ADVANCES IN

# COMPUTER VISION AND IMAGE PROCESSING

*Editor:* THOMAS S. HUANG

*VOLUME 1* • 1984

## IMAGE RECONSTRUCTION FROM INCOMPLETE OBSERVATIONS

# ADVANCES IN COMPUTER VISION AND IMAGE PROCESSING

*A Research Annual*

IMAGE RECONSTRUCTION FROM INCOMPLETE OBSERVATIONS

*Editor:* THOMAS S. HUANG
*Coordinated Science Laboratory*
*University of Illinois*

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VOLUME 1 · 1984

JAI PRESS INC.

*Greenwich, Connecticut* *London, England*

*36 Sherwood Place*
*Greenwich, Connecticut 06830*

*JAI PRESS INC.*
*3 Henrietta Street*
*London WC2E 8LU*
*England*



*ISBN: 0-89232-280-2*

*Manufactured in the United States of America*

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*Volume 1* · 1984

## IMAGE RECONSTRUCTION FROM INCOMPLETE OBSERVATIONS

# LIST OF CONTRIBUTORS

*R. H. T. Bates* — Electrical and Electronic Engineering, University of Canterbury, New Zealand

*J. R. Fienup* — Radar and Optics Division, Environmental Research Institute of Michigan

*W. R. Fright* — Electrical and Electronic Engineering, University of Canterbury, New Zealand

*M. H. Hayes* — School of Electrical Engineering, Georgia Institute of Technology

*D. A. Hayner* — Gould Research Center, Illinois

*T. S. Huang* — Coordinated Science Laboratory, University of Illinois

*W. K. Jenkins* — Coordinated Science Laboratory, University of Illinois

*J. L. C. Sanz* — IBM Research Laboratory, San Jose, California

*H. J. Trussell* — Department of Electrical Engineering, North Carolina State University

*R. Y. Tsai* — IBM T. J. Watson Research Center, Yorktown Heights, New York

# PREFACE

The goal of this series is to present in-depth treatment of topics of current interest in computer vision and image processing. The terms computer vision and image processing are used in their broad sense to include image coding, enhancement, restoration and understanding, as well as the analysis of three-dimensional time-varying scenes. Computer vision and image processing have important applications in diverse areas such as robotics, industrial automation, medical diagnosis, and defense-related problems. These applications draw concepts and techniques from many different disciplines including multi-dimensional signal processing, pattern recognition, and artificial intelligence. We aim to have each volume of this series concentrate on a special topic or several closely related topics.

This inaugural volume of our series concentrates on signal and image reconstruction from observations which are in one sense or another incomplete. Chapters 1 and 2 treat the problem of extrapolation: Given part of a signal, we aim to recover the whole. Chapters 3, 4, and 5 treat the problem of phase retrieval: Given the magnitude of the Fourier transform of a signal, we aim to recover the phase angle. Chapter 3 also discusses the related problem of recovering the magnitude given the phase.

In reconstructing a signal from incomplete observations, it is of the utmost importance to utilize a priori knowledge about the signal which we may possess. The important question of what and how a priori knowledge can be used is addressed in Chapter 6. This chapter also describes a number of recent techniques in image restoration. In fact, this chapter by Trussell, together with an earlier work of Frieden ("Image Enhancement and Restoration," in *Picture Processing and Digital Filtering*, ed. by T. S. Huang, Springer-Verlag, 1979) should provide the readers with an excellent overview of the area of image restoration.

The last chapter of this volume, Chapter 7, treats a problem of signal reconstruction from multiple incomplete observations. Specifically, several low-resolution images are combined to construct a higher-resolution image. Included as part of the algorithm is a novel technique for registering multiple images.

Although in this series we publish mainly invited papers, suitable unsolicited contributions may also be published after careful reviewing. Potential contributors are advised to contact the editor before submitting their manuscripts.

Thomas S. Huang
*Series Editor*

# CONTENTS

Chapter 1

# SUPPORT-LIMITED SIGNAL AND IMAGE EXTRAPOLATION

Jorge L. C. Sanz and Thomas S. Huang

## OUTLINE



**Advances in Computer Vision and Image Processing, vol. 1, pages 1–82**

**ISBN: 0-89232-280-2**

## ABSTRACT

In this paper, the problem of bandlimited multidimensional signal extrapolation is studied.

We will first briefly review some important past work, and then new contributions to this area will be presented. Four models under which the extrapolation problem can be posed are introduced. These models are useful for understanding the relationships among several extrapolation algorithms. An important unification for iterative extrapolation algorithms and also for noniterative procedures which are known in the engineering literature will be shown. The technique we follow in our approach provides as a by-product some new iterative algorithms which give faster extrapolations.

Also, the basic issues of discretization and noise are extensively discussed. Some new algorithms to cope with noise in the given signal and approximation results concerning discretization of the known signal are presented.

In addition, we will give abundant numerical simulation results by means of which we hope to provide insight into the effectiveness and applicability of the new techniques, as well as preliminary comparisons with other procedures.

## 1. INTRODUCTION

Band-limited signal extrapolation (or equivalently, support-limited spectrum extrapolation) is a key problem in signal reconstruction and restoration.

An important motivation in reconstruction is the missing-cone problem in computer tomography as applied to nondestructive testing. As is well known, a three-dimensional structure can be reconstructed if two-dimensional x-ray images (projections) of it are given over all viewing angles. However, in nondestructive testing, it often happens that it is physically impossible to get images over certain angle ranges. In theory, support-limited spectrum extrapolation could be used to supply the missing information.

Another application is the following signal restoration problem. Let us suppose that we are given a piece of a $n$-dimensional signal $g(t)$, $t \in A \subseteq \mathbb{R}^n$. In addition, we assume that $g$ is obtained from some other signal $f(x)$, $x \in \Omega$, through a linear space variant system,

$$g(t) = \int_{\Omega} K(t,x)\, f(x)\, dx, \qquad t \in \mathbb{R}^n, \tag{1}$$

where $K$ is known. The goal is to recover the "real" object $f(x)$, $x \in \Omega$, from a finite set of samples of the "observed" signal $g(t)$, when $t \in A$. This problem is very well known in the engineering literature [8] and has been extensively studied in the mathematical literature [1].

A very important case is obtained from (1) when $K(t,x) = k(t - x)$, $t,x \in \mathbb{R}^n$. In that case, $g$ and $f$ satisfy the following relationship:

$$\hat{g}(w) = \hat{k}(w)\widehat{J_\Omega f}(w), \qquad w \in \mathbb{R}^n,$$

where $\hat{}$ denotes the Fourier transform and $J_\Omega f(x) = f(x)$ if $x \in \Omega$, and 0 elsewhere. It is clear that

$$\frac{\hat{g}(w)}{\hat{k}(w)} = \widehat{J_\Omega f}(w), \qquad \text{for all } w\text{: } \hat{k}(w) \neq 0.$$

Let us assume that $\hat{k}(w) \neq 0$ for $w \in N$, where $N$ contains a nonvoid open set of $\mathbb{R}^n$. Since $J_\Omega f$ has compact support (if $\Omega$ is compact) then $\widehat{J_\Omega f}$ is analytic; so the knowledge of $\widehat{J_\Omega f}(w)$, when $w \in N$, will be enough to determine $\widehat{J_\Omega f}(w)$ for any other $w \in \mathbb{R}^n$. In many applications, $\widehat{J_\Omega f}(w)$, $w \in \mathbb{R}^n$ will describe by itself all the information that we need from $J_\Omega f$. If this is not the case, then we should proceed to compute $J_\Omega f$ from $\widehat{J_\Omega f}(w)$, $w \in \mathbb{R}^n$. However, we have assumed, so far, that $\hat{g}(w)$ is known exactly for all $w \in \mathbb{R}^n$. This will not be the case if $g(t)$ is observed on the set $A \neq \mathbb{R}^n$, or on a finite subset of $A$ only. This shows that if we can improve our knowledge of $g$ [i.e., to know $g(t)$, when $t \notin A$] we will obtain a better knowledge of $\hat{g}$ [i.e., to compute $\hat{g}(w)$ more accurately].

In many cases, $\hat{k}(w) = 0$ if $w \notin N$, where $N$ is assumed to be compact. Therefore $\hat{g}(w) = 0$, $w \notin N$, which assures that $g$ will also be an analytic function. This means that the set of values $g(t)$, $t \in A$ will determine $g(t)$,

$t \notin A$. This shows that the solution $f$ to equation

$$g(z) = \int_{\Omega} k\,(z - x)\, f(x)\, dx$$

can be approached by solving a continuation problem for two analytic functions: given $g(z)$, $z \in A$, and $J\hat{\Omega}f(w)$, $w \in N$. It is important to note that the continuation of $J\hat{\Omega}f$ can be stated in the sense of Eq. (1) since

$$J\hat{\Omega}f(w) = \int_{\Omega} e^{-2\pi i x w}\, f(x)\, dx$$

and $e^{-2\pi i x w}$ plays the role of $K(w,x)$. This latter continuation problem shows an example of the importance of considering space-variant kernels. However, in this paper we will concern ourselves with the analytic continuation of band-limited functions only. That is to say when $g$ is given as in (1) with $K(w,x) = e^{-2\pi i w x}$.

One motivation for the continuation problem we give above is the restoration of $f$. However, there is another motivation: In many cases, we are interested in obtaining knowledge of $g(x)$, when $x \notin A$, and $x$ is "close" to $A$. Some examples of this situation are known in multidimensional signal processing. Let us suppose we apply a filter to a given image. The filter ideally performs over an infinite extent image. However, in practice we are given only a piece of the image. When the filter is applied to points close to or on the border of the real image, inaccuracies will result if we assume some arbitrary numbers for the unknown values of the image outside the border (e.g., the image is assumed to be periodic; or a constant number is assumed for the unknown values). The filter would improve its performance if we could fill in the unknown values of the image with some extrapolated information. Thus, small amounts of extrapolation (i.e., to extrapolate a small region beyond the boundary of $A$) can be of great help.

We now state the band-limited signal extrapolation problem in a more precise way. Let us suppose that $g$: $\mathbb{R}^n \to C$ ($C$ denotes the set of complex numbers) is a multidimensional signal which is of finite-energy; that is,

$$\int_{\mathbb{R}^n} |\, g\,|^2\, dx < \infty.$$

In that case, $g$ has a continuous Fourier transform $\hat{g}$ which satisfies

$$\int_{\mathbb{R}^n} |\, \hat{g}\,|^2\, dw = \int_{\mathbb{R}^n} |\, g\,|^2\, dx$$

and therefore $\hat{g}$ is also of finite-energy. Let us assume that $g$ is band-limited to some bounded set $\Omega \subseteq \mathbb{R}^n$. This means that $\hat{g}$ satisfies

$$\hat{g}(w) = 0, w \notin \Omega.$$

It is well known that if $g$ is a finite-energy band-limited function, then $g$ is an analytic function on $C^n$ [7].

Now, if we are given a piece of the function $g$, say $g: A \to C$, where $A$ is a nonvoid open subset of $\mathbb{R}^n$, then we will be able to recover the values $g(x)$ for $x \notin A$ because they are uniquely determined by $\{g(x), x \notin A\}$. We can now state that the band-limited signal extrapolation problem is (under the conditions stated above):

$$\text{given} \quad g(x), \qquad x \in A, \tag{2a}$$

$$\text{find} \quad g(x), \qquad x \notin A. \tag{2b}$$

To end this introductory section we will briefly outline the contents of this paper. In Section 2, continuous band-limited extrapolation is considered. Some known algorithms for solving this problem are reviewed. In Section 3, four basic models for extrapolation are presented. These models are useful in understanding the relationship between the continuous extrapolation problem and some discrete algorithms. In Section 4, these discrete algorithms are considered, certain generalizations are presented, and the relationship between discrete and continuous algorithms is established. We also include some numerical comparison among these techniques. Section 5 presents a unification of iterative algorithms for extrapolation. This is possible because of a general theory for solving linear equation in Hilbert spaces. Section 6 concerns another main issue in band-limited signal extrapolation: noise. Different approaches to this problem, algorithms, and abundant numerical simulation results are discussed. In Section 7, two-step discrete procedures are also unified. We will show that these techniques are special cases of certain procedures designed for solving a more general problem. Finally, in Section 8, several new directions for future research are outlined.

## 2. CONTINUOUS EXTRAPOLATION

### 2.1 Problem Statement

Let $g: \mathbb{R}^n \to C$ be a $\Omega$-band-limited finite-energy function. The *continuous-continuous* band-limited signal extrapolation is, as was stated in Section 1,

$$\text{given} \quad g(x), \qquad x \in A, \tag{2a}$$

$$\text{find} \quad g(x), \qquad x \notin A. \tag{2b}$$

We will always suppose that $A \subseteq \mathbb{R}^n$ is a nonvoid open set. It is clear that the solution to problem (2a)–(2b) is unique. This is simply because $g$ is an analytic function. In fact, since $\hat{g}$ is also a finite energy function and $\hat{g}(w) = 0$, $w \notin \Omega$, then

$$g(x) = \int_{\Omega} \hat{g}(w) e^{-2\pi i w x} \, dw. \tag{3}$$

This formula ensures that $g$ is analytic. In addition some bounds for $g$ in terms of the smoothness of its Fourier transform can be derived [7].

### 2.2 Algorithms for Continuous Extrapolation

Taking into account the analyticity of $g$, we may try to use expansions in power series to calculate $g(x)$ when $x \notin A$. However, these expansions always involve the computation of high-order derivatives. The reason for avoiding such an approach is twofold. First of all the derivatives of $g$ are *not* observable that is to say, we just know $g(x)$, $x \in A$. On the other hand, estimations of the derivatives from the given data are always very sensitive to noise (and the noisy case is precisely the situation we encounter in practice).

One of the first attempts to solve the extrapolation problem was made by Slepian and Pollack [27]. These authors used the Prolate Spheroidal Wave Functions, which are the eigenvectors of the following integral equation:

$$\lambda \phi(x) = \int_{-a}^{a} \operatorname{sinc}_{\Omega}(x - t) \phi(t) \, dt, \qquad x \in A = [-a, a],$$

where $\operatorname{sinc}_{\Omega}$ denotes the function whose Fourier transform is the indicator of $[-\Omega, \Omega]$. Let us call $\{(\lambda_i, \phi_i): i = 0, 1, \ldots\}$ the family of eigenfunctions and their corresponding eigenvalues $\lambda_i$, such that

$$\int_{-a}^{a} \phi_i(x) \phi_j(x) \, dx = \delta_{ij} \lambda_i.$$

Then, the function $g$ can be written as

$$g(x) = \sum_{j=0}^{\infty} c_j \, \phi_j(x), \qquad x \in \mathbb{R}, \tag{4}$$

where

$$c_j = \frac{1}{\lambda_j} \int_{-a}^{a} g(x) \phi_j(x) \, dx. \tag{5}$$

It is observed that $c_j$ is computed by using $g(x)$, $x \in A$, only and therefore the extrapolation (4) is obtained from $g(x)$, $x \in A$.

This technique has been very helpful in understanding the extrapolation problem. However, it has several drawbacks. In addition to the large computational efforts required to calculate $\lambda_i$ and $\phi_i$, a major difficulty arises in the computation of those eigenvectors corresponding to the "smaller" eigenvalues. The number of eigenvectors which are necessary to get a good approximation in (3) depends on $g$ and $\lambda_i$'s and cannot be determined readily from $g(x)$, $x \in [-a,a]$. Another disadvantage of this approach is that the series (3) will not converge if $g$ is contaminated with noise; though only a finite number of terms in the summation (3) can be tried as a "smoothing" technique, the determination of that number seems to be a very difficult matter.

It is worth pointing out that the technique described above is for one-dimensional band-limited signals. For the multidimensional case, attempts to generalize the one-dimensional approach were made in [28]. However, some limitations related to the form of the known region $A$ seem unavoidable.

Another well-known technique for solving the extrapolation problem is that of [5] and [15]. This iterative procedure is known in the engineering literature as the Papoulis–Gerchberg algorithm and is given by the following formula:

$$g_0 = 0$$

$$g_n = \mathrm{sinc}_\Omega * (J_A g + (I - J_A)g_{n-1}), \qquad n \geq 1, \tag{6}$$

where $J_A$ denotes truncation to the set $A$ and $*$ denotes convolution. In Reference [15], it was shown that $g_n$ converges to $g$ uniformly over the real line $\mathbb{R}$. The proof makes use of the Prolate Spheroidal Wave Functions, and therefore it is valid only for one-dimensional signals.

It is easily seen [25] that the recursion (6) is equivalent to the following formula:

$$g_0 = 0$$

$$g_n = g_{n-1} + \mathrm{sinc}_\Omega * J_A(g - g_{n-1}), \qquad n > 1. \tag{7}$$

Again, formula (6) or (7) is not convergent when $g$ is corrupted with noise. However, $g_n$ might provide a good approximation to the extrapolation after a certain number of iterations. But unfortunately, no criterion for stopping the recursion is known.

Cadzow proved the convergence of (7) by means of a rather different argument which does not make use of the Prolates [3]. This approach can be used to prove the convergence of (7) for multidimensional signals. However, in Section 5 (see also [17]) we will show that this well-known Papoulis–Gerchberg algorithm [(6) or (7)] is a special case of a quite general iterative procedure given by Landweber in 1951 [14].