Linear Programming with Statistical Applications

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To Marlene, Lori, and Philip

Preface

For many years I have felt the need for an introductory text in linear programming with applications in statistics and other disciplines. This book is based in part on course notes for a two semester sequence in mathematical programming for advanced undergraduate and first-year graduate students in economics, statistics, operations research, mathematics, and business, which I taught at Iowa State University.

In the book, I have emphasized the importance of the relationship between theory and computational techniques. The first three chapters emphasize the basic concepts and properties of linear programming through a hypothetical production example and a scheduling problem. The graphic method is used to provide a conceptual understanding of certain salient properties of linear programming. Thus, the book begins at an elementary level to help the student more easily grasp the main ideas of linear programming that form the core of mathematical programming. With this approach, topics such as sensitivity analysis, parametric programming, and other computational techniques are covered in a manner easily understood by students from any discipline.

The theory of linear duality is presented next and is followed by topics such as the Kuhn-Tucker theory. Other topics covered in linear programming emphasize statistics. Chapters 6 and 7 cover the transportation and assignment problems. Chapter 8 discusses goal programming under a preemptive priority structure.

The appendix discusses the use of IBM's Mathematical Programming System, MPSX, and/or Management Science's System, MPS-III. This material distinguishes this book from other linear programming texts. Since most students, especially at the graduate level, want a computer package to solve research problems, and most colleges and universities have IBM computers or compatible computers, the book discusses some relevant computer packages. Procedures related to the text, such as sensitivity analysis and parametric programming, are included.

For a one-semester undergraduate course Chapters 1–3, 6–8, and Appendix A can be covered, as well as various parts of the remaining chapters as time allows. In Chapter 4, for instance, linear duality can be studied without the proof of Lemma 4.6 or Appendix B, which discusses relevant concepts over convex cones. The first half of the text emphasizes linear programming models and computational techniques; it also covers the theoretical derivation of the simplex method and certain related linear programming topics, such as alternative criteria in curve fitting, and chance-constraint programming.

Students should have some knowledge of Chapters 1–3 or a good background in linear programming and matrix theory if the text is used in a graduate course, which would cover Chapters 4–8.

I am indebted to my family for helping me to pursue all of my objectives in writing this text. I would like to thank Darlene Wicks for her superb technical typing. My indebtedness also extends to many, not least of all Iowa State University, the statistical laboratory, and the computation center, which provided facilities, intellectual stimulation, and encouragement, without which this text might never have been written. I owe special thanks to Oscar Kempthorne and H. T. David for their constant encouragement and valuable comments after critically reading parts of this manuscript. I am also indebted to H. T. David and D. J. Soults who introduced me to this field of study and who taught me more than I know how to acknowledge. The author is indebted to Carla Tollefson for her valuable suggestions in the final editing of this text.

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Linear Programming with Statistical Applications

1 Introduction

1.1 Historical Perspective of Mathematical Programming

One of the first questions asked by various early scientists was: "How can one determine the best or optimal values in some restricted space so that these values will yield the greatest (or least) possible numerical value to some mathematical expression?"

These mathematical problems could take many forms, depending on the problem at hand, but the general mathematical programming problem can be expressed as

$$\begin{aligned} &\text{maximize (or minimize)} \quad f(\mathbf{x}) \\ &\text{such that} \quad g_i(\mathbf{x}) \leq b_i \qquad i = 1, 2, \ldots, m \end{aligned} \tag{1.1}$$

That is, $f(\mathbf{x})$ and each $g_i(\mathbf{x})$ are real-valued functions of $\mathbf{x}$ (in E^n), where the vector $\mathbf{b}$ is known. Special cases of (1.1) are linear programming models that can be written as

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{such that} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_j \geq 0 \qquad j = 1, 2, \ldots, n \end{aligned}$$

or

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}'\mathbf{x} \\ \text{such that} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

This was one of the earliest models to be considered. In particular, Fourier, a French mathematician, had formulated these types of models for use in mechanics and probability theory around 1826. In 1939, the Russian mathematician Kantorovich formulated production problems as linear programming problems and suggested a possible way of solving such models.

The general area of research in linear-programming-type problems came into its own during World War II when large-scale military operations required careful planning of logistic support. This led to mathematical techniques known as operations research techniques. Linear programming was one of these techniques. The primary contributor in establishing procedures to solve linear programming models was George Dantzig. His general algorithm, known as the simplex algorithm, was developed in 1947. Subsequently, interest in this area grew, and in 1949 T. C. Koopmans organized in Chicago the Cowles Commission Conference on Linear Programming. The papers presented in this conference were collected by Koopmans in 1951 in the book entitled *Activity Analysis of Production and Allocation.*

Charnes, Cooper, and Henderson wrote the first textbook on linear programming. Since then the number of texts in this area has grown tremendously, with applications in agriculture, economics, engineering, statistics, mathematics, and business.

Karush in 1939 and Kuhn and Tucker in 1950 addressed themselves to the nonlinear mathematical programming problem. Their work has proven to be a classic. It has led the way to solutions of quadratic programming problems—i.e., models with linear constraints and objective functions of the form $\mathbf{c}'\mathbf{x} + \mathbf{x}'D\mathbf{x}$. Moreover, the Karush–Kuhn-Tucker theory has led to the area of duality through the saddle value approach. That is, associated with a mathematical programming problem is another problem that, when solved, yields the optimal solution to the original problem, and conversely. The saddle value problem is formulated with Lagrangian functions for inequality-type restrictions.

Since the late 1940s and early 1950s, the area of mathematical programming has branched to many types of models. For one, geometric programming in the late 1960s has come into its own. These problems have proven quite popular in engineering-type models. The early pioneers in this area were Duffin, Peterson, and Zener.

1.2 Examples of Programming Problems

Consider the linear regression model

$$y_i = \beta_0 + \beta_1 x_i + d_i$$

where y_i is the ith observation on a dependent variable, x_i is the ith observation on an independent variable, and d_i is a random disturbance. Then the classical least squares theory leads to estimates of β_0 and β_1 that will minimize

$$Q(\beta_0, \beta_1) = \sum_i (y_i - \beta_0 - \beta_1 x_i)^2 = (\mathbf{y} - X\boldsymbol{\beta})'(\mathbf{y} - X\boldsymbol{\beta})$$

or minimize the sum of squares of errors. $Q(\beta_0, \beta_1)$ is a quadratic function of β_0 and β_1; i.e., $Q(\beta_0, \beta_1) = \Sigma_i y_i^2 + \mathbf{c}'\boldsymbol{\beta} + \boldsymbol{\beta}'D\boldsymbol{\beta}$ where

$$\mathbf{c}' = -2\mathbf{y}'X \qquad \text{and} \qquad D = X'X$$

In this case $\boldsymbol{\beta}$ is unrestricted, so one can find estimates that minimize $Q(\beta_0, \beta_1)$ by solving the well-known normal equations

$$X'X\boldsymbol{\beta} = X'\mathbf{y}$$

In certain cases, certain limits or constraints are imposed on β_0, β_1, or both. For example, assume the researcher desires a positive intercept and a value of the slope between ℓ and u. Then we must

$$\begin{aligned} \text{minimize} \quad & Q(\beta_0, \beta_1) \\ \text{subject to} \quad & \ell \le \beta_1 \le u \\ & \beta_0 \ge 0 \end{aligned}$$

which is now a quadratic programming problem.

Another researcher with the same set of measurements might also want to find estimates of β_0 and β_1 for the above linear model, but believes it is more "reasonable" to determine values of β_0 and β_1 that will minimize the sum of absolute deviations; i.e.,

$$\text{minimize} \quad \sum_i |y_i - \beta_0 - \beta_1 x_i|$$

Whether or not we have restrictions imposed on our model, the values of β_0 and β_1 that minimize the sum of absolute deviations can be reformulated as a linear programming problem. In particular, let d_{1i} denote the positive deviations (and d_{2i} the negative deviations) above (below) the fitted line for the ith observation. Then for any i,

$$y_i = \beta_0 + \beta_1 x_i + d_{1i} - d_{2i}$$

where at most one d_{1i} or d_{2i} can be nonzero by the nature of the problem. Also, $d_{1i} + d_{2i}$ is the absolute deviation between the fitted equation and y_i. We therefore have the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & \sum_i (d_{1i} + d_{2i}) \\ \text{such that} \quad & X\beta + \mathbf{I}\mathbf{d}_1 - \mathbf{I}\mathbf{d}_2 = \mathbf{y} \\ & \mathbf{d}_1, \mathbf{d}_2 \geq 0 \end{aligned}$$

which will yield the best estimates of β_0 and β_1 under the criterion of minimizing the sum of absolute deviations.

Another example of a programming problem is that of an investor trying to determine a security portfolio from a set of n securities that will hopefully provide at least a certain return at the least possible risk. Here, we assume the variance based on past performances of a security is a measure of the risk that the realized rate of return deviates from the expected rate of return, μ_i. The covariance, σ_{ij}, of their returns provides a measure of the correlation between the rates of return on securities i and j. When σ_{ij} is positive, this would imply that the returns on the securities will usually go up and down together. Therefore, a rational investor would not invest too heavily in a set of securities that seem to move together. In this case, we would do well to diversify our portfolio; sacrificing return for risk reduction and measuring covariance between securities will enable us to achieve this required diversification.

To minimize risk arising from variability within a security and risk from an undiversified portfolio, we would minimize the function $\mathbf{x}'S\mathbf{x}$. S is the covariance matrix determined from past performances of the securities. Terms of the type $x_i^2 s_{ii}$ account for variability within a security, and terms of the type $x_i x_j s_{ij}$ account for the covariance between securities. If the investor wants at least a rate of return of p percent, based on d dollars, then we have the following quadratic programming problem:

$$\begin{aligned} \text{minimize} \quad & \mathbf{x}'S\mathbf{x} \\ \text{such that} \quad & \Sigma \mu_i x_i \geq p \\ & \Sigma x_i = 1 \\ & x_i \geq 0 \qquad i = 1, 2, \ldots, n \end{aligned}$$

where x_i denotes the amount of the d dollars to be invested in security i.

1.3 Plan of Book

The objective of this text is to discuss, first, linear programming models and computational techniques that are highly useful in areas of economics and business applications. Second, linear duality in the spirit of the Kuhn-Tucker theory is presented. These concepts will be applied to various statistical problems. In particular, generalized properties of regression estimators under the criteria of minimizing the sum of absolute deviations, L_1, are given in Chapter 5. Computational procedures based on these properties are subsequently presented.

Chapter 6 considers the transportation problem. Computational algorithms are presented. Moreover, it is shown how this problem is related to the analysis of variance problem under L_1.

The final chapter on "Goal Programming" shows how to solve such problems using IBM's MPSX/370 Mathematical Programming System. This extension allows researchers to solve goal programming problems with as many as 10,000 rows and over 100,000 variables with a high degree of precision.

The text was written for a senior level undergraduate course or a first-year graduate course in linear programming with statistical applications. The theoretical derivation of the simplex method is covered, and certain related linear programming topics, such as sensitivity analysis and parametric programming, are discussed.

2 Linear Programming

2.1 Introduction

Linear programming is by far the most widely used optimization technique. Linear programming deals with the problem of determining feasible plans that are optimal with respect to a certain agreed upon linear objective function; in particular, it determines a plan that maximizes or minimizes some linear function over all possible feasible plans. The *feasible* plans are such that they must satisfy certain restrictions that are usually in the form of a system of linear inequalities. Hence, linear programming is defined in terms of a mathematical model composed of linear functions, where the word "programming" is used as a synonym for planning. Thus, linear programming involves choosing activities (plans, schedules, allocations) in such a way as to obtain an optimal program. Here an *optimal* program is defined as a feasible plan that maximizes (or minimizes) the objective function from among all possible feasible plans.

The great variety of problems to which linear programming can be applied is indeed remarkable. It is used in curve-fitting problems under different criteria such as minimizing the sums of squares, minimizing the maximum deviation, or minimizing the sum of absolute deviations. It is used in portfolio or investment problems, in transportation problems, in allocation problems, in production scheduling, in game problems, and in numerous other areas. In particular, consider the problem of finding $x_1, x_2, \ldots, x_n$ that

maximizes (or minimizes) the linear function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

$$
\begin{aligned}
\text{subject to}\quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1\\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2\\
& \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots\\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m\\
& \qquad\qquad\qquad x_1, x_2, \ldots, x_n \ge 0
\end{aligned}
$$

where a_{ij}, b_i, and c_j are known constants. The unknown variables x_j ($j = 1, 2, \ldots, n$) can be solved by linear programming techniques.

Expressed in matrix notation, the problem is to find a vector $\mathbf{x}$ in E^n that

$$\begin{aligned} &\text{maximizes (minimizes)} \quad && \mathbf{c}'\mathbf{x} \\ &\text{subject to} && A\mathbf{x} \leq \mathbf{b} \\ & && \mathbf{x} \geq 0 \end{aligned} \tag{2.1}$$

The next section illustrates some problems that can be expressed in terms of a linear programming model. Section 2.3 considers some geometric interpretations of linear programming that will give valuable insight to certain fundamental properties and concepts. Section 2.4 discusses a procedure based on the Gauss-Jordan technique for obtaining a consistent solution of a system of linear equations and determining a vector that maximizes the specified objective function. The theoretical derivation of the simplex procedure is deferred until later in the chapter.

2.2 Linear Programming Models

EXAMPLE 2.1 *Activity-analysis problem* (Karlin 1959, p. 174)

A manufacturer has the option of using one or more of four types of production processes. The first and second processes yield items *A*, and the third and fourth processes yield items *B*. The inputs for each of these processes are labor measured in work-weeks, pounds of raw material *X*, and boxes of raw material *Y*. Since each process varies as to its input requirements, the profits of the processes differ, even for processes producing the same item. Now suppose further that the manufacturer has fixed amounts of labor, raw material *X*, and boxes of raw material *Y*. Hence, the manufacturer, in deciding on a week's production schedule, is limited in the range of possibilities by the available amounts of the three resources. The manufacturer wants to determine how much of the two products should be manufactured and which processes should be used to maximize profits.

Table 2.1 gives a breakdown of how much of each resource must be used to produce an item by each process, the profit associated with that item, and the limitation on each resource.

This problem has four decision variables and three restrictions. One is tempted to use the process (or processes) that produce items with the highest unit profit in order to maximize profits. However, the interdependence encountered in order to allocate the resources in the "best" manner makes an intuitive approach invalid, as shown later in the chapter. Presently we are only interested in the proper formulation of the linear programming model.