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ALGEBRAIC IDEAS IN ERGODIC THEORY

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Introduction

These notes were the basis of a series of ten CBMS lectures at the University of Washington, Seattle, in July 1989, whose theme was the influence of algebraic ideas on the development of ergodic theory. However, as anybody familiar with the subject will realize, any comprehensive exploration of this theme would fill a substantial book and several lecture courses, even if no proofs are included. In view of this I had to restrict myself to two specific topics, and even within these topics the shortage of space and time imposed severe restrictions on the material I could hope to cover.

The first of these topics is the influence of operator algebras on dynamics. The construction of factors from group actions on measure spaces introduced by F. J. Murray and J. von Neumann in the 1930s has, in turn, influenced ergodic theory by leading to H. A. Dye's notion of orbit equivalence, G. W. Mackey's study of virtual groups, and the investigation of ergodic and topological equivalence relations by W. Krieger, J. Feldman and C. C. Moore, A. Connes, and many others. The theory of operator algebras not only motivated the study of equivalence relations (or orbit structures), but it also provided some of the key ideas for the development of this particular branch of ergodic theory. The first four sections of these notes are devoted to ergodic equivalence relations, their properties, and their classification, and present occasional glimpses of the operator-algebraic context from which many of the ideas and techniques arose. Ergodic theorists tend to regard ergodic equivalence relations as a subject set apart from the main body of their field; for this reason I have included a large number of examples which (I hope) show that equivalence relations provide a very natural setting for many classical constructions and classification problems. Many of these examples are drawn from the context of Markov shifts; this was partly motivated by the fact that the CBMS meeting followed on from a workshop on dynamics with significant emphasis on coding theory, and partly by the ease and naturalness with which some of the most useful invariants in coding theory can be derived and interpreted from the point of view of equivalence relations.

The last three sections of these notes are dedicated to higher dimensional

Markov shifts, a difficult field of research with no indication yet of a satisfactory general theory. This lack of progress is all the more remarkable when compared with the richness of the theory in one dimension; it is due to a variety of reasons, the most famous of which is that any reasonably general definition of higher dimensional Markov shifts immediately leads to the problem that it may be undecidable whether the shift space is nonempty. A second reason is that none of the techniques which have been so successful for one dimensional Markov shifts, and some of which were described in the preceding sections, appear to be applicable here. Section 5 is devoted to elementary examples of such shifts and to the surprising difficulties these examples present. However, if one makes the (very restrictive) assumption that the Markov shift carries a group structure, then many of these difficulties can be resolved, and one has the beginnings of a successful analysis which turns out to encompass the theory of expansive $\mathbb{Z}^d$-actions by automorphisms of compact groups, and which exhibits an intriguing interplay between commutative algebra and dynamics (Sections 6–7).

The attentive reader will have noticed that there were ten lectures, but that there are only seven sections in these notes. Since seven and ten have no common factor, I should explain how the lectures were organized: section one was covered in two lectures, section two in three lectures, sections three, four, and five in one lecture each, and the remaining two sections were covered in two lectures after some of the material from section seven had already been presented earlier by D. Lind in greater detail in a research seminar.

Copies of a preliminary version of these lecture notes had been distributed to all members of the audience before the beginning of the lecture series, and this enabled me to be more selective in the material I presented in the talks and to make occasional references to the notes for unexplained background or further details.

These notes have benefitted from many conversations with experts both before and during the conference. Special thanks are due to W. Parry and C. Sutherland for reading critically an earlier draft of these notes (to the latter also for a number of discussions on equivalence relations on odometers and Markov shifts), to D. Rudolph for explaining to me the tilings described in Section 5, to D. Berend for informing me about D. Masser's result [**Mas**], and to B. Kitchens for helping me to correct a mistake in Example 2.2(4), and to him and his friends for feeding me seafood in order to build up my strength for the talks. However, my sincerest thanks go to D. Lind and S. Tuncel, who organized both the workshop and the CBMS conference in an exemplary manner, and whose warm hospitality made my stay in Seattle greatly enjoyable.

1. Operator Algebras and Dynamical Systems

The connection between operator algebras and ergodic theory goes back almost to the origins of the two subjects. In 1936, F. J. Murray and J. von Neumann [**MurN1**] introduced the *group measure space construction* of von Neumann algebras from group actions on measure spaces and showed that these algebras are *factors*, i.e. have trivial centres, if and only if the group actions are ergodic. This construction (with later refinements) remains one of the most important sources of examples of operator algebras. A striking early result in von Neumann algebras was the proof of the uniqueness of the hyperfinite II_1-factor in 1943 [**MurN2**], which implied in particular that the factors constructed from any two measure-preserving, ergodic automorphisms on probability spaces[1] are isomorphic. Building on earlier work by E. Hopf [**Hop**], H. A. Dye investigated the reason for the existence of such an isomorphism and found that the von Neumann algebra arising from the group measure space construction only depends on the *orbits* of the group (or the single transformation) acting on the measure space, and that any two measure-preserving, ergodic automorphisms of probability spaces have isomorphic orbits [**Dye1**]. In [**Dye2**] he showed that every measure-preserving action of a countable Abelian group (cf. [**Abel**]) on a probability space has the same orbits as a suitably chosen *single* automorphism of the measure space, thereby providing a proof for an announcement in [**MurN2**] that every measure-preserving, ergodic action of an infinite Abelian group on a probability space again leads to the hyperfinite II_1-factor. This emphasis on *orbits* rather than *group actions* was brought to its logical conclusion in [**FelM1**] and [**FelM2**], where the acting group is suppressed completely and the orbit space is replaced by an equivalence relation.[2] Hand in hand with this development

[1] A measure space $(X, \mathscr{S}, \mu)$ is *standard* if $(X, \mathscr{S})$ is a standard Borel space (i.e. Borel isomorphic to a Borel subset of $\mathbb{R}$ with the induced Borel structure) and μ is a σ-finite measure on $\mathscr{S}$. The elements of $\mathscr{S}$ are the *Borel sets*. If there is no danger of confusion we sometimes write (X, μ) instead of $(X, \mathscr{S}, \mu)$. All measure spaces are assumed to be standard and all measures σ-finite.

[2] The study of equivalence relations is a special case of G. W. Mackey's investigation of virtual groups [**Mac1**], [**Mac2**], [**Ram1**]. For a discussion of equivalence relations from a geometric point of view we refer to [**Con5**], and topological equivalence relations are discussed in [**Ren**].

came the investigation of those properties of a nonsingular action $g \to T_g$ of a countable[3] group G on a measure space $(X, \mathcal{S}, \mu)$ which are carried by the orbit structure, independently of the way in which these orbits are generated.

The theory of operator algebras not only led to the study and partial classification of orbit spaces (or countable equivalence relations), but also provided many of the ideas central to the investigation of the properties carried by the orbit structure. These ideas have since been stripped of their origins and can be discussed and used without any reference to (or knowledge of) operator algebras, but I shall try to offer a glimpse of the context they originated from in order to provide motivation, and to point the way to an area where there are likely to be many more hidden treasures.

1.1. DEFINITION **[FelM1]**. Let $(X, \mathcal{S})$ be a standard Borel space. A Borel set $\mathbf{R} \subset X \times X$ is a (*countable* or *discrete*) *Borel equivalence relation* on X if $\mathbf{R}$ is an equivalence relation, and if, for every $x \in X$, the *equivalence class* $\mathbf{R}(x) = \{x' \in X : (x, x') \in \mathbf{R}\}$ of x is countable. A Borel equivalence relation $\mathbf{R}$ on X is *finite* if $\mathbf{R}(x)$ is finite for every $x \in X$. If $\mathbf{S}$ is a second Borel equivalence relation on X then $\mathbf{S}$ is a *subrelation* of $\mathbf{R}$ if $\mathbf{S}(x) \subset \mathbf{R}(x)$ for every $x \in X$.

Let $\mathbf{R}$ be a Borel equivalence relation on X and let μ be a measure on $\mathcal{S}$. Then $\mathbf{R}$ is (μ-)*nonsingular* if $\mu(\mathbf{R}(A)) = 0$ for every $A \in \mathcal{S}$ with $\mu(A) = 0$, where $\mathbf{R}(A) = \bigcup_{x \in A} \mathbf{R}(x)$ denotes the *saturation*[4] of a set $A \subset X$. A nonsingular equivalence relation $\mathbf{R}$ is *transitive* if $\mu(X \setminus \mathbf{R}(x)) = 0$ for some $x \in X$ and *intransitive* otherwise. An intransitive equivalence relation $\mathbf{R}$ is (*properly*) *ergodic* if $\mu(\mathbf{R}(A)^c) = 0$ whenever $A \in \mathcal{S}$ and $\mu(A) > 0$. Two nonsingular equivalence relations $\mathbf{R}, \mathbf{S}$ on $(X, \mathcal{S}, \mu)$ are *equal* (mod μ) if there exists a μ-null set $N \in \mathcal{S}$ such that $\mathbf{R} \setminus (N \times N) = S \setminus (N \times N)$. Two nonsingular equivalence relations $\mathbf{R}$ and $\mathbf{S}$ on measure spaces $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are *isomorphic* if there exists a measure space isomorphism $\psi : (X, \mathcal{S}, \mu) \to (Y, \mathcal{T}, \nu)$ such that $(\psi \times \psi)(\mathbf{R}) = \mathbf{S}$ (mod ν), and the map ψ is an *isomorphism*[5] of $\mathbf{R}$ and $\mathbf{S}$.

1.2. EXAMPLES. (1) *Equivalence relations and group actions.* Let $T : g \to T_g$ be a nonsingular action[6] of a countable group G on $(X, \mathcal{S}, \mu)$. Then

[3]The restriction to actions of countable groups and to countable equivalence relations is inessential, but has the advantage of technical simplicity. Ingredients for extending the theory to actions of locally compact, second countable groups and the associated equivalence relations can be found in **[FelHM]** and **[FelR]**.

[4]Corollary 2 in **[Kur**, §39, VII] implies that $\mathbf{R}(A) \in \mathcal{S}$ for every $A \in \mathcal{S}$.

[5]If $\mathbf{R} = \mathbf{S}$ then ψ is an *automorphism* of $\mathbf{R}$, and ψ is an *inner automorphism* of $\mathbf{R}$ if $\psi \in [\mathbf{R}]$.

[6]If $(X, \mathcal{S}, \mu)$ is a measure space, a surjective Borel map $V : X \to X$ is a *nonsingular endomorphism* of X if, for every $B \in \mathcal{S}$, $\mu(B) = 0$ if and only if $\mu(V^{-1}B) = 0$ (in order for μV^{-1} to be σ-finite it may be necessary to replace μ by an equivalent probability measure).

$\mathbf{R}^T = \{(x, T_g x): x \in X, g \in G\}$ is a nonsingular equivalence relation, and $\mathbf{R}^T$ is transitive or ergodic if and only if T is transitive or ergodic. Every nonsingular equivalence relation $\mathbf{R}$ on $(X, \mathcal{S}, \mu)$ is of the form $\mathbf{R} = \mathbf{R}^T$ for some nonsingular action T of a countable group on $(X, \mathcal{S}, \mu)$ **[FelM1]**. We define the *full* group $[\mathbf{R}]$ of $\mathbf{R}$ as the group of all nonsingular automorphisms V of $(X, \mathcal{S}, \mu)$ with $(x, Vx) \in \mathbf{R}$ for μ-a.e. $x \in X$. Then there exists a Borel map $(x, x') \to \rho_{\mathbf{R},\mu}(x, x') = d\mu(x)/d\mu(x')$ from $\mathbf{R}$ to $\mathbb{R}^+$ such that, for every $V \in [\mathbf{R}]$, $(d\mu/d\mu V)(x) = \rho_{\mathbf{R},\mu}(x, Vx)$ for μ-a.e. $x \in X$. The map $\rho_{\mathbf{R},\mu}$ is the *Radon–Nikodym derivative* of the equivalence relation $\mathbf{R}$. If $\rho_{\mathbf{R},\mu} \equiv 1$ then $\mathbf{R}$ *preserves* μ (or μ is $\mathbf{R}$-*invariant*). Consider the σ-finite measures $\mu_{\mathbf{R}}^{(L)}$ and $\mu_{\mathbf{R}}^{(R)}$ on $\mathbf{R}$ defined by

$$\mu_{\mathbf{R}}^{(L)}(B) = \int |\{x \in X: (x, x') \in R \cap B\}| d\mu(x') \tag{1.1}$$

and

$$\mu_{\mathbf{R}}^{(R)}(B) = \int |\{x' \in X: (x, x') \in R \cap B\}| d\mu(x) \tag{1.2}$$

for every Borel set $B \subset \mathbf{R}$, where $|S|$ denotes the cardinality of a set S. These measures are equivalent (i.e. have the same null sets), and

$$d\mu_{\mathbf{R}}^{(R)}/d\mu_{\mathbf{R}}^{(L)} = \rho_{\mathbf{R},\mu}. \tag{1.3}$$

If we are only interested in the measure class of $\mu_{\mathbf{R}}^{(L)}$ we write $\mu_{\mathbf{R}}$ to denote either $\mu_{\mathbf{R}}^{(L)}$ or $\mu_{\mathbf{R}}^{(R)}$.

(2) *Induced equivalence relations.* Let $\mathbf{R}$ be a Borel equivalence relation on $(X, \mathcal{S})$, $B \in \mathcal{S}$, and let $\mathbf{R}_B = \mathbf{R} \cap (B \times B)$ be the equivalence relation *induced* by $\mathbf{R}$ on B. If μ is a measure on X and $\mu(B) > 0$ then $\mathbf{R}_B$ is obviously a nonsingular equivalence relation on $(B, \mathcal{S}_B, \mu_B)$, where $\mathcal{S}_B =$

If V is invertible (up to a null set) then V is a *nonsingular automorphism* of $(X, \mathcal{S}, \mu)$. In both cases the measure μ is said to be *quasi-invariant* under V, and μ is *invariant* if $\mu V^{-1} = \mu$. A *nonsingular action* T of a group G on $(X, \mathcal{S}, \mu)$ is a map $g \to T_g$ from G into the group $\operatorname{Aut}(X, \mathcal{S}, \mu)$ of nonsingular automorphisms of $(X, \mathcal{S}, \mu)$ such that $T_g T_{g'} = T_{gg'}$ μ-a.e., for all $g, g' \in G$. The action T is *measure preserving* if μ is invariant under every T_g, $g \in G$, and T is *ergodic* if every $B \in \mathcal{S}$ with $\mu(B \Delta T_g B) = 0$ for all $g \in G$ satisfies that either $\mu(B) = 0$ or $\mu(X \backslash B) = 0$. A second nonsingular action T' of G on a measure space $(X', \mathcal{S}', \mu')$ is *conjugate* to T if there exists a nonsingular isomorphism $\psi: (X, \mathcal{S}, \mu) \to (X', \mathcal{S}', \mu')$ such that, for every $g \in G$, $\psi T_g = T'_g \psi$ μ-a.e.

If the group G is locally compact and second countable then every nonsingular action T of G on $(X, \mathcal{S}, \mu)$ is conjugate to a nonsingular action T' of G on a measure space $(X', \mathcal{S}', \mu')$ with the property that the map $(g, x) \to T'_g x$ from $G \times X'$ to X' is Borel, and $T'_g T'_{g'} = T'_{gg'}$ for all $g, g' \in G$ (cf. **[Var]**). In this case the *orbit* $T_G x = \{T_g x: g \in G\}$ is a Borel set for every $x \in X'$, and T' (as well as T) is called *transitive* if there exists a point $x \in X'$ with $\mu(X \backslash T'_G x) = 0$. The actions T and T' are *properly ergodic* if they are ergodic and not transitive.

$\{A \cap B: A \in \mathcal{S}\}$ and μ_B is the restriction of μ to $\mathcal{S}_B$. If $\mathbf{R}$ is transitive or ergodic, then the same is true for $\mathbf{R}_B$.

Let V be a nonsingular automorphism of $(X, \mathcal{S}, \mu)$, and let $\mathbf{R}^V = \{(x, V^k x): k \in \mathbb{Z}\}$ be the equivalence relation induced by the $\mathbb{Z}$-action $n \to V^n$ on X. The automorphism V is *conservative* if $\mu(B) = 0$ for every $B \in \mathcal{S}$ with $\mu(B \cap V^k B) = 0$ for all $0 \neq k \in \mathbb{Z}$. If V is conservative then $(\mathbf{R}^V)_B = \mathbf{R}^{V_B}$ (mod μ) for every $B \in \mathcal{S}$ with $\mu(B) > 0$, where V_B is the automorphism induced by V on B:

$$V_B x = V^{m(x)} x \ \mu\text{-a.e.} \quad \text{with } m(x) = \begin{cases} 0 \text{ if } \{k > 0: T^k x \in B\} = \emptyset, \\ \min\{k > 0: T^k x \in B\} \text{ otherwise} \end{cases}.$$

(3) *Equivalence relations on Markov shifts.* Let $P = (P(i, j), 1 \leq i, j \leq k)$ be a nonnegative, irreducible matrix[7], and let $X_P = \{(x_n) \in \{1, \dots, k\}^{\mathbb{Z}}: P(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{Z}\}$ be the *Markov shift space* (or simply *Markov shift*) defined by P. Then X_P is a closed, shift invariant subset of $\{1, \dots, k\}^{\mathbb{Z}}$, and we write σ_P for the restriction of the shift $(\sigma x)_n = x_{n+1}$ on $\{1, \dots, k\}^{\mathbb{Z}}$ to X_P. We define a Borel equivalence relation $\mathbf{R}^P$ on X_P by setting $(x, x') \in \mathbf{R}^P$ if and only if there exist $m, m', n, n' \geq 0$ with

$$x_{-m-s} = x'_{-m'-s} \quad \text{and} \quad x_{n+s} = x'_{n'+s} \tag{1.4}$$

for every $s \geq 0$ and denote by $\mathbf{S}^P \subset \mathbf{R}^P$ the subrelation consisting of all pairs $(x, x') \in \mathbf{R}^P$ satisfying (1.4) with $m = m'$ and $n = n'$. The equivalence relation $\mathbf{S}^P$ was defined in **[Kri5]**, but was already implicit in **[Hed]**, and $\mathbf{R}^P$ is taken from **[Sch7]**. All these equivalence relations are shift invariant.

The matrix P can be converted into a stochastic[8] matrix as follows: let λ be the unique maximal eigenvalue of P, and let v be a nonzero right eigenvector for P with eigenvalue λ. Then the matrix $\mathbf{P}$ defined by $\mathbf{P}(i, j) = \lambda^{-1} \cdot P(i, j) \cdot v(j)/v(i)$ for $i, j = 1, \dots, k$, is stochastic, and we define the *Markov measure* μ_P on the Borel field $\mathcal{S}$ of X_P by setting

$$\mu_P(C) = \overline{p}(i_0)\mathbf{P}(i_0, i_1) \cdots \mathbf{P}(i_{n-1}, i_n) \tag{1.5}$$

for every cylinder set $C = [i_0, \dots, i_n]_r = \{x \in X_P: x_{r+k} = i_k \text{ for } k = 0, \dots, n\}$, where $\overline{p} = (p(1), \dots, p(k))$ is the unique vector satisfying $\overline{p}P = \overline{p}$ and $\sum_{i=1,\dots,k} \overline{p}(i) = 1$. Of particular interest is the measure of *maximal*

[7] A $k \times k$ matrix P is *nonnegative* if all its entries are nonnegative, and a nonnegative matrix P is *irreducible* if there exists, for every (i, j) with $1 \leq i, j \leq k$, an $n \geq 1$ with $P^n(i, j) > 0$, where P^n is the nth power of P under matrix multiplication. If there exists an $n \geq 1$ with $P^n(i, j) > 0$ for all i, j then P is *aperiodic*. Although we shall always assume aperiodicity, this assumption is inessential and can be removed at the expense of a slight technical complication.

[8] A nonnegative $k \times k$ matrix P is *stochastic* if its row sums are all equal to 1.

entropy $m_P = \mu_{P^0}$, where P^0 is the 0-1-matrix compatible[9] with P **[Par1]**. The equivalence relation $\mathbf{R}^P$ is nonsingular and ergodic with respect to μ_P, and

$$(1.6)\qquad \rho_{\mathbf{R}^P,\mu_P}(x, x') = \left(\prod_{-m\le i<n} P(x_i, x_{i+1})\right)\Big/\left(\prod_{-m'\le i<n'} P(x'_i, x'_{i+1})\right)$$

whenever $(x, x') \in \mathbf{R}^P$ satisfies (1.4). Proposition 44 in **[ParT1]** implies that μ_P is the measure of maximal entropy if and only if $\rho_{\mathbf{S}^P,\mu_P} \equiv 1$. We always assume that the space X_P and the equivalence relations $\mathbf{R}^P$, $\mathbf{S}^P$ are furnished with the measure μ_P.

The equivalence relations $\mathbf{R}^P$ and $\mathbf{S}^P$ are preserved under certain kinds of isomorphism of the shift spaces. If P and Q are stochastic matrices, the Markov shifts σ_P and σ_Q are *metrically conjugate* if there exists a measure-preserving isomorphism $\varphi: X_P \to X_Q$ such that $\varphi \cdot \sigma_P = \sigma_Q \cdot \varphi$ μ_P-a.e. The isomorphism φ is *finitary* if there exist null sets $N_P \subset X_P$, $N_Q \subset X_Q$, and Borel maps a_φ, $m_\varphi: X_P \to \mathbb{N}$, $a_{\varphi^{-1}}$, $m_{\varphi^{-1}}: X_Q \to \mathbb{N}$ such that

$$(\varphi(x))_0 = (\varphi(y))_0 \quad \text{for all } x, y \in X_P \setminus N_P \\ \text{with } x_n = y_n \text{ for } -m_\varphi(x) \le n \le a_\varphi(x)$$

and

$$(\varphi^{-1}(x))_0 = (\varphi^{-1}(y))_0 \quad \text{for all } x, y \in X_Q \setminus N_Q \\ \text{with } x_n = y_n \text{ for } -m_{\varphi^{-1}}(x) \le n \le a_{\varphi^{-1}}(x).$$

A finitary isomorphism $\varphi: X_P \to X_Q$ has *finite expected code lengths* if the functions a_φ, m_φ, $a_{\varphi^{-1}}$, $m_{\varphi^{-1}}$ can all be chosen to be integrable.

Under our assumptions, σ_P and σ_Q are metrically conjugate if and only if they have the same (metric) entropy $h_{\mu_P}(\sigma_P) = -\sum_{i,j} \overline{p}(i)P(i,j) \cdot \log(\overline{p}(i)P(i,j))) = h_{\mu_Q}(\sigma_Q)$ **[FriO]**. In **[KeaS]** M. Keane and M. Smorodinsky prove that, if σ_P and σ_Q are metrically conjugate, then they are also finitarily conjugate. However, W. Parry **[Par6]** and W. Krieger **[Kri5]** found obstructions to the existence of isomorphisms with finite expected code lengths, which turn out to be connected with the equivalence relations $\mathbf{R}^P$ and $\mathbf{S}^P$ defined above.

For every $x \in X_P$ we denote by $W^s(x) = \{y \in X_P: x_n = y_n$ for all sufficiently large $n \in \mathbb{N}\}$ and $W^u(x) = \{y \in X_P: x_{-n} = y_{-n}$ for all sufficiently large $n \in \mathbb{N}\}$ the *stable* and *unstable* sets of x. A measure-preserving isomorphism $\varphi: X_P \to X_Q$ is *hyperbolic* (more precisely: *preserves the hyperbolic structure*) if there exist null sets $N_P \subset X_P$ and $N_Q \subset X_Q$ such that

[9] Two nonnegative $k \times k$ matrices P, Q are *compatible* if, for all $1 \le i, j \le k$, $P(i,j) > 0$ if and only if $Q(i,j) > 0$. Any two compatible matrices P, Q satisfy that $X_P = X_Q$, $\mathbf{R}^P = \mathbf{R}^Q$, and $\mathbf{S}^P = \mathbf{S}^Q$, but the measures μ_P and μ_Q are in general (mutually) singular.

$\varphi(W^u(x)\backslash N_P) = W^u(\varphi(x))\backslash N_Q$ and $\varphi(W^s(x)\backslash N_P) = W^s(\varphi(x))\backslash N_Q$ for every $x \in X_P$. Since $\mathbf{S}^P(x) = W^u(x) \cap W^s(x)$ and $\mathbf{R}^P(x) = \bigcup_{m,n\in\mathbb{Z}} W^u(\sigma_{P^m}x) \cap W^s(\sigma_{P^n}x)$ for every $x \in X_P$, every hyperbolic isomorphism $\varphi: X_P \to X_Q$ is an isomorphism of the equivalence relations $\mathbf{S}^P \subset \mathbf{R}^P$ and $\mathbf{S}^Q \subset \mathbf{R}^Q$.

By **[Kri5]** every shift-commuting, finitary isomorphism $\varphi: X_P \to X_Q$ with finite-expected code lengths is hyperbolic. If φ is finitary, but only the functions m_φ and a_φ are integrable, then $(\varphi\times\varphi)(\mathbf{R}^P) \subset \mathbf{R}^Q$ and $(\varphi\times\varphi)(\mathbf{S}^P) \subset \mathbf{S}^Q$. According to **[Sch8]** a hyperbolic isomorphism $\varphi: X_P \to X_Q$ is finitary if and only if there exist null sets $N_P \subset X_P$, $N_Q \subset X_Q$, and Borel maps a^*_φ, $m^*_\varphi: X_P \to \mathbb{N}$, $a^*_{\varphi^{-1}}$, $m^*_{\varphi^{-1}}: X_Q \to \mathbb{N}$, such that

$$(1.7)\quad \begin{aligned} (\varphi(x))_n = (\varphi(x'))_n \quad &\text{for all } n \ge 0 \ (n \le 0) \text{ whenever } x, x' \in X_P\backslash N_P \\ &\text{and } x_m = x'_m \text{ for all } m \ge -m^*_\varphi(x) \quad (m \le a^*_\varphi(x)), \end{aligned}$$

and

(1.8)
$$\begin{aligned} (\varphi_{\varphi^{-1}}(x))_n = (\varphi^{-1}(x'))_n \quad &\text{for all } n \ge 0 \ (n \le 0) \text{ whenever } x, x' \in X_Q\backslash N_Q \\ &\text{and } x_m = x'_m \text{ for all } m \ge -m^*_{\varphi^{-1}}(x) \quad (m \le a^*_{\varphi^{-1}}(x)). \end{aligned}$$

(4) *Equivalence relations on one-sided Markov shifts.* In the notation of the example (3), let $Y_P = \{x = (x_n) \in \{1, \dots, k\}^{\mathbb{N}}: P(x_n, x_{n+1}) > 0$ for every $n \in \mathbb{N}\}$ be the *one-sided Markov shift space* (or *Markov shift*) defined by P, and denote the shift on Y_P again by σ_P. We set

$$\mathbf{R}'^P = \{(x, x') \in Y_P \times Y_P: \sigma_P^m(x) = \sigma_P^n(x') \quad \text{for some } m, n \ge 0\}$$

and

$$\mathbf{S}'^P = \{(x, x') \in Y_P \times Y_P: \sigma_P^n(x) = \sigma_P^n(x') \quad \text{for some } n \ge 0\},$$

and we furnish Y_P, R'^P, and $\mathbf{S}'^P$ with the probability measure $\nu_P = \mu_P \cdot \pi^{-1}$, where $\pi: X_P \to Y_P$ is the projection obtained by forgetting negative coordinates.[10] Then $\mathbf{S}'^P \subset \mathbf{R}'^P$, the equivalence relations $\mathbf{R}'^P$ and $\mathbf{S}'^P$ are both nonsingular and ergodic with respect to ν_P, and

(1.9)
$$\rho_{\mathbf{R}^P, \nu_P}(x, x') = \Big(\bar{p}(x_0)\cdot \prod_{0\le i<n} P(x_i, x_{i+1})\Big) \Big/ \Big(\bar{p}(x'_0)\cdot \prod_{0\le i<n'} P(x'_i, x'_{i+1})\Big)$$

whenever $\sigma_P^n(x) = \sigma_P^{n'}(x')$. Finally we note that $\pi(\mathbf{R}^P(x)) = \mathbf{R}'^P(\pi(x))$ and $\pi(\mathbf{S}^P(x)) = \mathbf{S}'^P(\pi(x))$ for every $x \in X_P$, and that $\mathbf{R}'^P$ and $\mathbf{S}'^P$ are invariant under the shift σ_P.

[10] If m_P is the measure of maximal entropy on X_P then we denote the measure $m_P \cdot \pi^{-1}$ on Y_P again by m_P.

(5) *Equivalence relations of an endomorphism.* Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $V: X \to X$ a nonsingular, surjective Borel map such that $V^{-1}(\{x\})$ is countable for every $x \in X$ (i.e. V is a nonsingular, *countable-to-one* endomorphism of $(X, \mathcal{S}, \mu)$). We define Borel equivalence relations

$$\mathbf{R}^V = \{(x, x') \in X \times X : V^m x = V^n x' \quad \text{for some } m, n \geq 0\}$$

and

$$\mathbf{S}^V = \{(x, x') \in X \times X : V^m x = V^m x' \quad \text{for some } m \geq 0\}$$

and assume that the equivalence relations $\mathbf{R}^V$ and $\mathbf{S}^V$ are nonsingular with respect to μ (this is not automatic!). Example (4) corresponds to the special case where $X = Y_P$ and $V = \sigma_P$. Another well-known example is obtained by setting $X = [0, 1]\backslash\mathbb{Q}$ with its usual Borel field $\mathcal{S}$. The measure $d\mu(x) = (1+x)^{-1}dx$ on $\mathcal{S}$ is invariant under the Borel endomorphism $Vx = x^{-1} \pmod 1$ of X, and the equivalence relations $\mathbf{R}^V$ and $\mathbf{S}^V$ are nonsingular and ergodic on $(X, \mathcal{S}, \mu)$. The endomorphism V is called the *continued fraction* transformation, since the continued fraction expansion

$$x = [a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

of $x \in \mathbb{R}_+\backslash\mathbb{Q}$ is given by $a_n = (V^{n-1}x)^{-1} - V^n x$ for every $n \geq 1$. The map $\varphi: x \to y = (a_1, a_2, \dots)$ from X to $Y = \mathbb{N}^{\times \mathbb{N}^{\times}}$ is a Borel isomorphism, and $\varphi \cdot V = \sigma \cdot \varphi$, where σ denotes the shift $(\sigma y)_k = y_{k+1}$ on Y. It is well known (and not difficult to verify) that two points $x, x' \in X$ satisfy that $(x, x') \in \mathbf{R}^V$ (or, equivalently, that $(\varphi(x), \varphi(x')) \in \mathbf{R}^\sigma$) if and only if there exists a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ such that $x' = (ax+b)/(cx+d)$. Although equivalence relations of endomorphisms are not usually associated with group actions in any canonical way, this shows that $\mathbf{R}^V = (\mathbf{R}^T)_X$, where T is the action of $\mathrm{GL}(2, \mathbb{Z})$ on $\mathbb{R}\backslash\mathbb{Q}$ defined by $T_g x = (ax+b)/(cx+d)$ for every $x \in \mathbb{R}\backslash\mathbb{Q}$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ **[HardW]**, **[CorFS]**.

The continued fraction transformation just described is typical in the sense that, if V is an arbitrary, countable-to-one, *measure-preserving* endomorphism of a probability space $(X, \mathcal{S}, \mu)$, and if $\mathbf{R}^V$ and $\mathbf{S}^V$ are μ-nonsingular, then $\mathbf{R}^V$ and $\mathbf{S}^V$ can always be realized as in example (3) as the equivalence relations on a one-sided shift space (possibly with infinite alphabet). However, if V is only assumed to be nonsingular, then $\mathbf{R}^V$ and $\mathbf{S}^V$ may have some very unexpected properties. Here is an example where

$\mathbf{R}^V$ preserves μ: take $X = \mathbb{R}_+$, μ = Lebesgue measure on $\mathbb{R}_+$, and let $V:\mathbb{R}_+ \to \mathbb{R}_+$ be defined by setting

$$V0 = 0, \quad \text{and} \quad Vx = \sum_{k>k_0(x)} a_k(x) \cdot 10^{-k}$$

for every

$$x = \sum_{k\geq k_0(x)} a_k(x) \cdot 10^{-k} \in \mathbb{R}_+,$$

where $a_{k_0(x)}(x) > 0$ and $0 \leq a_k(x) \leq 9$ for every $k \in \mathbb{Z}$. In other words, V replaces the leading digit in the decimal expansion of a point x by 0. Then V is nonsingular and ergodic on (X, μ), and $\mathbf{R}^V$ preserves μ.

This kind of pathological behaviour can obviously not occur if an endomorphism V of $(X, \mathcal{S}, \mu)$ can be extended[11] to a nonsingular, (properly) ergodic automorphism W of a measure space $(Y, \mathcal{T}, \nu)$. There are many unresolved problems and phenomena in this area, and we refer to **[EigS]** for some recent results.

In order to describe how a nonsingular equivalence relation gives rise to a von Neumann algebra we consider a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and denote by $\mathbf{B}(H)$ the algebra of all bounded linear operators $A: H \to H$ with norm $\|\cdot\|$. The *adjoint* A^* of an operator $A \in \mathbf{B}(H)$ is defined by $\langle Av, w\rangle = \langle v, A^*w\rangle$, $v, w \in H$. A set $\mathscr{A} \subset \mathbf{B}(H)$ is *self adjoint* if $A^* \in \mathscr{A}$ for every $A \in \mathscr{A}$. A C^**-algebra* is a norm closed, self adjoint subalgebra of $\mathbf{B}(H)$ which contains the identity operator 1 on H. A C^*-algebra $\mathscr{A} \subset \mathbf{B}(H)$ is a *von Neumann algebra* if $\mathscr{A}$ is closed in the *strong topology*, i.e. in the smallest topology on $\mathbf{B}(H)$ in which all the maps $A \to \|Av\|$, $v \in H$, are continuous. If $\mathscr{A} \subset \mathbf{B}(H)$ is a self-adjoint subset then its *commutant* $\mathscr{A}' = \{B \in \mathbf{B}(H): AB = BA \text{ for every } A \in \mathscr{A}\}$ is a von Neumann algebra, and $\mathscr{A}$ is a von Neumann algebra if and only if $\mathscr{A} = \mathscr{A}'' = (\mathscr{A}')'$. A von Neumann algebra $\mathscr{A}$ is a *factor* if $\mathscr{A} \cap \mathscr{A}' = \mathbb{C} \cdot 1$, and $\mathscr{A} = \mathbf{B}(H)$ if and only if $\mathscr{A}' = \mathbb{C} \cdot 1$.

The most simple-minded construction of a von Neumann algebra $\mathscr{A}$ from a nonsingular equivalence relation $\mathbf{R}$ on a measure space $(X, \mathcal{S}, \mu)$ is obtained by setting $H = L^2(X, \mu)$ and $\mathscr{A} = (\{U_V: V \in [\mathbf{R}]\} \cup L^\infty(X, \mu))'' \subset \mathbf{B}(H)$, where U_V is the unitary operator[12] $U_V f = (d\mu V/d\mu)^{1/2} \cdot (f \cdot V)$, $f \in H$, $V \in [\mathbf{R}]$, and where every $h \in L_\infty(X, \mu)$ is regarded as a multiplication operator in $\mathbf{B}(H)$. It is not difficult to verify that $L^\infty(X, \mu) \subset \mathbf{B}(H)$ is a maximal Abelian subalgebra[13], and that $\mathscr{A} \cap \mathscr{A}' = \{f \in L^\infty(X, \mu)$:

[11] An automorphism W on $(Y, \mathcal{T}, \nu)$ *extends* V if there exists a nonsingular, surjective map $\psi: Y \to X$ such that $\psi \cdot W = V \cdot \psi$ ν-a.e.

[12] An operator $U \in \mathbf{B}(H)$ is *unitary* if $U^* = U^{-1}$ (or, equivalently, if $UU^* = U^*U = 1$).

[13] If $\mathscr{A} \subset \mathbf{B}(H)$ is a von Neumann algebra, a subalgebra $\mathscr{B} \subset \mathscr{A}$ is *maximal Abelian* if $\mathscr{B}' \cap \mathscr{A} = \mathscr{B}$.

$f \cdot V = f$ μ-a.e. for all $V \in [\mathbf{R}]$. In particular, $\mathscr{A}$ is a factor if and only if $\mathbf{R}$ is ergodic, and $\mathscr{A} = \mathbf{B}(H)$ in this case.

In order to construct a more interesting von Neumann algebra from a nonsingular equivalence relation $\mathbf{R}$ on $(X, \mathscr{S}, \mu)$ we follow **[FelM1]** (see also **[Ver1]**) and set $H = L^2(\mathbf{R}, \mu_{\mathbf{R}}^{(L)})$ (cf. (1.1)). For every $h \in L^\infty(X, \mu)$ we obtain multiplication operators $M_h, M'_h \in \mathbf{B}(H)$ by setting $(M_h f)(x, x') = h(x)f(x, x')$, $(M'_h f)(x, x') = f(x, x')h(x')$, $f \in H$, and we put $\mathscr{M}(\mathbf{R}) = \{M_h : h \in L^\infty(X, \mu)\}$ and $\mathscr{M}'(\mathbf{R}) = \{M'_h : h \in L^\infty(X, \mu)\}$. For $V \in [\mathbf{R}]$ (Example 1.2(1)) we define unitary operators $L_V, L'_V \in \mathbf{B}(H)$ by $(L_V f)(x, x') = f(V^{-1}x, x')$ and $(L'_V f)(x, x') = f(x, V^{-1}x') \cdot \rho_{\mathbf{R},\mu}(V^{-1}x', x')^{1/2}$, $f \in H$. Then $V \to L_V$ and $V \to L'_V$ are homomorphisms from $[\mathbf{R}]$ into $\mathbf{B}(H)$, and $L_V^{-1} \cdot \mathscr{M}(\mathbf{R}) \cdot L_V = L'_{V^{-1}} \cdot \mathscr{M}(\mathbf{R}) \cdot L'_V = \mathscr{M}(\mathbf{R})$ for every $V \in [\mathbf{R}]$. The algebra $\mathscr{A}(\mathbf{R}) = (\mathscr{M}(\mathbf{R}) \cup \{L_V : V \in [\mathbf{R}]\})''$ is called the *von Neumann algebra of the equivalence relation* $\mathbf{R}$. If $\mathbf{R}$ is ergodic, then $\mathscr{A}(\mathbf{R})$ is a factor, $\mathscr{M}(\mathbf{R}) \subset \mathscr{A}(\mathbf{R})$ is maximal Abelian, and $\mathscr{A}(\mathbf{R}) = \mathscr{N}(\mathscr{M}(\mathbf{R}))''$, where $\mathscr{N}(\mathscr{M}(\mathbf{R}))$ is the normalizer of $\mathscr{M}(\mathbf{R})$.[14]

Since the (isomorphism class[15] of the) von Neumann algebra $\mathscr{A}(\mathbf{R})$ is unaffected if we replace μ by an equivalence measure $\nu \sim \mu$ we can assume without loss in generality that $\mu(X) = 1$. The unit vector $\omega \in H$ given by $\omega(x, x') = \delta_{x,x'}$ [16] is *cyclic* under $\mathscr{A}(\mathbf{R})$ (i.e. $\mathscr{A}(\mathbf{R})\omega$ is dense in H), and we set, for every $A \in \mathscr{A}(\mathbf{R})$,

$$\eta_\mu(A) = \langle A\omega, \omega \rangle. \tag{1.10}$$

Then $\eta_\mu : \mathscr{A}(\mathbf{R}) \to \mathbb{C}$ is a *(faithful) state*, i.e. a bounded linear functional such that $\eta_\mu(1) = 1$ and $\eta_\mu(A^*A) > 0$ whenever $0 \neq A \in \mathscr{A}(\mathbf{R})$, and $\eta_\mu(M_h) = \int h\, d\mu$ for all $h \in L^\infty(X, \mu)$. If $\mathbf{R}$ preserves μ, then η_μ is a (normalized) *trace* on $\mathscr{A}(\mathbf{R})$, i.e. $\eta_\mu(AB) = \eta_\mu(BA)$ for all $A, B \in \mathscr{A}(\mathbf{R})$. If μ is an infinite $\mathbf{R}$-invariant measure on X we can formally define ω and η_μ as above, but $\omega \notin H$, and $\eta_\mu(A)$ cannot be defined for all elements of $\mathscr{A}(\mathbf{R})$. The *domain* $D(\eta_\mu) = \{A \in \mathscr{A}(\mathbf{R}) : \eta_\mu(A^*A) < \infty\}$ of η_μ is a dense subalgebra of $\mathscr{A}(\mathbf{R})$, $\eta_\mu(AB) = \eta_\mu(BA)$ for all $A, B \in D(\eta_\mu)$, and η_μ is a *semifinite trace* on $\mathscr{A}(\mathbf{R})$.

1.3. EXAMPLES. (1) *The von Neumann algebra of a free group action.* Let

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[14] The *normalizer* of $\mathscr{M}(\mathbf{R})$ is the set of all unitary operators $U \in \mathscr{A}(\mathbf{R})$ with $U^{-1} \cdot \mathscr{M}(\mathbf{R}) \cdot U = \mathscr{M}(\mathbf{R})$. One checks easily that $\{M_h, M'_h : h \in L^\infty(X, \mu)\}'' = L^\infty(\mathbf{R}, \mu_{\mathbf{R}})$, $\mathscr{A}(\mathbf{R}) = (\mathscr{M}'(\mathbf{R}) \cup \{L'_V : V \in [\mathbf{R}]\})'' \subset \mathscr{A}(\mathbf{R})'$ and $\mathscr{A}(\mathbf{R}) \subset (\mathscr{A}'(\mathbf{R}))'$. From this it is clear that $\mathscr{A}(\mathbf{R})$ is a factor with nontrivial commutant, $\mathscr{M}(\mathbf{R}) \subset \mathscr{A}(\mathbf{R})$ is maximal Abelian, and $\mathscr{A}(\mathbf{R}) = \mathscr{N}(\mathscr{M}(\mathbf{R}))''$.

[15] A *homomorphism* $\Phi : \mathscr{A} \to \mathscr{B}$ of C^*-algebras is a linear map satisfying $\Phi(1) = 1$, $\Phi(A^*) = \Phi(A)^*$, and $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathscr{A}$, and an *isomorphism* is a bijective homomorphism.

[16] $\delta_{\alpha,\beta}$ is the Kronecker delta: $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$, and $\delta_{\alpha,\beta} = 0$ otherwise.

T be a nonsingular, free[17] action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$. Then we can write $\mathbf{R}$ as $G \times X$ and H as $L^2(G \times X, \lambda \times \mu)$, where λ is the counting measure on G. The algebra $\mathscr{A}(\mathbf{R})$ is generated by the operators $(M_h f)(g, x) = h(T_g x) f(g, x)$, $h \in L^\infty(X, \mu)$, and $(L_{g'} f)(g, x) = f(g'^{-1} g, x)$, $g \in G$. This is the Murray-von Neumann group measure space construction in its original setting [**MurN1**].

(2) *The von Neumann algebra of a transitive equivalence relation.* Let $X = \{0, \dots, n-1\}$, $\mu(\{i\}) = 1/n$, $i = 0, \dots, n-1$, and let $\mathbf{R} = X \times X$. Then $\mu_{\mathbf{R}}^{(L)}(i, j) = 1/n$ for all $(i, j) \in \mathbf{R}$, $H \cong \mathbb{C}^{n^2}$, and $\mathscr{A}(\mathbf{R})$ is generated by the operators $(M_i f)(j, k) = \delta_{i,j} f(j, k)$ and $(L_i f)(j, k) = f(j - i \pmod n, k)$, $i = 0, \dots, n-1$. In order to realize $\mathscr{A}(\mathbf{R})$ explicitly we define basis elements $v_{i,j} \in H$, $0 \le i, j < n$ by $v_{i,j}(i', j') = \delta_{(i,j),(i',j')}$ and write e_i for the ith unit vector in $\mathbb{C}^n$. The map $v_{i,j} \to e_i \otimes e_j$ extends linearly to an isomorphism of H and $\mathbb{C}^n \otimes \mathbb{C}^n$ and sends $\mathscr{A}(\mathbf{R})$ to $M_n(\mathbb{C}) \otimes 1_n$, where $M_n(\mathbb{C})$ denotes the algebra of all complex $n \times n$ matrices and 1_n is the $n \times n$ identity matrix. In this picture η_μ is the trace $A \otimes 1_n \to \mathrm{tr}(A)/n$.

If $X = \mathbb{N}$, $\mu(\{i\}) = 1$ for all $i \in \mathbb{Z}$, and $\mathbf{R} = \mathbb{N} \times \mathbb{N}$, the above construction gives $\mathscr{A}(\mathbf{R}) = \mathbf{B}(l^2(\mathbb{N})) \otimes 1 \subset \mathbf{B}(l^2(\mathbb{N}) \otimes l^2(\mathbb{N})) \simeq \mathbf{B}(l^2(\mathbb{N}^2))$, where 1 denotes the identity in $\mathbf{B}(l^2(\mathbb{N}))$.

Finally, let X be a finite set, $\mu(\{x\}) = |X|^{-1}$ for every $x \in X$, and let $\mathbf{R} \subset X \times X$ be an equivalence relation. If $B_1, \dots, B_k$ denotes the set of distinct $\mathbf{R}$-equivalence classes in X (on each of which $\mathbf{R}$ is transitive), then $\mathscr{A}(\mathbf{R}) \simeq \mathscr{A}(\mathbf{R}_{B_1}) \oplus \cdots \oplus \mathscr{A}(\mathbf{R}_{B_k}) \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, where $n_i = |B_i|$ for $i = 1, \dots, k$, and η_μ is the normalized trace on $\mathscr{A}(\mathbf{R})$.

(3) *C^*-algebras associated with Markov shifts.* If R is a nonsingular equivalence relation on a measure space $(X, \mathcal{S}, \mu)$, where X is a compact, metrizable space and $\mathcal{S}$ is the Borel field of X, and if $[\mathbf{R}]$ has a distinguished countable subgroup of homeomorphisms of X (which happens, for example, if $\mathbf{R} = \mathbf{R}^T$ for a nonsingular action T of a countable group G by homeomorphisms of X), then we can associate a *separable* C^*-algebra $\mathscr{C}(\mathbf{R}) \subset \mathscr{A}(\mathbf{R})$ with the equivalence relation $\mathbf{R}$. From the point of view of dynamics, this construction has been particularly useful in the context of Markov shifts (a general construction of C^*-algebras from *topological equivalence relations* is described in [**Ren**]). Let X_P and $\mathbf{R}^P$ be defined as in example 1.2(3). For every $(x, x') \in \mathbf{R}^P$ satisfying (1.4) we choose the integers m, m', n, n' occurring there as small as possible and put $D(x, x') = \max\{m, m', n, n'\}$. For every $M \ge 0$, the set

[17] A nonsingular action T of G on $(X, \mathcal{S}, \mu)$ is *free* if $\mu(\{x \in X : T_g x = x\}) = 0$ for every $g \ne 1$ in G. It is an open problem whether every nonsingular, ergodic equivalence relation $\mathbf{R}$ on $(X, \mathcal{S}, \mu)$ is of the form $\mathbf{R} = \mathbf{R}^T \pmod \mu$ for a *free* action T of some countable group G on $(X, \mathcal{S}, \mu)$.

$\mathbf{R}^P(M) = \{(x, x') \in \mathbf{R}^P : D(x, x') \leq M\} \subset X \times X$ is closed and hence compact. Furthermore, if $M_1 < M_2$, then $\mathbf{R}^P(M_1) \subset \mathbf{R}^P(M_2)$, and $\mathbf{R}^P(M_1)$ is a compact, open topological subspace of $\mathbf{R}^P(M_2)$. This allows us to furnish $\mathbf{R}^P = \bigcup_{M \geq 0} \mathbf{R}^P(M)$ with that topology in which every $\mathbf{R}^P(M)$ is compact and open, and $\mathbf{R}^P$ is locally compact, second countable space in this topology. Let $[[\mathbf{R}^P]]$ be the group[18] of all homeomorphisms V of X_P which satisfy the following conditions: (1) $\{(Vx, x) : x \in X\} \subset \mathbf{R}^P(M)$ for some $M \geq 1$; (2) the set $\{x : Vx = x\} \subset X_P$ is open. Then $[[\mathbf{R}^P]]x = \mathbf{R}^P(x)$ for every $x \in X$. If $\mathbf{S}^P \subset \mathbf{R}^P$ is the subrelation introduced in Example 1.2(3) then $\mathbf{S}^P(M) = \{(x, x') \in \mathbf{S}^P : D(x, x') \leq M\} \subset X \times X$ is an equivalence relation for every $M \geq 0$, and $[\mathbf{S}^P]^{(M)} = \{V \in [[\mathbf{S}^P]] : (Vx)_k = x_k$ for all $x \in X_P$ and $|k| \geq M\}$ is a finite group with $[\mathbf{S}^P]^{(M)}x = \mathbf{S}^P(M)(x)$ for every $x \in X_P$. We topologize $\mathbf{S}^P = \bigcup_{M \geq 0} \mathbf{S}^P(M)$ as above so that each $\mathbf{S}^P(M)$ is a compact, open subset of $\mathbf{S}^P$, set $[[\mathbf{S}^P]] = \bigcup_{M \geq 0} [\mathbf{S}^P]^{(M)} \subset [[\mathbf{R}^P]]$, and note that $[[\mathbf{S}^P]]x = \mathbf{S}^P(x)$ for every $x \in X$.

We begin by associating a C^*-algebra with the equivalence relation $\mathbf{S}^P$. For every $M \geq 0$ we write $C(X_P)^{(M)}$ for the space of complex valued functions on X_P which depend only on the coordinates $x_{-M}, \ldots, x_M$. Let $C(X_P)$ be the set of continuous, complex valued functions on X_P, and let $C_c(\mathbf{S}^P)$ denote the continuous functions with compact support on $\mathbf{S}^P$. Every $V \in [[\mathbf{S}^P]]$ and $h \in C(X_P)$ defines a linear operator on $C_c(\mathbf{S}^P)$ by $(L_V f)(x, x') = f(V^{-1}x, x')$ and $(M_h f)(x, x') = h(x) f(x, x')$, $f \in C_c(\mathbf{S}^P)$. Consider the algebras $\mathscr{F}(\mathbf{S}^P)$ and $\mathscr{F}(\mathbf{S}^P)^{(M)} \subset \mathscr{F}(\mathbf{S}^P)$ generated by $\{L_V, M_h : V \in [[\mathbf{S}^P]], h \in C(X_P)\}$ and $\{L_V, M_h : V \in [\mathbf{S}^P]^{(M)}, h \in C(X_P)^{(M)}\}$, respectively, where $M \geq 0$. As we have seen in Example (2), $\mathscr{F}(\mathbf{S}^P)^{(M)}$ is a direct sum of certain finite dimensional matrix algebras $M_k(\mathbb{C})$. The completion of $\mathscr{F}(\mathbf{S}^P)$ in the operator norm $\|\cdot\|_P$ on the Hilbert space $H_P = L^2(\mathbf{S}^P, (\mu_P)^{(L)}_{\mathbf{S}^P})$ is the desired C^*-algebra[19] $\mathscr{C}(\mathbf{S}^P)$ of the equivalence relation $\mathbf{S}^P$. The algebra $\bigcup_{M \geq 0} \mathscr{F}(\mathbf{S}^P)^{(M)}$ is dense in $\mathscr{C}(\mathbf{S}^P)$, and $\mathscr{C}(\mathbf{S}^P)$ is an AF-algebra.[20]

For the corresponding one-sided algebras $\mathscr{F}(\mathbf{S}'^P)$ and $\mathscr{C}(\mathbf{S}'^P)$ (cf. example 1.2(4)) we set $D(y, y') = \max\{m, n\}$ if $(y, y') \in \mathbf{S}'^P$ and $\sigma_P^n(y) =$

[18] The countable group $[[\mathbf{R}_P]]$ is an *ample group* in the sense of **[Kri7]**.

[19] If Q is an arbitrary stochastic matrix compatible with P, and if $\|\cdot\|_Q$ is the operator norm on $\mathscr{F}(\mathbf{S}^P)$ acting on the Hilbert space $H_Q = L^2(\mathbf{S}^P, (\mu_Q)^{(L)}_{\mathbf{S}})$, then $\|\cdot\|_P = \|\cdot\|_Q$, and the completions of $\mathscr{F}(\mathbf{S}^P)$ in the two norms $\|\cdot\|_P$ and $\|\cdot\|_Q$ coincide (cf. **[Dix]**)

[20] A C^*-algebra $\mathscr{A}$ is an *AF-algebra* if there exists an increasing sequence $(\mathscr{A}_n, n \geq 1)$ of finite-dimensional subalgebras of $\mathscr{A}$ such that $\bigcup_n \mathscr{A}_n$ is dense in $\mathscr{A}$ (cf. **[Bra]**— AF stands for approximately finite dimensional).