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ABSTRACT

P-adic logarithms are used to translate localization sequences, involving multiplicative groups of units, to simpler additive descriptions of $D(\mathbb{Z}G)$ and $Wh'(G)$ for a p-group G. When G is a 2-group, applications include explicit computations of $D(\mathbb{Z}G)$ in many cases, a general formula for $|D(\mathbb{Z}G)|$, a description of the Tate cohomology of $Wh'(G)$ under the involution induced by an orientation $G \longrightarrow \{\pm 1\}$, and results on representing elements of $Wh'(G)$ by units.

Key words and phrases: Whitehead groups, projective class group, p-adic logarithm, p-group, localization sequences

INTRODUCTION

For a finite group G, various localization sequences have been used to describe the groups

$$D(\mathbb{Z}G) = \mathrm{Ker}[K_0(\mathbb{Z}G \longrightarrow K_0(\mathfrak{M})]$$

(where $\mathfrak{M} \supseteq \mathbb{Z}G$ is a maximal order in $\mathbb{Q}G$); and

$$Wh'(G) = Wh(G)/SK_1(\mathbb{Z}G) = K_1(\mathbb{Z}G)/\langle \pm g, SK(\mathbb{Z}G)\rangle = K_1(\mathbb{Z}G)/\mathrm{torsion}.$$

These are simplest when G is a p-group for some prime p, but even in that case one must work with kernels and cokernels of maps between multiplicative groups; and calculations can be quite complicated even when generators and relations are known for all groups involved. The goal of this paper is to work out a procedure for using logarithms to translate the localization sequences into sequences involving additive groups. This procedure is sketched in Section 0.

The most interesting consequences dealt with here all involve 2-groups. For a 2-group G, the main results include:

(1) A formula for $|D(\mathbb{Z}G)|$ (Theorem 5.5). For example (Theorem 5.7), if G is abelian, $|G| = 2^N$, and r_k denotes the number of cyclic subgroups of order 2^k, then $|D(\mathbb{Z}G)| = 2^M$ where

$$M = \tfrac{1}{2}[3N + r_1(N-4) + r_2(2N-8) + r_3(3N-15) + r_4(5N-27) + \dots].$$

(2) If $\mathbb{R}G$ is isomorphic to a product of matrix rings over $\mathbb{R}$, then $D(\mathbb{Z}G) \cong 2A_{\mathbb{Q}}(G)$ (Proposition 5.2). Here, $A_{\mathbb{Q}}(G)$ denotes the Artin cokernel (i.e., the rational representation ring of G modulo the subgroup generated by cyclic induction), and $2A_{\mathbb{Q}}(G)$ means multiples of 2.

(3) Formulas are derived which relate $D(\mathbb{Z}[G\times C_2^k])$ (any $k > 0$) to $D(\mathbb{Z}G)$ and related functors of G (Theorem 6.2). In particular, descriptions are given of $D(\mathbb{Z}[G \times C_2^k])$ when G is a dihedral 2-group.

(4) For any $x\in Wh'(G)$, x^2 is a product of elements induced up from cyclic subgroups of G, and x is such a product if G is abelian (Theorem 4.3).

(5) For any $x\in Wh'(G)$, $x^2 = [u]$ for some unit $u\in(\mathbb{Z}G)^*$ (Proposition 4.6). An example is also constructed of a 2-group G such that not every element of $Wh'(G)$ is represented by a unit (Theorem 4.7).

(6) Formulas for $\hat{H}^i(\mathbb{Z}/2;Wh'(G))$ $(i=0,1)$ are given, when $\mathbb{Q}G$ is a product of matrix rings over fields, and the $\mathbb{Z}/2$-action is induced by the antiinvolution $g\longmapsto w(g)\cdot g^{-1}$ on $\mathbb{Z}G$, for any given $w\colon G\longrightarrow\{\pm1\}$ (Theorem 4.8).

The general algebraic machinery used to study $D(\mathbb{Z}G)$ and $Wh'(G)$ for p-groups G are developed in Sections 1 to 3 -- and summarized in Section 0. Sections 4 and 5 concentrate on $Wh'(G)$ and $D(\mathbb{Z}G)$, respectively, in the 2-group case. The case of p-groups for odd regular p is studied briefly in Section 7; many of the results on 2-groups listed above are already known (or not applicable) in the odd p-group case.

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CHAPTER 0. SUMMARY OF METHODS

For the sake of easy reference, we first collect the main algorithms for describing $\hat{\mathbb{Z}}_p \otimes Wh'(G)$ and $D(\mathbb{Z}G)$ for a p-group G, as well as the necessary definitions. Throughout this section, p denotes a fixed prime.

<u>Definition 0.1</u>. For any p-group G, set:

(a) $Wh'(\hat{\mathbb{Z}}_pG) = K_1(\hat{\mathbb{Z}}_pG)/(SK_1(\hat{\mathbb{Z}}_pG) \times \langle \lambda g: \lambda \in tors(\hat{\mathbb{Z}}_p)^*, g \in G \rangle)$

(b) $W_p(\mathbb{Q}G) = K_1(\hat{\mathfrak{M}}_p)_{(p)}/K_1(\mathfrak{M})\hat{}$

where $\mathfrak{M} \subseteq \mathbb{Q}G$ is any maximal order containing $\mathbb{Z}G$, and $K_1(\mathfrak{M})\hat{}$ is the p-adic closure of the image of $K_1(\mathfrak{M})$ in $K_1(\hat{\mathfrak{M}}_p)_{(p)}$.

(c) $\hat{R}_{\mathbb{Q}}(G) = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(G)$, $\hat{R}_{\mathbb{C}/\mathbb{R}}(G) = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (R_{\mathbb{C}}(G)/R_{\mathbb{R}}(G))$,

where $R_K(G)$ denotes the K-representation ring of G.

(d) $H_0(G;\hat{\mathbb{Z}}_pG)$ denotes the free $\hat{\mathbb{Z}}_p$-module with basis the set of conjugacy classes in G: i.e, the 0-th homology group when G acts on $\hat{\mathbb{Z}}_pG$ via conjugation.

(e) $tors_p K_1(\mathbb{Q}G)$ denotes the subgroup of elements of p-power order.

With this notation, there is for any G an exact localization sequence

$$0 \longrightarrow \hat{\mathbb{Z}}_p \otimes Wh'(G) \longrightarrow Wh'(\hat{\mathbb{Z}}_pG) \xrightarrow{\varphi} W_p(\mathbb{Q}G) \xrightarrow{\partial} D(\mathbb{Z}G) \longrightarrow 0 ,$$

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where the first two maps are induced by the inclusions $\mathbb{Z}G \subseteq \hat{\mathbb{Z}}_pG \subseteq \hat{\mathfrak{A}}_p$.

The naturality of the above exact sequence, with respect to group homomorphism, tranfer maps, and the standard involution $g \to g^{-1}$, is dicussed in Section 1. Note in particular that the boundary map to $D(\mathbb{Z}G)$ possibly commutes with the involution only up to sign (depending on which convention is chosen for the involution on $D(\mathbb{Z}G)$).

The idea now is to approximate $Wh'(\hat{\mathbb{Z}}_p(G)$ by $H_0(G;\hat{\mathbb{Z}}_pG)$, and $W_p(\mathbb{Q}G)$ by $\hat{R}_{\mathbb{Q}}(G) \times \hat{R}_{\mathbb{C}/\mathbb{R}}(G)$, so that φ can be approximated by an additive homomorphism whose kernel and cokernel are much easier to describe. More specifically, we show:

<u>Theorem</u> <u>0.2</u>. (Theorems 3.7 and 3.8) Set $\epsilon = (-1)^{p-1}$, and assume $p = 2$ or p is an odd regular prime. Then for any p-group G, the following diagram is commutative with exact rows and columns:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{\mathbb{Z}}_p\otimes Wh'(G) & \longrightarrow & Wh'(\hat{\mathbb{Z}}_pG) & \xrightarrow{\varphi} & W_p(\mathbb{Q}G) & \xrightarrow{\partial} & D(\mathbb{Z}G) & \longrightarrow & 0 \\
 & & \cong\downarrow \mathrm{Id} & & \downarrow\Gamma & & \downarrow\eta & & \downarrow\bar{\eta} & & \\
0 & \longrightarrow & \hat{\mathbb{Z}}_p\otimes Wh'(G) & \xrightarrow{\hat{\Gamma}} & H_0(G;\hat{\mathbb{Z}}_pG) & \xrightarrow{\theta} & \hat{R}_{\mathbb{Q}}(G)\times\hat{R}_{\mathbb{C}/\mathbb{R}}(G) & \longrightarrow & \mathrm{Coker}(\theta) & \longrightarrow & 0 \\
 & & & & \downarrow\omega & (2) & \downarrow\nu & & \downarrow\bar{\nu} & & \\
(1) & & 0 & \longrightarrow & \langle\epsilon\rangle \times G^{ab} & \xrightarrow{\iota} & \mathrm{tors}_pK_1(\mathbb{Q}G) & \longrightarrow & \mathrm{tors}_pK_1(\mathbb{Q}G)/\{\pm g\} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

The homomorphisms in (1) are defined as follows. Here, for any $r \geq 0$, $\xi_r = \exp(2\pi i/p^r)$ and $K_r = \mathbb{Q}(\xi_r)$.

(a) $\Gamma(u) = \log(u) - \frac{1}{p}\Phi(\log(u))$ for any $u \in 1+J(\hat{\mathbb{Z}}_pG)$ ($J(\hat{\mathbb{Z}}_pG)$ the Jacobson radical); where $\Phi(\sum a_ig_i) = \sum a_ig_i^p$ for any $a_i \in \hat{\mathbb{Z}}_p$, $g_i \in G$.

(b) $\theta(g) = \left(\mathrm{Ind}_{\langle g\rangle}^{G}([\mathbb{Q}]),\ \mathrm{Ind}_{\langle g\rangle}^{G}(\sum_{r=0}^{k}[\mathbb{C}(g \to \xi_r)])\right)$ for any $g\in G$ of order p^k.

(c) $\nu([W],0) = [\epsilon,W] \in K_1(\mathbb{Q}G)$ for any $\mathbb{Q}G$-module W.

$\nu(0,[\mathbb{C}\otimes_{K_r} V]) = [\epsilon\xi_r, V|_{\mathbb{Q}G}] \in K_1(\mathbb{Q}G)$ for any $r \geq 1$ and any $K_r[G]$-module V.

(d) $\omega(\sum a_i g_i) = \Pi(\epsilon g_i)^{a_i}$ for any $a_i\in\hat{\mathbb{Z}}_p$, $g_i\in G$.

(e) ι is induced by the inclusion $\langle\epsilon\rangle \times G \subseteq (\mathbb{Q}G)^*$.

(f) η is the unique natural homomorphism such that $\eta\circ\varphi = \theta\circ\Gamma$.

For simplicity, this result has been stated here only in the case where p is regular. The situation for irregular p is slightly more complicated, and is described (in part) in Theorems 3.7 and 3.8 below. We also remark that $J(\hat{\mathbb{Z}}_pG)$ is the kernel of the mod-p augmentation $\hat{\mathbb{Z}}_pG \to \mathbb{Z}/p\mathbb{Z}$.

The diagram in Theorem 0.2 is, of course, natural with respect to group homomorphisms: this is immediate from the definitions of the maps involved. But it is also natural with respect to transfer (restriction) maps for injections $i : H \to G$, and with respect to the involutions which arise in surgery theory. The appropriate formulas for restriction or involution on $H_0(G;\hat{\mathbb{Z}}_pG)$ are in particular worth noting.

(1) A restriction map

$$\mathrm{Res}_H^G : H_0(G;\hat{\mathbb{Z}}_pG) \to H_0(H;\hat{\mathbb{Z}}_pH)$$

is defined for any pair $H \subseteq G$ of finite p-groups (see Theorem 1.4); when $[G:H] = p$, this takes the form

$$\mathrm{Res}_H^G(g) = \begin{cases} \sum_{i=0}^{p-1} x^i g x^{-i} & \text{if } g\in H,\ x\in G\setminus H \\ g^p & \text{if } g\in G\setminus H. \end{cases}$$

(2) If G is a 2-group, if $W: G \to \{\pm 1\}$ is any homomorphism (i. e., any "orientation" of G), and if $f: g \to \omega(g)g^{-1}$ is the induced antiinvolution, then f_* is defined on $H_0(G;\hat{\mathbb{Z}}_pG)$ by setting

$$f_*(g) = \begin{cases} g^{-1} & \text{if } \omega(g) = 1 \\ g^{-2} - g^{-1} & \text{if } \omega(g) = -1. \end{cases}$$

The chief problem in the above construction lies in defining

$$\eta_G : W_p(\mathbb{Q}G) \longrightarrow \hat{R}_{\mathbb{Q}}(G) \times \hat{R}_{\mathbb{C}/\mathbb{R}}(G) .$$

When p is odd, or when $p = 2$ and $\mathbb{Q}G$ is a product of matrix algebras over fields, then $\hat{R}_{\mathbb{C}/\mathbb{R}}(G)$ is torsion free, and η_G can be defined directly using logarithms. In other cases, however, there are elements in $K_1(\hat{\mathfrak{M}}_p)$ of finite order which must be sent non-trivally into $\hat{R}_{\mathbb{C}/\mathbb{R}}(G)$, and we have been unable to discover a direct formula for doing this.

The appearance of $\mathrm{tors}_p(K_1(\mathbb{Q}G))$ to describe $\mathrm{Coker}(\eta)$ seems almost coincidental, and our only real explanation is that is just happens to work. It is, however, intriguing to note that η and υ are induced by the exact sequence

$$K_1(\hat{\mathfrak{M}}_p)_{(p)} \xrightarrow{\tilde{\eta}} \hat{R}_{\mathbb{Q}}(G) \times \hat{R}_{\mathbb{C}/\mathbb{Q}}(G) \xrightarrow{\tilde{\upsilon}} \mathrm{tors}_p K_1(\mathbb{Q}G) \longrightarrow 0$$

($\mathfrak{M} \supseteq \mathbb{Z}G$ is maximal, see Theorem 3.6); and that

$$\mathrm{Ker}(\tilde{\eta}) \cong \mathrm{tors}_p K_1(\mathfrak{M}) \cong \mathrm{tors}_p K_1(\mathbb{Q}G) \cong \mathrm{Coker}(\hat{\eta}).$$

This is similar to the situation for Γ: if Γ is regarded as being defined on $K_1'(\hat{\mathbb{Z}}_pG)_{(p)}$, then

$$\mathrm{Ker}(\Gamma) \cong \mathrm{tors}_p K_1'(\mathbb{Z}G) \cong \langle \epsilon \rangle \times G^{ab} \cong \mathrm{Coker}(\Gamma).$$

With the above diagram, $Wh'(G)$ and $D(\mathbb{Z}G)$ can now be studied via the isomorphism

$$\hat{\Gamma}_G:\ \hat{\mathbb{Z}}_p\otimes Wh'(G) \longrightarrow \mathrm{Ker}[\theta_G : H_0(G;\hat{\mathbb{Z}}_pG) \longrightarrow \hat{R}_{\mathbb{Q}}(G)\times\hat{R}_{\mathbb{C}/\mathbb{R}}(G)]$$

and the short exact sequence

$$0 \longrightarrow D(\mathbb{Z}G) \xrightarrow{\bar{\eta}} \mathrm{Coker}(\theta_G) \xrightarrow{\bar{\nu}} \mathrm{tors}_p K_1(\mathbb{Q}G/\{\pm G\}) \longrightarrow 0.$$

These descriptions depend on a good understanding of the square

$$\begin{array}{ccc} H_0(G;\hat{\mathbb{Z}}_pG) & \xrightarrow{\theta_G} & \hat{R}_{\mathbb{Q}}(G)\times\hat{R}_{\mathbb{C}/\mathbb{R}}(G) \\ \Big\downarrow{\scriptstyle\omega_G} & & \Big\downarrow{\scriptstyle\nu_G} \\ \langle\epsilon\rangle\times G^{ab} & \xrightarrow{\iota_G} & \mathrm{tors}_p K_1(\mathbb{Q}G)\ ; \end{array} \qquad (0.3)$$

in particular, of θ_G and ν_G (as well as $\mathrm{tors}_p K_1(\mathbb{Q}G)$). Many of the results in Sections 4, 5, and 6 are proven by using the above definitions directly. But for results involving specific groups, more efficient methods and formulas are useful.

If $\mathbb{Q}G = \prod_{i=1}^k A_i$, where the A_i are simple, then we can split $\theta_G = \prod\theta_{A_i}$, $\nu_G = \prod\nu_{A_i}$, and $\iota_G = \prod\iota_{A_i}$. For any simple $A \subseteq \mathbb{Q}G$, θ_A, ν_A, and ι_A sit in a commutative square

$$\begin{array}{ccc} H_0(G;\hat{\mathbb{Z}}_pG) & \xrightarrow{\theta_A} & \hat{R}_{\mathbb{Q}}(A)\times\hat{R}_{\mathbb{C}/\mathbb{R}}(A) = \hat{\mathbb{Z}}_p\otimes_{\mathbb{Z}}\left[K_0(A)\times\dfrac{K_0(\mathbb{C}\otimes A)}{K_0(\mathbb{R}\otimes A)}\right] \\ \Big\downarrow{\scriptstyle\Gamma_G} & & \Big\downarrow{\scriptstyle\nu_A} \\ \langle\epsilon\rangle\times G^{ab} & \xrightarrow{\iota_A} & \mathrm{tors}_p K_1(A)\ . \end{array} \qquad (0.4)$$

Also, $\mathrm{tors}_p K_1(A)$ is described explicitly by:

Theorem 0.5. For any simple $\mathbb{Q}$-algebra A with center K, the reduced norm homomorphism defines an isomorphism

$$\mathrm{nr}:\ K_1(A) \xrightarrow{\cong} K_+^* = \{u \in K^*:\ u>0 \text{ for any } K \subseteq \mathbb{R} \text{ such that } \mathbb{R}\otimes_K A \text{ is a matrix algebra over } \mathbb{H}\}\ .$$

<u>Proof</u>. See Weil [32], Chapter XI, §3, Proposition 3. □

One way to explicitly describe the maps θ_A and υ_A in 0.4 is by comparison with the so-called "basic" p-groups. When p is odd, these include only the cyclic groups

$$C_{p^n} = \langle a \mid a^{p^n} = e \rangle$$

for $n \geq 0$. If $p = 2$, in addition to the cyclic groups, we have

Dihedral: $D(2^n) = \langle a,b \mid a^{2^{n-1}} = e = b^2;\ bab^{-1} = a^{-1}\rangle$
$(n \geq 4)$

Quaternion: $Q(2^n) = \langle a,b \mid a^{2^{n-1}} = e;\ a^{2^{n-2}} = b^2;\ bab^{-1} = a^{-1}\rangle$
$(n \geq 3)$

Semi-dihedral: $SD(2^n) = \langle a,b \mid a^{2^{n-1}} = e = b^2;\ bab^{-1} = a^{2^{n-2}-1}\rangle$
$(n \geq 4)$.

Note that $D(8)$, the dihedral group of order 8, is not basic.

Recall the relation between the irreducible rational representations of a finite group G and the Wedderburn decomposition of $\mathbb{Q}G$. Write $\mathbb{Q}G = \Pi_{i=1}^k A_i$, where the A_i are simple, and let V_i be the irreducible A_i-module. Then $V_1,\ldots,V_k$ are the irreducible $\mathbb{Q}G$-representations. To each V_i, we can associate a division algebra $D_i = \mathrm{End}_{\mathbb{Q}G}(V_i)$ $(=\mathrm{End}_{A_i}(V_i))$. Then $A_i \cong \mathrm{End}_{D_i}(V_i)$, a matrix algebra over D_i.

For each basic p-group G, there is a unique simple summand A_G of $\mathbb{Q}G$ upon which G acts effectively. The significance of these groups is as follows. If G is any p-group, and $\mathbb{Q}G = \Pi_{i=1}^k A_i$, where each A_i is simple with irreducible representation V_i, then there exist for each i subgroups

$H_i \triangleleft K_i \subseteq G$ with the property that A_i and A_{K_i/H_i} have the same associated division algebra (and $V_i = \mathrm{Ind}_{K_i}^G(W_i)$ if W_i is the irreducible A_{K_i/H_i}-module). Furthermore, for any functor X on $\mathbb{Q}$-algebras satisying certain conditions (including the cases $X = \hat{R}_{\mathbb{Q}} \times \hat{R}_{\mathbb{C}/\mathbb{R}}$ and $X = W_p$) , there is a commutative diagram

$$\begin{array}{ccccc} X(\mathbb{Q}G) & \xrightarrow{\sum_i \mathrm{Trf}_{K_i}^G} & \bigoplus_i X(\mathbb{Q}K_i) & \xrightarrow{\oplus\ \mathrm{proj}} & \bigoplus_i X(\mathbb{Q}[K_i/H_i]) \\ \downarrow \cong & & & & \downarrow \\ \bigoplus_i X(A_i) & & \xrightarrow[\cong]{\oplus(\text{isomorphism})} & & \bigoplus_i X(A_{K_i/H_i}). \end{array}$$

A more precise statement of this is given in Proposition 2.5.

In the theorem and table to follow we describe square (0.4) for G a basic p-group and $A = A_G$.

<u>Theorem</u> <u>0.6</u>. Fix a basic p-group $G \neq 1$ and let $A = A_G$ be the effective simple component of $\mathbb{Q}G$. Recall $G = \langle a\rangle$ or $G = \langle a,b\rangle$, where a generates a cyclic group of order $|G|$ or $\frac{1}{2}|G|$. Define

(a) $p^n = |a|$ and $z = a^{p^{n-1}}$. Note z generates the unique central subgroup of order p in G.

(b) W is the simple A-module, and $w = [W]$ is the corresponding generator of $\hat{R}_{\mathbb{Q}}(A)$ $(\cong \mathbb{Z}_p)$.

(c) $v_i = \mathrm{Ind}_{\langle a\rangle}^G([\mathbb{C}(a \to \xi_n^{1/i})]) \in \hat{R}_{\mathbb{C}/\mathbb{R}}(G)$ for any $i \in \mathbb{Z}$ with $(i,p) = 1$ $(\xi_n = \exp(2\pi i/p^n))$.

(d) $X_n = \Big[\bigoplus_{(i,p)=1} \hat{\mathbb{Z}}_p(v_i)\Big] / \langle v_i - v_{i+p^n},\ v_i + v_{-i} : (i,p) = 1 \rangle$. Note that $X_n \cong (\hat{\mathbb{Z}}_p)^{(1/2)\varphi(p^n)}$.

(e) $A \cong M_r(D)$ for some division algebra D. Here, $r = 1$ if G is cyclic or quaternionic; $r = 2$ if G is dihedral or semi-dihedral.

(f) $H_0(G;\hat{\mathbb{Z}}_pG)$ is the free $\hat{\mathbb{Z}}_p$-module on the conjugacy classes of elements of G. The conjugacy classes consist of the following sets:

$\{a^m\}$ (G cyclic);

$\{a^m, a^{-m}\}$, $\{a^{2i}b\}$, $\{a^{2i+1}b\}$ (G dihedral or quaternionic);

$\{a^{2m+1}, za^{-(2m+1)}\}$, $\{a^{2m}, a^{-2m}\}$, $\{a^{2i}b\}$, $\{a^{2i+1}b\}$ (G semi-dihedral).

Here, in each case, $m \in \mathbb{Z}$ is fixed and $i \in \mathbb{Z}$ varies. We abuse notation and write θ of an element rather than a conjugacy class. We describe θ on at least one representative for each conjugacy class.

(g) $\theta_A(a^i)$ is listed in the table below only when $p \nmid i$. To compute $\theta(a^i)$ if $p \mid i$, apply inductively the formula

$$\theta(a^{pi}) = \sum_{j=0}^{p-1} \theta(a^i z^j)$$

Otherwise, $\hat{R}_{\mathbb{C}/A}(A)$, $\text{tors}_p K_1(A)$, θ_A, υ_A, and ι_A are described by the following:

Table 0.7

G	1	C_2	C_{p^n} $(p^n>2)$	$D(2^{n+1})$	$Q(2^{n+1})$	$SD(2^{n+1})$
$D\cong$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}(\xi_n)$	$\mathbb{Q}(\xi_n+\xi_n^{-1})$	$\mathbb{Q}(\xi_n,j)$	$\mathbb{Q}(\xi_n-\xi_n^{-1})$
$\hat{R}_{\mathbb{C}/\mathbb{R}}(A)$	0	0	X_n	0	$X_n/2X_n$ $\cong(\mathbb{Z}/2)^{2^{n-2}}$	$X_n/\langle v_i-v_{2^{n-1}-i}\rangle$ $\cong(\hat{\mathbb{Z}}_2)^{2^{n-3}}$
$\theta(e)$	w	w	w	$2w$	w	$2w$
$\theta(a^i)$ $(i,p)=1$	$-$	0	v_i	0	v_i	v_i
$\theta(b)$	$-$	$-$	$-$	w	$\sum v_i$ (i odd)	w
$\theta(ab)$	$-$	$-$	$-$	w	$\sum v_i$ $(0<i<2^{n-1})$	0
$\mathrm{tors}_p K_1(A)$	$\langle\epsilon\rangle$	$\{\pm1\}$	$\langle\xi_n\rangle$	$\{\pm1\}$	$\{1\}$	$\{\pm1\}$
$\nu(w)$	ϵ	-1	ϵ	$-$	1	-1
$\nu(v_i)$	$-$	$-$	$\epsilon\xi_n^i$	$-$	1	-1
$\iota(\epsilon)$	ϵ	-1	ϵ	1	1	1
$\iota(a)$	$-$	-1	ξ_n	1	1	-1
$\iota(b)$	$-$	$-$	$-$	-1	1	-1

CHAPTER 1. LOCALIZATION SEQUENCES AND LOGARITHMS

There are by now various localization sequences in use for describing $D(\mathbb{Z}G)$ (see, e.g [4] or [29]). We start by presenting the one to be used here.

Throughout the paper, p always denotes a fixed prime. If $R = \mathbb{Z}$ or $\hat{\mathbb{Z}}_p$, and G is any finite p-group, we define

$$Wh(RG) = K_1(RG)/\langle \pm g \rangle;$$

$$Wh'(RG) = K_1(RG)/(SK_1(RG) \times \{\lambda g : \lambda \in \mathrm{tors}\, R^*,\ g \in G\}).$$

As usual, we write $Wh(G) = Wh(\mathbb{Z}G)$ and $Wh'(G) = Wh'(\mathbb{Z}G)$. Note that when $R = \hat{\mathbb{Z}}_p$, we do not divide out by all units in R (in contrast to, e.g., [30] and [14]). By [30], $Wh'(RG)$ is a free R-module of known (finite) rank.

If A is any finite dimensional semisimple $\mathbb{Q}$-algebra, and if $\mathfrak{M} \subseteq A$ is a maximal $\mathbb{Z}$-order, then $K_1(\hat{\mathfrak{M}}_p)$ $(\hat{\mathfrak{M}}_p = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathfrak{M})$ is the product of a pro-p-group and a finite group (see [28]). In particular, $K_1(\hat{\mathfrak{M}}_p)_{(p)}$ can be regarded as a $\hat{\mathbb{Z}}_p$-module. Using this structure, define

$$W_p(A) = \mathrm{Coker}\left[\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} K_1(\mathfrak{M}) \longrightarrow K_1(\hat{\mathfrak{M}}_p)_{(p)}\right].$$

That this is independent of $\mathfrak{M}$ follows from:

Proposition 1.1 Let $\mathfrak{M}$ be a maximal order in a finite dimensional simple $\mathbb{Q}$-algebra A. Let $K \subseteq A$ be the center, and let $R \subseteq K$ be the ring of integers. Let

$$nr: K_1(\mathfrak{M}) \longrightarrow R^* \quad \text{and} \quad (nr)_p: K_1(\hat{\mathfrak{M}}_p) \longrightarrow (\hat{R}_p)^*$$

denote the reduced norm maps. Then $(nr)_p$ is onto, $\mathrm{Ker}((nr)_p) = SK_1(\hat{\mathfrak{M}}_p)$ is finite of order prime to p, and