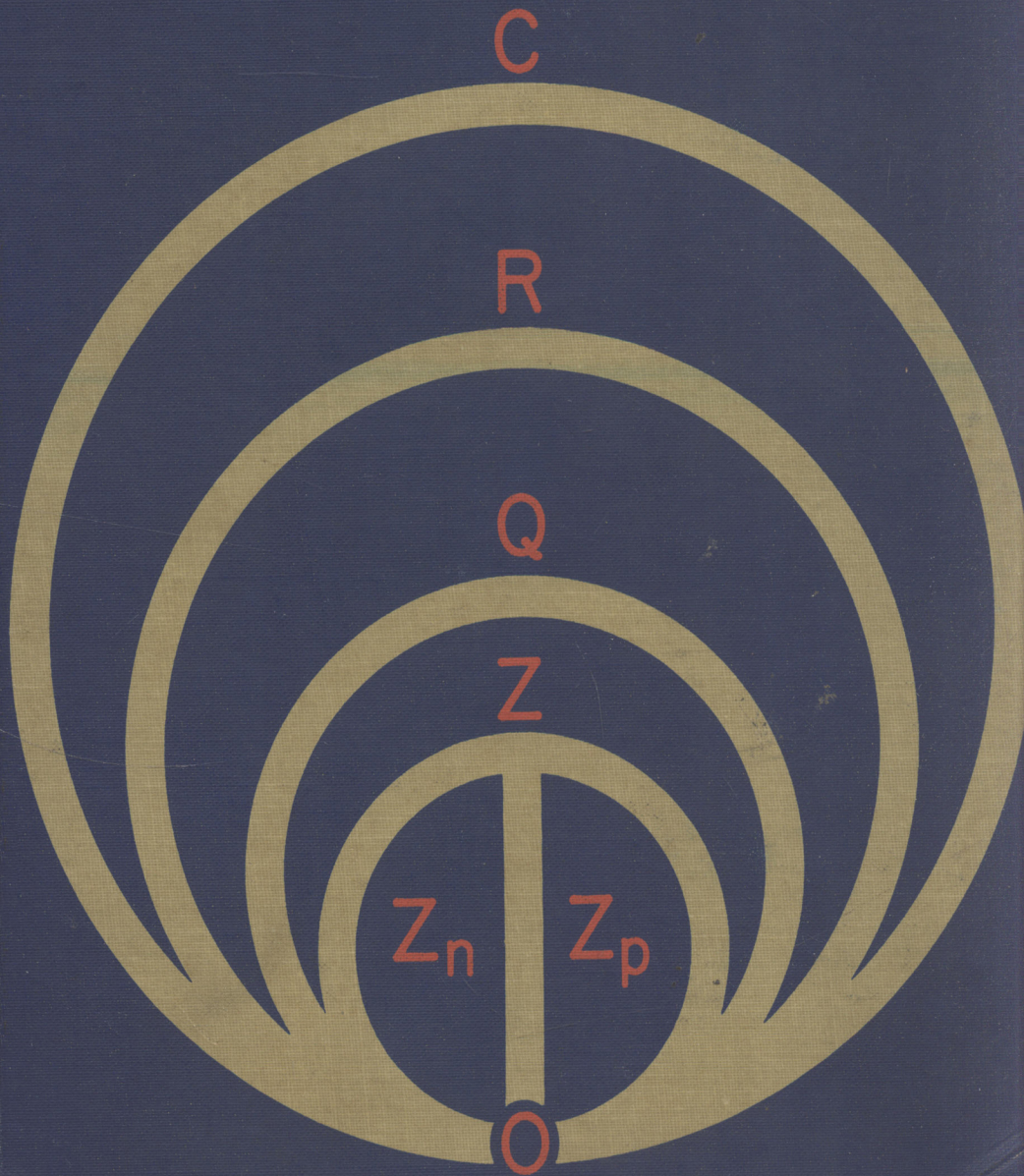


Alphonse J. Jackowski

John B. Sbrega

fundamentals of MODERN MATHEMATICS



fundamentals of MODERN MATHEMATICS

MODERN MATHEMATICS

PRENTICE-HALL, INC., ENGLEWOOD CLIFFS, NEW JERSEY

fundamentals of MODERN MATHEMATICS

Jackowski/Sbrega

© 1970 by Prentice-Hall, Inc., Englewood Cliffs, New Jersey

*All rights reserved. No part of this book may be
reproduced in any form or by any means without permission
in writing from the publisher.*

Current printing (last digit):

10 9 8 7 6 5 4 3 2 1

13-341172-9

Library of Congress Catalog Card Number 75-95705

Printed in the United States of America

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LTD., *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

fundamentals of

Alphonse J. Jackowski

*Westfield State College
Westfield, Massachusetts*

John B. Sbrega

*Westfield State College
Westfield, Massachusetts*

*Affectionately dedicated
to our wives, Kitty and Rita,
without whose patience and understanding
this book would not have been possible*

preface

This text is designed for a two-semester course in the fundamental concepts of mathematics. It is strictly a content and not a methods text. Its purpose is two-fold: (1) to provide prospective teachers with the mathematical preparation necessary to teach the modern elementary and junior high school curricula, and (2) to provide the liberal arts student with a terminal course wherein structure and unifying concepts are emphasized. It is also suitable for inservice training programs for teachers.

The authors have been influenced and guided by the recommendations of the CUPM, the CEEB, the SMSG, and other agencies concerned with curriculum improvement. This text is a compromise of these recommendations. A conscious effort has been made to keep the text readable from the point of view of a student with two years of high school mathematics preparation. Understanding is stressed throughout and is enhanced by a balance of intuition and rigor. The structural concepts of group, ring, integral domain, and field are introduced in a natural way and only after simple and familiar examples have been discussed in detail.

A distinguishing feature is the insertion of numerous and frequent sets of exercises. This permits immediate application of newly acquired concepts and reinforces learning. Optional sections and exercises are indicated by an asterisk and may be used for enrichment at the discretion of the instructor.

Chapter 1 introduces the student to the nature of mathematics and the essentials of a deductive system. Chapter 2 presents the elements of set theory used throughout the remainder of the text. Chapter 3 discusses the nature of a mathematical proof and illustrates the various methods of proof and disproof in detail. Chapter 4 is a cursory and intuitive preview of number systems which prepares the student for the detailed treatment of the properties of number systems in the ensuing chapters. Beginning with the set of natural numbers, the successive extensions to the integers, the rational numbers, and the real numbers are motivated by employing only the closure axiom with respect to the four arithmetic operations. The extension of the set of real numbers to the set of complex numbers is based on the need for algebraic closure. In Chapter 5 the properties of the natural numbers are treated. The concept of a mathematical (algebraic) system is introduced and is used to discuss the axioms for the set of natural numbers under addition and multiplication. This is followed by a discussion of the order properties. The principle of finite induction is presented as an optional topic. Chapter 6 treats the properties of the set of integers. The concepts of a group, a ring, and an integral domain are introduced in a familiar setting here. Examples of finite groups and topics from elementary number theory are included. The section concerning the congruence of integers modulo n provides an opportunity to enhance the understanding of the structural concepts presented at the beginning of the chapter. Chapter 7 presents a development of the field of rational numbers using the integral domain of integers as a basis. A discussion of finite fields is included. Chapter 8 consists of an intuitive treatment of the field of real numbers and Chapter 9 completes the discussion of the field of complex numbers begun in Chapter 4. Chapter 10 consists of a detailed treatment of numeration systems. It contains a review of the Hindu-Arabic system and an introduction to systems involving base two, base five, and base twelve. Chapter 11 is a careful development of the foundations of Euclidean geometry. A sequence of theorems recommended by the Commission on Mathematics of the CEEB is presented in Chapter 12. This serves as an efficient preparation for the coordinate geometry of Chapter 13. Chapter 14 introduces the concepts of precision and accuracy and develops the use of approximate numbers and the mensuration formulas for the common geometric figures of two and three dimensions. In the final chapter, which treats relations and functions, the emphasis is on fundamental properties rather than on manipulative techniques.

A preliminary edition of this text was class-tested for two years at Westfield State College, Westfield, Massachusetts.

The authors wish to thank President Leonard J. Savignano and Dean Edward S. Townsend of Westfield State College for their administrative support and encouragement. We are particularly indebted to our colleagues

in the department of mathematics at Westfield who have read parts of the manuscript, have made valuable suggestions, and have class-tested the preliminary edition.

ALPHONSE J. JACKOWSKI

JOHN B. SBREGA

Westfield, Massachusetts

contents

1

introduction 1

- 1.1 *THE NATURE OF MATHEMATICS* 1
- 1.2 *DEDUCTIVE AND INDUCTIVE REASONING* 2
- 1.3 *THE NATURE OF A DEDUCTIVE SYSTEM* 2
- 1.4 *THE NEED FOR UNDEFINED TERMS* 3
- 1.5 *DEFINITIONS IN MATHEMATICS* 3
- 1.6 *AXIOMS AND POSTULATES* 4
- 1.7 *THEOREMS* 5
- 1.8 *HINTS ON HOW TO STUDY MATHEMATICS* 5
- EXERCISES* 6

2

elementary set theory 7

- 2.1 *THE CONCEPT OF A SET* 7
- 2.2 *ONE-TO-ONE CORRESPONDENCE* 9
- EXERCISES* 10

- 2.3 *EQUIVALENT SETS AND CARDINALITY* 10
- 2.4 *THE N-FACTORIAL CONCEPT* 12
EXERCISES 13
- 2.5 *SUBSETS, THE UNIVERSAL SET, DEFINITION OF AN INFINITE SET* 14
EXERCISES 17
- 2.6 *THE POWER SET* 17
- 2.7 *UNION AND INTERSECTION* 18
EXERCISES 20
- 2.8 *COMPLEMENTATION* 21
EXERCISES 22
- 2.9 *ORDERED PAIRS, CARTESIAN PRODUCTS* 23
EXERCISES 24
- 2.10 *EQUIVALENCE RELATIONS* 25
EXERCISES 26

3

methods of proof 27

- 3.1 *STATEMENTS* 27
- 3.2 *VARIABLES, CONSTANTS, OPEN SENTENCES* 28
- 3.3 *TRUTH SETS, NEGATIONS, EQUIVALENT OPEN SENTENCES* 29
EXERCISES 31
- 3.4 *IMPLICATIONS* 32
- 3.5 *CONVERSES, CONTRAPOSITIVES* 34
EXERCISES 37
- 3.6 *DIRECT PROOF* 38
- 3.7 *INDIRECT PROOF* 42
- 3.8 *PROOF BY USE OF THE CONTRAPOSITIVE* 44
- 3.9 *METHODS OF DISPROOF* 45
EXERCISES 46

4

number systems 48

- 4.1 *THE COUNTING NUMBERS* 48
- 4.2 *THE POSITIVE AND NEGATIVE INTEGERS AND ZERO* 49
- 4.3 *THE RATIONAL NUMBERS* 50
- 4.4 *THE REAL NUMBERS* 50
- 4.5 *THE COMPLEX NUMBERS* 54
EXERCISES 57

5

the natural numbers 61

- 5.1 OPERATIONS, MATHEMATICAL SYSTEMS 62
- 5.2 EQUALITY IN \mathbb{Z}_p 63
- 5.3 FUNDAMENTAL PROPERTIES OF $(\mathbb{Z}_p, +)$ 63
EXERCISES 65
- 5.4 THE PROPERTIES OF (\mathbb{Z}_p, \cdot) 66
- 5.5 THE DISTRIBUTIVE LAWS FOR $(\mathbb{Z}_p, +, \cdot)$ 67
- 5.6 THE ORDER PROPERTIES OF \mathbb{Z}_p 69
EXERCISES 71

6

the integers 73

- 6.1 EQUALITY IN \mathbb{Z} 73
- 6.2 FUNDAMENTAL PROPERTIES OF $(\mathbb{Z}, +)$ 74
- 6.3 SUBTRACTION IN \mathbb{Z} 75
EXERCISES 75
- 6.4 THE PROPERTIES OF (\mathbb{Z}, \cdot) 76
- 6.5 DISTRIBUTIVE LAWS FOR $(\mathbb{Z}, +, \cdot)$ 76
- 6.6 PROPERTIES OF ZERO, DIVISION IN \mathbb{Z} 77
- 6.7 ORDER PROPERTIES OF $(\mathbb{Z}, +, \cdot)$ 79
EXERCISES 80
- 6.8 THE CONCEPT OF A GROUP 81
EXERCISES 84
- 6.9 RINGS, INTEGRAL DOMAINS 85
EXERCISES 87
- 6.10 PRIME AND COMPOSITE NUMBERS 88
- 6.11 THE FUNDAMENTAL THEOREM OF ARITHMETIC 90
- 6.12 THE INFINITUDE OF PRIMES 90
EXERCISES 91
- 6.13 GREATEST COMMON DIVISOR 92
- 6.14 DIVISION ALGORITHM, EUCLIDEAN ALGORITHM 93
- 6.15 LEAST COMMON MULTIPLE 95
EXERCISES 97
- 6.16 CONGRUENCE OF INTEGERS MODULO n 97
- 6.17 ADDITION AND MULTIPLICATION IN \mathbb{Z}_n 100
EXERCISES 103
- 6.18 THE STRUCTURE OF $(\mathbb{Z}_n, \oplus, \odot)$ 103
EXERCISES 105
- 6.19 CASTING OUT NINES 107
EXERCISES 111

7

the rational numbers 112

- 7.1 *EQUALITY, ADDITION, AND MULTIPLICATION IN \mathbb{Q}* 113
- 7.2 *THE PROPERTIES OF $(\mathbb{Q}, +)$* 115
- 7.3 *SUBTRACTION IN \mathbb{Q}* 118
 - EXERCISES* 118
- 7.4 *THE PROPERTIES OF (\mathbb{Q}, \cdot)* 119
- 7.5 *DISTRIBUTIVE LAWS FOR $(\mathbb{Q}, +, \cdot)$* 121
- 7.6 *THE FIELD OF RATIONAL NUMBERS $(\mathbb{Q}, +, \cdot)$* 123
 - EXERCISES* 124
- 7.7 *ORDER IN $(\mathbb{Q}, +, \cdot)$* 125
 - EXERCISES* 130
- 7.8 *DENUMERABILITY OF \mathbb{Q}* 130
- 7.9 *FINITE FIELDS* 131
 - EXERCISES* 131

8

the real numbers 133

- 8.1 *THE FIELD OF REAL NUMBERS* 134
- 8.2 *ORDER IN \mathbb{R}* 135
- 8.3 *THE SUBSET OF IRRATIONAL NUMBERS* 135
 - EXERCISES* 136
- 8.4 *THE REAL NUMBERS AS INFINITE DECIMALS* 137
- 8.5 *THE COMPLETENESS PROPERTY* 141
- 8.6 *NONDENumerABILITY OF \mathbb{R}* 142
- 8.7 *INTERVALS OF REAL NUMBERS, ABSOLUTE VALUE* 143
 - EXERCISES* 145

9

the complex numbers 147

- 9.1 *EQUALITY, ADDITION, AND MULTIPLICATION IN \mathbb{C}* 148
- 9.2 *THE PROPERTIES OF $(\mathbb{C}, +)$* 148
 - EXERCISES* 150
- 9.3 *THE PROPERTIES OF (\mathbb{C}, \cdot)* 151
- 9.4 *DISTRIBUTIVE LAWS FOR $(\mathbb{C}, +, \cdot)$* 155
- 9.5 *THE FIELD PROPERTIES OF $(\mathbb{C}, +, \cdot)$* 156
- 9.6 *INTEGRAL POWERS OF i* 157

- 9.7 LACK OF ORDER IN C 158
EXERCISES 159
- 9.8 GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS 160
EXERCISES 161

10

numeration systems 162

- 10.1 NUMBERS, NUMERALS, EXPONENTS 163
- 10.2 THE HINDU-ARABIC SYSTEM (DECIMAL SYSTEM) 164
EXERCISES 167
- 10.3 PRELIMINARY CONCEPTS CONCERNING NONDECIMAL SYSTEMS 168
- 10.4 THE QUINARY SYSTEM 170
EXERCISES 175
- 10.5 FUNDAMENTAL OPERATIONS IN BASE FIVE 176
EXERCISES 180
- 10.6 QUINARY FRACTIONS AND FRACTIONS (BASE FIVE) 181
EXERCISES 188
- 10.7 THE BINARY SYSTEM (BASE TWO) 189
- 10.8 FUNDAMENTAL OPERATIONS IN BASE TWO 192
EXERCISES 194
- 10.9 THE DUODECIMAL SYSTEM (BASE TWELVE) 195
- 10.10 FUNDAMENTAL OPERATIONS IN BASE TWELVE 198
EXERCISES 201

11

geometry 203

- 11.1 PRIMITIVE GEOMETRY 204
- 11.2 STRUCTURE OF GEOMETRY 205
- 11.3 UNDEFINED TERMS 207
- 11.4 PRELIMINARY DEFINED TERMS 208
EXERCISES 210
- 11.5 DISTANCE: BETWEENNESS 210
EXERCISES 214
- 11.6 HALF-LINE, RAY, SEGMENT, HALF-PLANE, HALF-SPACE 215
EXERCISES 221
- 11.7 PROPERTIES OF POINTS, LINES, AND PLANES 222
EXERCISES 226
- 11.8 PROPERTIES OF SEGMENTS 227
- 11.9 ANGLES 228
EXERCISES 232
- 11.10 SIMPLE CLOSED CURVE, PLANE REGION 233
- 11.11 POLYGONS 236
EXERCISES 242

- 11.12 *SIMPLE CLOSED SURFACE, SPACE REGION* 243
- 11.13 *POLYHEDRONS* 245
- 11.14 *EULER'S LAW* 249
EXERCISES 249
- 11.15 *CYLINDRICAL AND PRISMATIC SURFACES* 250
- 11.16 *CONICAL AND PYRAMIDAL SURFACES* 252
EXERCISES 255

12

a sequence of theorems 256

- 12.1 *THE ISOSCELES TRIANGLE THEOREM* 257
- 12.2 *THE EXTERIOR ANGLE THEOREM* 258
- 12.3 *THE ALTERNATE INTERIOR ANGLES THEOREM* 259
EXERCISES 261
- 12.4 *THE ANGLE SUM THEOREM* 262
- 12.5 *THE ANGLE-ANGLE SIMILARITY THEOREM (A.A.)* 263
- 12.6 *THE RIGHT TRIANGLE SIMILARITY THEOREM* 264
- 12.7 *THE PYTHAGOREAN THEOREM* 265
EXERCISES 267

13

coordinate geometry 271

- 13.1 *THE NATURE OF COORDINATE GEOMETRY* 271
- 13.2 *DIRECTED LINES AND SEGMENTS* 273
- 13.3 *HORIZONTAL AND VERTICAL SEGMENTS* 275
EXERCISES 277
- 13.4 *THE DISTANCE FORMULA* 278
- 13.5 *POINT OF DIVISION* 279
EXERCISES 281
- 13.6 *SLOPE* 282
- 13.7 *PARALLEL AND PERPENDICULAR LINES* 283
EXERCISES 287
- 13.8 *EQUATIONS OF LINES* 288
- 13.9 *THE CIRCLE* 293
EXERCISES 294
- 13.10 *THE GRAPH OF AN EQUATION* 295
EXERCISES 301
- 13.11 *CONICS* 303
EXERCISES 309
- 13.12 *THE EQUATION OF A LOCUS* 310
EXERCISES 311
- 13.13 *ANALYTIC PROOFS* 312
EXERCISES 315

14

mensuration 317

- 14.1 *APPROXIMATE NUMBERS 317*
EXERCISES 319
- 14.2 *PRECISION OF A MEASUREMENT, ABSOLUTE ERROR 320*
- 14.3 *ACCURACY OF A MEASUREMENT, RELATIVE ERROR 323*
- 14.4 *SIGNIFICANT FIGURES 324*
- 14.5 *ROUNDING OFF 327*
EXERCISES 329
- 14.6 *COMPUTATIONS WITH APPROXIMATE NUMBERS 330*
- 14.7 *ADDITION AND SUBTRACTION OF APPROXIMATE NUMBERS 331*
- 14.8 *MULTIPLICATION AND DIVISION OF APPROXIMATE NUMBERS 334*
- 14.9 *SCIENTIFIC NOTATION 338*
EXERCISES 339
- 14.10 *MEASURE OF A RECTANGULAR REGION 341*
- 14.11 *MENSURATION FORMULAS—PLANE REGIONS 344*
EXERCISES 351
- 14.12 *MEASURE OF A RECTANGULAR SOLID 353*
- 14.13 *MENSURATION FORMULAS—SPACE REGIONS 355*
EXERCISES 362

15

relations and functions 366

- 15.1 *RELATIONS 366*
EXERCISES 368
- 15.2 *FUNCTIONS 369*
EXERCISES 371
- 15.3 *ONE-TO-ONE FUNCTIONS, ONTO FUNCTIONS, INVERSE FUNCTIONS 373*
EXERCISES 378
- 15.4 *ALGEBRA OF FUNCTIONS 380*
- 15.5 *TYPES OF FUNCTIONS 383*
EXERCISES 387

selected answers 389

index 413

1

introduction

1.1

The Nature of Mathematics

Mathematics has many aspects and its generality and usefulness are apparent in many forms of human endeavor. The scientist uses the language of mathematics to formulate his theories in concise form. The engineer finds mathematics an essential tool for design and construction. The manufacturer turns to mathematics to regulate and to control the quality of his products. Furthermore, since the advent of the computer, we find mathematics making a tremendous impact on such diverse fields as biology, business, economics, medicine, and even music. Likewise, we find mathematics playing an ever-increasing role in the more mundane phases of everyday life.

Although mathematics is often divided into two main categories, applied mathematics and pure mathematics, mathematics itself is essentially abstract and its power and generality are inherent in its abstract nature. The kind of mathematics we have discussed thus far is applied mathematics, and one involved in such work is called an applied mathematician. On the other hand, the pure mathematician studies mathematics in its own right and finds great aesthetic appeal in its logical structure and abstract systems.