SPECTRAL THEORY OF LINEAR OPERATORS

H. R. Dowson
Department of Mathematics
University of Glasgow
Glasgow, Scotland

1978

ACADEMIC PRESS
LONDON NEW YORK SAN FRANCISCO
A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS INC. (LONDON) LTD.
24/28 Oval Road
London NW1

United States Edition published by
ACADEMIC PRESS INC.
111 Fifth Avenue
New York, New York 10003

Library of Congress Catalog Card Number: 77-79301
ISBN: 0-12-220950-8

PRINTED IN GREAT BRITAIN BY
PAGE BROS (NORWICH) LTD, NORWICH

L.M.S. MONOGRAPHS

Editors: P. M. COHN *and* G. E. H. REUTER

1. Surgery on Compact Manifolds *by* C. T. C. Wall, F.R.S.
2. Free Rings and Their Relations *by* P. M. Cohn
3. Abelian Categories with Applications to Rings and Modules *by* N. Popescu
4. Sieve Methods *by* H. Halberstam and H.-E. Richert
5. Maximal Orders *by* I. Reiner
6. On Numbers and Games *by* J. H. Conway
7. An Introduction to Semigroup Theory *by* J. M. Howie
8. Matroid Theory *by* D. J. A. Welsh
9. Subharmonic Functions, Volume 1 *by* W. K. Hayman and P. B. Kennedy
10. Topos Theory *by* P. T. Johnstone
11. Extremal Graph Theory *by* B. Bollabás
12. Spectral Theory of Linear Operators *by* H. R. Dowson

Published for the London Mathematical Society
by Academic Press Inc. (London) Ltd.

Preface

During the period 1958–1971 there appeared the three volumes of the monumental treatise "Linear Operators" by N. Dunford and J. T. Schwartz. This work contains a wealth of material. The purpose of my book is to develop this research further in one particular direction: namely the study of various classes of linear operators on a complex Banach space which possess a rich spectral theory.

General spectral theory is developed in Part One. The material presented is very much influence by the aforementioned work of N. Dunford and J. T. Schwartz and by the book "Introduction to Functional Analysis" by A. E. Taylor.

Part Two contains two chapters. The first relies heavily on ideas of F. F. Bonsall and V. I. Lomonosov to simplify the usual presentation of the spectral theory of compact operators and of the theory of superdiagonal forms for compact operators due to J. R. Ringrose. In Chapter 3 we present the theory of Riesz operators initiated by A. F. Ruston and later developed by T. T. West.

Part Three is a very brief section containing all the properties of hermitian operators on a Banach space required for later sections of the book.

Part Four is the longest section of the book and is devoted to the theory of prespectral operators. It includes brief sections on spectral operators and normal operators on Hilbert space that contain proofs of the basic theorems significantly simpler than those found in existing textbooks.

In Part Five we develop the theory of well-bounded operators initiated by D. R. Smart and developed by J. R. Ringrose and others.

My thanks are due to Professor John Ringrose on two counts. First, as my research supervisor in the early 60s he first aroused my interest in many of the topics discussed in this book and second because Chapters 2, 15 and 16 are heavily dependent on his research papers on compact and well-bounded operators. I should also like to express my gratitude to Professor Earl

Berkson for many valuable discussions on hermitian operators, prespectral operators and well-bounded operators. I am indebted to Dr Philip Spain for submitting to me several of his unpublished manuscripts and for reading and criticizing an earlier version of Part Three of this book.

My thanks are due to Dr Trevor West for reading and criticizing an earlier version of Parts One and Two of this book. Also, I am very grateful to Dr Alastair Gillespie for reading and criticizing an earlier version of Part Four of this book and for helping me very considerably in writing Chapters 18 and 20 by sending to me several of his unpublished manuscripts.

Finally, I am much indebted to Miss Daphne Davidson for her patient and very careful work in producing the typescript of this volume.

University of Glasgow H. R. DOWSON
March, 1977

Note to the Reader

It will be assumed that the reader of this book has a basic knowledge of functional analysis as could be acquired from "Elements of functional analysis" by A. L. Brown and A. Page. We assume also a knowledge of Gelfand theory in commutative Banach algebras and of elementary spectral theory in general Banach algebras. Our standard reference for these topics is "Complete Normed Algebras" by F. F. Bonsall and J. Duncan. For the results on measure theory that we shall need, the reader is referred to "Measure Theory" by P. R. Halmos. It will be assumed also that the reader is familiar with the theories of vector-valued holomorphic functions and of integration of vector-valued functions presented in Chapter III of "Linear Operators" by N. Dunford and J. T. Schwartz. However, the deeper results, derived from this theory, on the representation of weakly compact linear mappings are specifically recalled.

Theorems, propositions, lemmas, corollaries, definitions and notes are numbered consecutively. For example, Theorem 4.17 refers to the seventeenth item in Chapter 4.

We have divided the bibliography into two sections. The works in "General background reading" are referred to in the text by a number alone. Other references are cited by giving the author's name followed by a number.

Some Terminology, Notation and Conventions Used Throughout This Book

Throughout, X denotes a non-zero complex Banach space, otherwise arbitrary unless the contrary is explicitly stated. H denotes a non-zero complex Hilbert space, otherwise arbitrary unless the contrary is explicitly stated. Operator means "bounded linear operator". The Banach algebra of operators on X is denoted by $L(X)$. The dual space of X is denoted by X^*. We write $\langle x, \phi\rangle$ for the value of the functional ϕ in X^* at the point x of X. In a Hilbert space setting this notation is also used for the inner product of two vectors. The adjoint of an operator T on X is denoted by T^*. A similar notation is used in a Hilbert space setting for the Hilbert adjoint of an operator. The *annihilator* $Y^\perp$ of a closed subspace Y of X is the set

$$\{\phi \in X^* : \langle y, \phi\rangle = 0 \quad \text{for all } y \text{ in } Y\}.$$

If Y is a closed subspace of H, then $Y^\perp$ denotes the orthogonal complement of Y.

R denotes the set of real numbers
C denotes the set of complex numbers
Z denotes the set of integers
N denotes the set of positive integers

Throughout, scalars and functions are complex-valued unless the contrary is explicitly stated. $\subseteq$ is used for "is contained in", while $\subset$ is reserved for "is strictly contained in". $\{a\}$ denotes the set consisting of the point a alone. $\chi(\tau; z)$ denotes the characteristic function of the set τ evaluated at the point z. Occasionally the characteristic function of τ is denoted by χ_τ. $C(K)$ denotes the Banach algebra of continuous complex-valued functions on the compact set K under the supremum norm. Σ_p denotes the σ-algebra of Borel subsets of the complex plane.

Let $A \subseteq X$. We denote the norm closure of A in X by $\bar{A}$ or cl A. If Y is a closed subspace of X, the quotient space of X by Y is denoted by X/Y. Let

$T \in L(X)$ and let $TY \subseteq Y$. The restriction of T to Y is denoted by $T|Y$ and the operator induced on X/Y by T is denoted by T_Y.

If $\mathscr{A} \subseteq L(X)$, then $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$ and $\mathscr{A}''$ denotes the bicommutant of $\mathscr{A}$. Let $T \in L(X)$. We write $N(T)$ for the null-space of the operator T and $R(T)$ for the range space TX. $\sigma(T)$ and $\rho(T)$ denote respectively the spectrum and resolvent set of T. $\mathscr{F}(T)$ denotes the family of complex functions analytic on some open neighbourhood of $\sigma(T)$. $\sigma_a(T)$, $\sigma_c(T)$, $\sigma_p(T)$, $\sigma_r(T)$ denote respectively the approximate point spectrum, the continuous spectrum, the point spectrum and the residual spectrum of T. $\nu(T)$ denotes the spectral radius of T.

Let $J = [a, b]$ be a compact interval of $\mathbf{R}$. Let $BV(J)$ be the Banach algebra of complex-valued functions of bounded variation on J with norm $|||\ |||$ defined by

$$|||f||| = |f(b)| + \operatorname{var}(f, J) \qquad (f \in BV(J)),$$

where $\operatorname{var}(f, J)$ is the total variation of f over J.

Let $AC(J)$ be the Banach subalgebra of $BV(J)$ consisting of absolutely continuous functions on J. For f in $AC(J)$

$$|||f||| = |f(b)| + \int_a^b |f'(t)|\,dt.$$

Let $NBV(J)$ be the Banach subalgebra of $BV(J)$ consisting of those functions f in $BV(J)$ which are normalized by the requirement that f is continuous on the left on $(a, b]$.

Contents

Part 1

GENERAL SPECTRAL THEORY

1. General Spectral Theory

In this chapter, we develop the spectral theory of a bounded linear operator on a non-zero complex Banach space. The concepts of spectrum and resolvent set are introduced and the various subdivisions of the spectrum are studied. A functional calculus for such operators is introduced. The theory of ascent and descent of linear operators is developed. Various results on the spectra of restrictions of operators to closed invariant subspaces are proved.

Let $T \in L(X)$.

Definition 1.1. The *resolvent set* $\rho(T)$ *of* T is the set of complex numbers λ for which $\lambda I - T$ is invertible in the Banach algebra $L(X)$.

Definition 1.2. The *spectrum* $\sigma(T)$ *of* T is defined to be $\mathbf{C}\backslash\rho(T)$.

Definition 1.3. The function

$$\lambda \to (\lambda I - T)^{-1} \qquad (\lambda \in \rho(T))$$

is called the *resolvent* of T.

THEOREM 1.4. *Let* $T \in L(X)$. *The resolvent set* $\rho(T)$ *is open. Also the function* $\lambda \to (\lambda I - T)^{-1}$ *is analytic in* $\rho(T)$.

Proof. Let λ be a fixed point in $\rho(T)$ and let μ be any complex number with $|\mu| < \|(\lambda I - T)^{-1}\|^{-1}$. We show that $\lambda + \mu \in \rho(T)$. Consider the series $\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)}$. Since $\|\mu(\lambda I - T)^{-1}\| < 1$ this series converges in the norm of $L(X)$. Denote its sum by $S(\mu)$. Then

$$[(\lambda + \mu) I - T] S(\mu) = (\lambda I - T) S(\mu) + \mu S(\mu) = I,$$

$$S(\mu)[(\lambda + \mu) I - T] = S(\mu)(\lambda I - T) + \mu S(\mu) = I.$$

It follows that $\lambda + \mu \in \rho(T)$ and the function $\mu \to S(\mu) = [(\lambda + \mu) I - T]^{-1}$ is analytic at the point $\mu = 0$.

COROLLARY 1.5. *Let $T \in L(X)$. If $d(\lambda)$ is the distance from λ to the spectrum $\sigma(T)$ then*

$$\|(\lambda I - T)^{-1}\| \geqslant \frac{1}{d(\lambda)} \qquad (\lambda \in \rho(T)).$$

Therefore $\|(\lambda I - T)^{-1}\| \to \infty$ as $d(\lambda) \to 0$, and the resolvent set is the natural domain of analyticity of the resolvent.

Proof. In the course of proving Theorem 1.4 it was shown that if $|\mu| < \|(\lambda I - T)^{-1}\|^{-1}$ then $\lambda + \mu \in \rho(T)$. Hence $d(\lambda) \geqslant \|(\lambda I - T)^{-1}\|^{-1}$, from which the statements follow.

THEOREM 1.6. *Let $T \in L(X)$. Then $\sigma(T)$ is compact and non-empty.*

Proof. For $|\lambda| > \|T\|$, the series $\sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ converges in the norm of $L(X)$. Let $S(\lambda)$ denote its sum. Then

$$(\lambda I - T) S(\lambda) = S(\lambda)(\lambda I - T) = I.$$

Hence

$$S(\lambda) = (\lambda I - T)^{-1} \qquad (|\lambda| > \|T\|).$$

It follows that $\sigma(T)$ is bounded. By Theorem 1.4, $\sigma(T)$ is closed. Hence $\sigma(T)$ is compact. It remains to show that the spectrum is non-empty. If $\sigma(T) = \varnothing$, then the resolvent of T is an entire function. Since $(\lambda I - T)^{-1}$ is readily seen to be analytic at infinity, it follows from Liouville's theorem that $(\lambda I - T)^{-1}$

is a constant. Hence the coefficient of λ^{-1} in the Laurent series $\sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ vanishes, so that $I = 0$, which contradicts the assumption $X \neq \{0\}$. This completes the proof.

Definition 1.7. Let $T \in L(X)$. The *spectral radius* $\nu(T)$ *of* T is defined by

$$\nu(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

PROPOSITION 1.8. *Let $T \in L(X)$. The spectral radius of T has the properties*

$$\nu(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \leqslant \|T\|.$$

Proof. In the course of proving Theorem 1.6 it was shown that

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} T^n/\lambda^{n+1} \qquad (|\lambda| > \|T\|)$$

and so $\nu(T) \leqslant \|T\|$. By Theorem 1.4, the resolvent is analytic in $\rho(T)$. Let $x \in X$, $y \in X^*$. Then the function $\lambda \to \langle(\lambda I - T)^{-1} x, y\rangle$ is analytic for $|\lambda| > \nu(T)$. Hence the singularities of this function all lie in the disc $\{\lambda : |\lambda| \leqslant \nu(T)\}$. Thus the series $\sum_{n=0}^{\infty} \langle \lambda^{-n-1} T^n x, y\rangle$ converges for $|\lambda| > \nu(T)$ and for such λ we have

$$\sup_n \left| \frac{\langle T^n x, y\rangle}{\lambda^{n+1}} \right| < \infty.$$

The principle of uniform boundedness shows that there is M_λ such that

$$\|T^n \lambda^{-n-1}\| \leqslant M_\lambda < \infty$$

and hence that

$$\limsup_n \|T^n\|^{1/n} \leqslant |\lambda|.$$

Since λ is an arbitrary number with $|\lambda| > \nu(T)$ it follows that

$$\limsup_n \|T^n\|^{1/n} \leqslant \nu(T).$$

To complete the proof we show that $v(T) \leq \liminf_n \|T^n\|^{1/n}$. Observe that if $\lambda \in \sigma(T)$ then $\lambda^n \in \sigma(T^n)$: for the factorization

$$(\lambda^n I - T^n) = (\lambda I - T)P(T) = P(T)(\lambda I - T)$$

shows that if $(\lambda^n I - T^n)$ has an inverse in $L(X)$ so will $\lambda I - T$. Thus $|\lambda|^n \leq \|T^n\|$, and hence

$$\sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T^n\|^{1/n}.$$

$$v(T) \leq \liminf_n \|T^n\|^{1/n}.$$

Definition 1.9. Let $T \in L(X)$. T is said to be *quasinilpotent* if and only if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$.

PROPOSITION 1.10. *Let $T \in L(X)$.*
(i) *T is quasinilpotent if and only if $v(T) = 0$.*
(ii) *T is quasinilpotent if and only if $\sigma(T) = \{0\}$.*

Proof. These results follow at once from Definition 1.7 and Proposition 1.8.

PROPOSITION 1.11. *Let $T \in L(X)$. The following identity, known as the resolvent equation, is valid for every pair of points λ, μ in $\rho(T)$.*

$$(\lambda I - T)^{-1} - (\mu I - T)^{-1} = (\mu - \lambda)(\lambda I - T)^{-1}(\mu I - T)^{-1}.$$

Proof. Observe that

$$(\mu I - T)(\lambda I - T)\{(\lambda I - T)^{-1} - (\mu I - T)^{-1}\} = (\mu I - T) - (\lambda I - T).$$

$$(\mu I - T)(\lambda I - T)\{(\lambda I - T)^{-1} - (\mu I - T)^{-1}\} = (\mu - \lambda)I.$$

Multiply both sides of this equation by $(\lambda I - T)^{-1}(\mu I - T)^{-1}$ to complete the proof.

Let $T \in L(X)$. There is an operator T^* in $L(X^*)$ called the *adjoint* of T such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad (x \in X, y \in X^*).$$

The map $T \to T^*$ is an isometric linear map of $L(X)$ into $L(X^*)$ with the additional property

$$(AB)^* = B^*A^* \qquad (A, B \in L(X)).$$

PROPOSITION 1.12. *Let $T \in L(X)$. The spectrum of T^* is equal to the spectrum of T. Moreover*

$$((\lambda I - T)^{-1})^* = (\lambda I^* - T^*)^{-1} \qquad (\lambda \in \rho(T)).$$

Proof. If T^{-1} exists and is in $L(X)$ then

$$T^*(T^{-1})^* = (T^{-1}T)^* = I^* = (TT^{-1})^* = (T^{-1})^*T^*.$$

Thus $(T^*)^{-1}$ exists, is in $L(X^*)$, and $(T^*)^{-1} = (T^{-1})^*$. Conversely, if $(T^*)^{-1}$ exists and is in $L(X^*)$ then, by what has already been proved, $(T^{**})^{-1}$ exists and is in $L(X^{**})$. Thus T^{**} is a homeomorphism of X^{**} onto itself. It is also an extension of T. Hence T is one-to-one and TX is closed. It only remains to show that $TX = X$. If $TX \neq X$, there is f in X^* with $f \neq 0$ and

$$\langle Tx, f\rangle = \langle x, T^*f\rangle = 0 \quad (x \in X).$$

Hence $T^*f = 0$, contradicting the assumption that T^* is one-to-one. The theorem follows easily.

We now introduce some important subsets of the spectrum.

Definition 1.13. Let $T \in L(X)$. Define

$$\sigma_p(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not one-to-one}\};$$

$$\sigma_c(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is one-to-one}, \; \overline{(\lambda I - T)X} = X \text{ but } (\lambda I - T)X \neq X\};$$

$$\sigma_r(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is one-to-one but } \overline{(\lambda I - T)X} \neq X\}.$$

$\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are called respectively the *point spectrum*, the *continuous spectrum* and the *residual spectrum* of T. Clearly $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint and

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$