Lecture Notes in Mathematics 1593

Jay Jorgenson & Serge Lang
Dorian Goldfeld

Explicit Formulas

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Explicit Formulas

for Regularized Products and Series

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Introduction

Explicit formulas in number theory were originally motivated by the counting of primes, and Ingham's exposition of the classical computations is still a wonderful reference [In 32]. Typical of these formulas is the Riemann-von Mangoldt formula

$$\sum_{p^n \leq x} \log p = x - \sum_{\rho} \frac{x^\rho}{\rho} - \zeta'_{\mathbf{Q}}/\zeta_{\mathbf{Q}}(0) - \frac{1}{2}\log(1 - x^{-2}).$$

Here the sum on the left is taken over all prime powers, and the sum on the right is taken over the non-trivial zeros of the Riemann zeta function.

Later, Weil [We 52] pointed out that these formulas could be expressed much more generally as stating that the sum of a suitable test function taken over the prime powers is equal to the sum of the Mellin transform of the function taken over the zeros of the zeta function, plus an analytic term "at infinity", viewed as a functional evaluated on the test function.

It is the purpose of these notes to carry through the derivation of the analogous so-called "explicit formulas" for a general zeta function having an Euler sum and functional equation whose fudge factors are of regularized product type. As a result, our general theorem applies to many known examples, some of which are listed in §7 of [JoL 93c]. The general Parseval formula from [JoL 93b] provides an evaluation of the "term at infinity", which we call the Weil functional. Also, as an example of our results, let us note that even in the well-studied case of the Selberg zeta function of a compact Riemann surface, our computations show that one may deal with a larger class of test functions than previously known.

For some time, analogies between classical analytic number theory and spectral theory have been realized. Minakshisundaram-Pleijel defined a zeta function in connection with the Laplacian on an arbitrary Riemannian manifold [MiP 49], and subsequently Selberg defined his zeta function [Se 56]. In [JoL 93a,b] we developed a general theory of regularized products and series applicable equally to the classical analytic number theory and to some of these analogous spectral situations. In particular, we proved the basic properties of the Weil functional at infinity in the context of regularized products and series, with a view to using the functional for the explicit formulas in this general context.

A fundamental class of zeta functions. In [JoL 93c] we defined a fundamental class of functions to which we could apply these properties and carry out analogues of results in analytic number theory. Roughly speaking, the functions Z in our class are those which satisfy the three conditions:

- there is a functional equation;
- the logarithm of the function admits a generalized Dirichlet series converging in some half plane (we call this Dirichlet series an Euler sum for Z);
- the fudge factors in the functional equation are of regularized product type.

The precise definition of our class of functions is recalled in Chapter II, §1. The explicit formula can be formulated and proved for functions in this class. In Chapter II, §1, we discuss the extent to which this class is a much broader class than a certain class defined by Selberg [Se 91]. Furthermore, certain applications require an even broader class of functions to which all the present techniques can be applied. We shall describe the need for such a class in greater detail below.

Just as we did for the analogue of Cramér's theorem proved in [JoL 93c], we emphasize that the explicit formula involves an inductive step which describes a relation between some of the zeros and poles of the fudge factors and some of those of the principal zeta function Z. Such a step can be viewed as a step in the ladder of regularized products, because our generalized Cramér theorem insures that a function Z in our class is also of regularized product type provided the fudge factors are of regularized product type.

If Z is a function in our class, and, for $\mathrm{Re}(s)$ sufficiently large, the expression

$$\log Z(s) = \sum_{\mathbf{q}} \frac{c(\mathbf{q})}{\mathbf{q}^s}$$

is the Euler sum for $\log Z(s)$, with a sequence $\{\mathbf{q}\}$ of real numbers > 1 tending to infinity, and complex coefficients $c(\mathbf{q})$, then such $\mathbf{q}$ play the role of prime powers. However, readers should keep in mind cases when $\mathbf{q}$ does not look at all like a prime power. For example, the general theory applies to the case when $Z(s)$ is a general

Dirichlet polynomial, up to an exponential fudge factor; a precise definition is given in Chapter II, §4. Such polynomials contain as special cases the local factors of more classical zeta functions and L-functions. In examples having to do with Riemannian geometry, $\log \mathbf{q}$ is the Riemannian distance between two points in the universal covering space.

The general version of Cramér's theorem in [JoL 93c] was carried out for the original Cramér's test function $\phi_z(s) = e^{sz}$. One can also view this version as a special case of an explicit formula with more general test functions. This is carried out in Chapter II. In [JoL 93c], §7 we gave a number of examples for our Cramér-type theorem. To these we are adding not only the general Dirichlet polynomials as mentioned above, but also Fujii-type L-functions, obtained from a zeta function by inserting what amounts to a generalized character as coefficient of the Dirichlet series defining the zeta function (see the papers by Fujii listed in the bibliography). In Chapter V we show both how to recover Fujii's theorems for the functions he considered, namely the Riemann zeta function and the Selberg zeta function for $PSL(2, \mathbf{Z})$, as well as an analogous theorem for the general zeta functions in our class, all as corollaries of our Cramér's theorem. Similarly, a result of Venkov, which relates the eigenvalues of the Laplacian relative to $PSL(2, \mathbf{Z})$ to the classical von Mangoldt function, will be generalized to any non-compact finite volume hyperbolic Riemann surface in [JoL 94]. The generalization involves another inductive type argument, using the fact that the fudge factor in the functional equation of the non-compact Selberg zeta function involves the determinant of the scattering matrix, which itself is in our class of functions since it has an Euler sum and a simple functional equation with constant fudge factors. In this case, the Euler sum exists whereas a classical Euler product does not. Thus, the general theory simultaneously contains previous results and gives new ones which were not proved previously by authors using such tools as the Selberg trace formula.

Analytic estimates for the proof. In addition to the Parseval formula of [JoL 93b], the proof of the general explicit formula relies on certain analytic estimates for regularized harmonic series, including the logarithmic derivatives of regularized products in strips. We gave such estimates already in [JoL 93a,b], but we need further such estimates which we present in Chapter I of the present work, using the technique of our generalized Gauss formula. Hard-core analytic estimates having thus been put out of the way, the rest of the work is then relatively formal. It is noteworthy that to each regularized product we associate naturally two nonnegative integers determined directly from the definition. Then the fundamental estimates of Chapter I show that the order of growth of the logarithmic derivatives of such products in strips is determined by these two integers. In the application to the evaluation of certain integrals involving test functions, one can then see that the order of decay of these test functions, needed to insure that the integrals converge, is also determined by these two integers. Our systematic approach both improves known estimates for the Selberg zeta function (cf. Chapter I, §4), and provides estimates for functions in our class which had not been considered previously.

Theta inversions. We shall postpone to still another work the application of explicit formulas to the counting of those objects which play the role of prime powers. Here we shall emphasize an entirely different type of application, obtained by taking Gaussian type functions as the test functions instead of other test functions which lead to the counting. Applying the general explicit formula to such Gaussians gives rise to relations which are vast generalizations of the classical Jacobi inversion formula for the classical Jacobi theta function, where t on one side gets inverted to $1/t$ on the other side of the formula. The classical Jacobi inversion formula is the relation

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-n^2 t} = \frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty} e^{-(2\pi n)^2/4t},$$

which holds for all $t > 0$. Here, $\log \mathbf{q} = 2\pi n$ where n is a positive integer. The zeta function $Z(s)$ giving rise to the above theta series is essentially the special Dirichlet polynomial

$$\sin(\pi i s) = -\frac{1}{2i}e^{\pi s}(1 - e^{-2\pi s}).$$

Thus, the most classical theta series appears in a new context, associated to a "zeta function" which looks quite different from those visualized classically.

The general context of Chapter IV and Chapter V allows a formulation of a theta inversion when the theta series is of type

$$\sum_k a_k e^{-\lambda_k t}$$

with various coefficients a_k. Theta inversion applies in certain cases when the sequence $\{\lambda_k\}$ is the sequence of eigenvalues of an operator. For example, as we will show in Chapter V, §4, such an inversion formula comes directly from considering the heat kernel on the compact quotient of an odd dimensional hyperbolic space which has metric with constant negative sectional curvature.

For certain manifolds, the theta inversion already gives rise to an extended class of zeta functions, which instead of an Euler sum may have a Bessel sum. For manifolds of even dimension, the class of functions having an Euler sum or Bessel sum is still not adequate, and it is necessary to define an even further extended class, which we shall describe briefly below. At this moment, it is not yet completely clear just how far an extension we shall need, but so far, whatever the extension of the fundamental class we have met, the techniques of [JoL 93a,b,c] and of Chapter I apply.

In [JoL 94], we show how the general explicit formula also applies to the scattering determinant of Eisenstein series. Here, the Euler sum exists, and scattering determinants are in the fundamental class.

An additive theory rather than multiplicative theory, and an extended class of functions. The conditions defining our fundamental class of functions are phrased in a manner still relatively close to the classical manner, involving the functions multiplicatively. However, it turns out that many essential properties of these functions involve only their logarithmic derivative, and thus give rise to an additive theory. For a number of applications, it is irrelevant that the residues are integers, and in some applications we are forced to deal with the more general notions of a regularized harmonic series (suitably normalized Mittag-Leffler expansions, with poles of order one) whose definition is recalled in Chapter I, §1. In general, the residues of such a series are not integers, so one cannot integrate back to realize this series as a logarithmic derivative of a meromorphic function. Even for the Artin L-functions, although they can be defined by an Euler product, it was natural for Artin to define them via their logarithmic derivative, and at the time, Artin could only prove that the residues were rational numbers. It took many years before the residues were finally proved to be integers. The systematic approach of [JoL 93a,b,c] in fact has been carried out so that it applies to this additive situation. The example of Chapter V, §4, shows why such an additive theory is essential.

Thus we are led to define not only the fundamental class of functions whose logarithmic derivative admits a Dirichlet series expression as mentioned above, but an extended class of functions where this condition is replaced by another one which will allow applications to more situations, starting with applications to various spectral theories as in [JoL 94]. Nevertheless, we still defined the fundamental class of functions having Euler sums, and we phrase some results multiplicatively, partly because at the present time, we feel that a complete change of notation with existing works would only make the present work less accessible, and partly because the class of functions admitting Euler sums is still a very important one including the classical functions of algebraic number theory and representation theory. However, we ask readers to keep in mind the additive rather than multiplicative formalism. Many sections, e.g. Chapter I and §1 and §2 of Chapter V, are written so that they apply directly to the additive situation.

Functions in the multiplicative fundamental class are obtained as Mellin transforms of theta functions having an inversion formula. Functions in the extended additive class are obtained as regularized harmonic series which are Gaussian transforms of such theta functions. For example, the (not regularized) harmonic series obtained from the heat kernel theta function in the special case of compact quotients of the three dimensional, complete, simply connected, hyperbolic manifold is essentially

$$\sum_k \frac{\phi_k(x)\phi_k(y)}{s(s-2)+\lambda_k}.$$

Observe how the presence of $s(s-2)$ in the series formally insures a trivial functional equation, that is invariance under $s \mapsto 2-s$.

Conversely, given a function in our extended additive class, one may go in reverse and see that the original theta inversion is only a special case of the general explicit formula valid for much more general test functions. The existence of an explicit formula with a more general test function will then allow us to obtain various counting results in subsequent publications.

Finally, let us note that many examples of explicit formulas using various test functions involving many examples of zeta functions have been treated in the literature, providing a vast number of papers on the subject. Most of the papers dealing with such explicit formulas are not directly relevant for what we do here, which is to lay out a general inductive "ladder principle" for explicit formulas in line with our treatment of Cramér's theorem. For instance, Deninger in [Den 93] emphasizes the compatibility of an explicit formula for the Riemann zeta function with a conjectural formalism of a Lefschetz trace formula. Such a formalism might occur in the presence of an operator whose eigenvalues are zeros of the zeta function. Our inductive hypotheses cover a wider class of functions than in [Den 93], and our treatment emphasizes another direction in the study of regularized products and series. Factors of regularized product type behave as if there were an operator, but no operator may be available.

We also mention Gallagher's attempt to unify a treatment of Selberg's trace formula with treatments of ordinary analytic number theory [Ga 84]. However, the conditions under which Gallagher proves his results are very restrictive compared to ours, and, in particular, are too restrictive to take into account the inductive ladder principle which we are following.

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