

themes in modern econometrics

# Statistics and Econometric Models

VOLUME TWO

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Christian Gourieroux  
and Alain Monfort

Translated by Quang Vuong

# Statistics and Econometric Models

VOLUME 2

*Testing, Confidence Regions, Model Selection,  
and Asymptotic Theory*

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# Contents

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|   |             |           |
|---|-------------|-----------|
| <b>14 Introduction to Tests of Hypotheses</b>   | <i>page</i> | <b>1</b>  |
| 14.1 Testing Theory and Modelling   |             | 1         |
| 14.2 Hypotheses   |             | 2         |
| 14.3 Examples   |             | 3         |
| 14.4 Definition of a Test   |             | 6         |
| 14.5 Types of Errors  |             | 7         |
| 14.6 Ordering Hypothesis Tests  |             | 9         |
| 14.7 Choice of a Test   |             | 12        |
| 14.7.1 Bayesian Principle   |             | 13        |
| 14.7.2 Neyman Principle   |             | 14        |
| 14.8 Exercises  |             | 15        |
| 14.9 References   |             | 16        |
| <b>15 Uniformly Most Powerful Tests</b>   |             | <b>17</b> |
| 15.1 Tests of Simple Hypotheses   |             | 17        |
| 15.1.1 Risk Diagram   |             | 17        |
| 15.1.2 Neyman–Pearson Theorem   |             | 22        |
| 15.1.3 Application to Exponential Families  |             | 26        |
| 15.2 Tests of One-Sided Hypotheses  |             | 28        |
| 15.2.1 Monotone Likelihood Ratio Families   |             | 28        |
| 15.2.2 Application to Exponential Families  |             | 30        |
| 15.2.3 Locally Uniformly Most Powerful Tests and Score Tests                                      |             | 31        |
| 15.3 Tests of Two-Sided Hypotheses  |             | 35        |
| 15.3.1 Case Where $\Theta_0 = \{\theta : \theta \leq \theta_1 \text{ or } \theta \geq \theta_2\}$ |             | 36        |
| 15.3.2 Case Where $\Theta_0 = \{\theta : \theta_1 \leq \theta \leq \theta_2\}$                    |             | 36        |
| 15.4 Exercises  |             | 37        |
| 15.5 References   |             | 39        |

|  |           |
|--|-----------|
| <b>16 Unbiased Tests and Invariant Tests</b>                                       | <b>41</b> |
| 16.1 Unbiased Tests  | 41        |
| 16.1.1 Unbiased Tests and $\alpha$ -Similar Tests                                  | 41        |
| 16.1.2 UMPU Tests in the Absence of Nuisance Parameters                            | 43        |
| 16.1.3 UMPU Tests in the Presence of Nuisance Parameters                           | 47        |
| 16.1.4 Locally Uniformly Most Powerful Unbiased Tests                              | 65        |
| 16.2 Invariant Tests   | 70        |
| 16.2.1 Models and Problems With Invariant Tests                                    | 70        |
| 16.2.2 Invariant and Maximal Invariant Tests                                       | 71        |
| 16.2.3 Application: Fisher Tests   | 72        |
| 16.3 Exercises   | 77        |
| 16.4 References  | 79        |
| <b>17 Likelihood Based Tests</b>   | <b>81</b> |
| 17.1 Wald, Score, and Likelihood Ratio Tests: Principles                           | 81        |
| 17.2 Classical Tests of $g(\theta) = 0$  | 82        |
| 17.2.1 Wald Test   | 84        |
| 17.2.2 The Score or Lagrange Multiplier Test                                       | 86        |
| 17.2.3 The Likelihood Ratio Test   | 90        |
| 17.2.4 Equivalent Tests and Independence Property                                  | 93        |
| 17.3 Examples of Tests of $g(\theta) = 0$  | 96        |
| 17.3.1 The Linear Model With Known Variance  | 96        |
| 17.3.2 The Linear Model With Unknown Variance                                      | 98        |
| 17.3.3 The Nonlinear Gaussian Regression Model                                     | 101       |
| 17.4 Classical Tests of $\theta = h(\gamma)$                                       | 103       |
| 17.4.1 A Wald-Type Test  | 103       |
| 17.4.2 Score Test  | 106       |
| 17.4.3 Likelihood Ratio Test   | 107       |
| 17.4.4 Equivalent Tests and Independence Property                                  | 108       |
| 17.5 Examples of Tests of a Null Hypothesis in Explicit Form: The Chi-Square Tests | 109       |
| 17.5.1 Tests of Goodness-of-Fit to a Family of Distributions                       | 109       |
| 17.5.2 Tests of Independence   | 111       |
| 17.5.3 Goodness-of-Fit Tests to a Distribution                                     | 113       |
| 17.6 Score Tests and Augmented Regressions   | 114       |
| 17.6.1 Omitted-Variable Tests in Nonlinear Regression Models                       | 114       |
| 17.6.2 Augmented Regression Principle  | 115       |

|           |  |            |
|-----------|--|------------|
| 17.6.3    | A Homoscedasticity Test  | 116        |
| 17.6.4    | A Test of Independence of the Errors                                       | 117        |
| 17.7      | Score Tests and Generalized Residuals                                      | 119        |
| 17.7.1    | Latent Exponential Model and Score Vector                                  | 119        |
| 17.7.2    | Application to the Probit Model  | 121        |
| 17.8      | Neyman or $C(\alpha)$ Tests  | 122        |
| 17.8.1    | Principle  | 122        |
| 17.8.2    | Example  | 125        |
| 17.9      | Hausman (Likelihood Based) Tests   | 125        |
| 17.9.1    | Test Description   | 125        |
| 17.9.2    | Examples   | 128        |
| 17.10     | Exercises  | 130        |
| 17.11     | References   | 133        |
| <b>18</b> | <b>General Asymptotic Tests</b>  | <b>135</b> |
| 18.1      | General Asymptotic Tests   | 135        |
| 18.1.1    | Tests Using the Constrained Maximization of $L_n(\theta)$                  | 136        |
| 18.1.2    | Tests Based on the Unconstrained Estimator $\hat{\theta}_n$<br>of $\theta$ | 145        |
| 18.1.3    | Tests Based on the Function $g(\hat{\theta}_n, a)$                         | 147        |
| 18.1.4    | An Example: Tests Based on Least Squares                                   | 150        |
| 18.2      | ALS Based Specification Tests  | 153        |
| 18.2.1    | General Principle  | 153        |
| 18.2.2    | Unknown Linear Constraints on Functions of the<br>Parameter $\theta$       | 155        |
| 18.2.3    | Examples   | 156        |
| 18.3      | GMM Based Specification Tests  | 160        |
| 18.3.1    | General Result   | 160        |
| 18.3.2    | Examples   | 163        |
| 18.4      | Hausman Specification Tests  | 166        |
| 18.4.1    | Principle  | 166        |
| 18.4.2    | Example: An Exogeneity Test Based on an Aug-<br>mented Regression          | 167        |
| 18.5      | Information Matrix Test  | 170        |
| 18.6      | Exercises  | 174        |
| 18.7      | References   | 176        |
| <b>19</b> | <b>Multiple Tests</b>  | <b>179</b> |
| 19.1      | The Linear Model   | 179        |
| 19.1.1    | Nested Hypotheses  | 179        |
| 19.1.2    | The Testing Procedure  | 180        |

## Contents

|           |  |            |
|-----------|--|------------|
| 19.1.3    | Interpretation in Terms of Correlation Coefficients                            | 183        |
| 19.2      | Implicit Form: The General Case  | 184        |
| 19.2.1    | Test of a Hypothesis When the Maintained Hypothesis is Already Constrained     | 184        |
| 19.2.2    | Sequential Tests of Nested Hypotheses in Implicit Form                         | 195        |
| 19.2.3    | Example  | 197        |
| 19.3      | Explicit Form: The General Case  | 198        |
| 19.3.1    | Test of a Hypothesis When the Maintained Hypothesis is Already Constrained     | 198        |
| 19.3.2    | Sequential Tests of Nested Hypotheses in Explicit Form                         | 200        |
| 19.3.3    | Example: Tests for a Common Root   | 200        |
| 19.4      | General Multiple Tests   | 201        |
| 19.4.1    | Tests of Nonnested Null Hypotheses in a Given Model                            | 202        |
| 19.4.2    | Tests of a Given Null Hypothesis Against Different Models                      | 205        |
| 19.5      | Exercises  | 210        |
| 19.6      | References   | 211        |
| <b>20</b> | <b>Set Estimation and Confidence Regions</b>                                   | <b>213</b> |
| 20.1      | Definitions  | 213        |
| 20.2      | Examples   | 214        |
| 20.2.1    | Interval Estimation of the Mean of a Normal Distribution With Known Variance   | 214        |
| 20.2.2    | Interval Estimation of the Mean of a Normal Distribution With Unknown Variance | 215        |
| 20.2.3    | Confidence Region for the Mean and the Variance of a Normal Distribution       | 215        |
| 20.2.4    | Confidence Region for the Parameters of a Linear Model                         | 217        |
| 20.3      | Conservative Confidence Regions  | 218        |
| 20.4      | Optimality of a Confidence Region  | 219        |
| 20.5      | Examples of UMPU Confidence Regions  | 223        |
| 20.5.1    | Interval Estimation of the Mean of a Normal Distribution With Unknown Variance | 223        |
| 20.5.2    | Confidence Interval for the Parameters of a Linear Model                       | 224        |
| 20.6      | Asymptotic Confidence Regions  | 224        |

|           |   |            |
|-----------|---|------------|
| 20.7      | Examples of Asymptotic Confidence Regions   | 227        |
| 20.8      | Bayesian Confidence Regions   | 228        |
| 20.9      | Exercises   | 229        |
| 20.10     | References  | 231        |
| <b>21</b> | <b>Inequality Constraints: Estimation and Testing</b>                                       | <b>233</b> |
| 21.1      | Some Examples   | 233        |
| 21.1.1    | Monotony Constraints  | 234        |
| 21.1.2    | Barycentric Interpretation of Some Parameters   | 234        |
| 21.1.3    | Positive Definiteness of Symmetric Matrices   | 235        |
| 21.1.4    | Stability Conditions  | 236        |
| 21.2      | Estimation Under Inequality Constraints   | 237        |
| 21.2.1    | Constrained Estimation of the Linear Model  | 237        |
| 21.2.2    | General Results   | 243        |
| 21.3      | $H_0 : \{g(\theta) = 0\}$ versus $H_1 : \{g(\theta) \geq 0 \text{ and } g(\theta) \neq 0\}$ | 247        |
| 21.3.1    | The Main Test Statistics  | 247        |
| 21.3.2    | Asymptotically Equivalent Optimization Problems   | 253        |
| 21.3.3    | Asymptotic Distribution of the Test Statistics Under the Null Hypothesis $H_0$              | 255        |
| 21.3.4    | Examples  | 261        |
| 21.4      | Tests of $H_0 : \{g(\theta) \geq 0\}$   | 266        |
| 21.4.1    | Some General Remarks  | 266        |
| 21.4.2    | A Special Case  | 267        |
| 21.4.3    | An Example  | 271        |
| 21.5      | Exercises   | 272        |
| 21.6      | References  | 273        |
| <b>22</b> | <b>Nonnested Models</b>   | <b>275</b> |
| 22.1      | Kullback Proximity and Pseudo True Values   | 276        |
| 22.1.1    | Definition  | 276        |
| 22.1.2    | Pseudo True Values and Proximities  | 278        |
| 22.2      | Tests of Nonnested Hypotheses   | 282        |
| 22.2.1    | Nonnested Hypotheses  | 282        |
| 22.2.2    | Generalized Wald Tests  | 283        |
| 22.2.3    | Generalized Score Tests   | 287        |
| 22.2.4    | Selection of Regressors in Linear Models  | 288        |
| 22.2.5    | Choice Among Qualitative Variable Models  | 290        |
| 22.2.6    | Cox Testing Procedure   | 293        |
| 22.2.7    | Using a Nesting Model   | 297        |



|           |  |            |
|-----------|--|------------|
| 22.3      | Model Selection Criteria   | 304        |
| 22.3.1    | Linear Models  | 304        |
| 22.3.2    | The Akaike Criterion   | 307        |
| 22.3.3    | Bayesian Criteria  | 312        |
| 22.4      | Appendix 1   | 313        |
| 22.5      | Appendix 2   | 317        |
| 22.6      | Exercises  | 320        |
| 22.7      | References   | 322        |
| <b>23</b> | <b>Asymptotic Efficiency</b>   | <b>325</b> |
| 23.1      | Asymptotic Comparison of Estimators  | 325        |
| 23.1.1    | First-Order Asymptotic Efficiency  | 326        |
| 23.1.2    | Higher-Order Expansions  | 334        |
| 23.1.3    | Edgeworth Expansion  | 343        |
| 23.2      | Asymptotic Comparison of Tests   | 345        |
| 23.2.1    | Statement of the Problem and Various Solutions                               | 345        |
| 23.2.2    | Concepts   | 350        |
| 23.2.3    | Bahadur's Relative Efficiency  | 354        |
| 23.2.4    | Pitman's Relative Efficiency   | 362        |
| 23.2.5    | Direct Approach Based on Local Alternatives                                  | 366        |
| 23.3      | Semiparametric Efficiency  | 370        |
| 23.3.1    | Information Contained in Identifying Conditions                              | 370        |
| 23.3.2    | Asymptotically Efficient Semiparametric Estimators                           | 376        |
| 23.3.3    | A Two-Step Method For Obtaining an Asymptotically Efficient Estimator        | 377        |
| 23.4      | References   | 380        |
| <b>24</b> | <b>Asymptotic Theory</b>   | <b>383</b> |
| 24.1      | Existence of an Extremum Estimator   | 383        |
| 24.2      | Consistency of an Extremum Estimator   | 386        |
| 24.2.1    | Almost Sure Consistency  | 386        |
| 24.2.2    | Some Consistency Results   | 387        |
| 24.2.3    | Conditions for the Uniform Almost Sure Convergence of the Objective Function | 388        |
| 24.2.4    | Consistency of Quasi Generalized Extremum Estimators                         | 392        |
| 24.3      | First-Order Conditions   | 394        |
| 24.3.1    | Unconstrained Extremum Estimators  | 394        |

|   |  |            |
|---|--|------------|
| 24.3.2  | Constrained Extremum Estimators  | 395        |
| 24.4  | Taylor Series Expansions   | 397        |
| 24.4.1  | Infinitely Small Quantities in Probability   | 398        |
| 24.4.2  | Taylor Series Expansion of the First-Order<br>Conditions and Asymptotic Normality                                      | 401        |
| 24.5  | References   | 404        |
| <b>Review of Linear Algebra and Matrix Calculus</b> |  | <b>405</b> |
| A.1   | Linear Algebra and Notations   | 405        |
| A.1.1   | Vector Spaces  | 405        |
| A.1.2   | Linear Mappings  | 406        |
| A.1.3   | Bases  | 406        |
| A.1.4   | Supplementary Subspaces  | 407        |
| A.1.5   | Matrix Representations   | 408        |
| A.1.6   | Matrix Operations  | 409        |
| A.1.7   | Change of Bases  | 411        |
| A.2   | Some Complements on Matrices   | 413        |
| A.2.1   | Trace of a Matrix  | 413        |
| A.2.2   | Partitioned Matrices   | 414        |
| A.3   | A Review of Quadratic Forms  | 420        |
| A.3.1   | Definition and Characterization  | 420        |
| A.3.2   | Positive Quadratic Forms   | 421        |
| A.3.3   | Symmetric Matrix Representations of a<br>Linear Mapping  | 423        |
| A.3.4   | Relationships among $\ker \mathbf{A}$ , $\text{Im } \mathbf{A}$ , $\ker \mathbf{A}'$ , and<br>$\text{Im } \mathbf{A}'$ | 423        |
| A.4   | Ordering Symmetric Matrices  | 424        |
| A.4.1   | Definition of an Order Relation  | 424        |
| A.4.2   | Comparing Ordered Spectra  | 425        |
| A.4.3   | Square Root of a Positive Semidefinite Matrix  | 427        |
| A.4.4   | A Geometric Interpretation   | 429        |
| A.5   | Projections  | 430        |
| A.5.1   | Definition and Characterization  | 431        |
| A.5.2   | Matrix Representations of Projections  | 433        |
| A.5.3   | Some Properties of Orthogonal Projections  | 434        |
| A.5.4   | An Optimal Property of Orthogonal Projections  | 436        |
| A.6   | Generalized Inverses   | 437        |
| A.6.1   | Definition and Construction  | 437        |
| A.6.2   | Invariance Properties  | 439        |
| A.6.3   | Solving Linear Systems of Equations  | 441        |

|   |            |
|---|------------|
| <b>Review of Probability</b>  | <b>443</b> |
| B.1 Review of Integration Theory  | 443        |
| B.1.1 Random Experiments  | 443        |
| B.1.2 Characterization of a Probability Distribution or a Measure                         | 444        |
| B.1.3 Integration with Respect to a Measure   | 446        |
| B.1.4 Density Functions   | 447        |
| B.1.5 Random Variables  | 448        |
| B.1.6 Properties of the Expectation of a Real Random Variable                             | 449        |
| B.1.7 Some Descriptive Measures of the Probability Distribution of a Real Random Variable | 450        |
| B.1.8 Some Probability Distributions on $\mathcal{R}$                                     | 450        |
| B.1.9 Random Vectors and Matrices   | 451        |
| B.2 Independence and Conditioning   | 458        |
| B.2.1 Conditional Probability Distributions   | 458        |
| B.2.2 Conditional Expectations  | 461        |
| B.2.3 Properties of the Conditional Expectation   | 462        |
| B.2.4 Linear Regression   | 465        |
| B.2.5 The Partial Variance Covariance Matrix  | 468        |
| B.2.6 Independence  | 470        |
| B.2.7 Conditional Independence  | 473        |
| B.3 Normal Distributions  | 475        |
| B.3.1 Univariate Normal Distributions   | 475        |
| B.3.2 Multivariate Normal Distributions   | 477        |
| B.3.3 Independence and Conditioning   | 480        |
| B.3.4 Truncated Normal Distributions  | 483        |
| B.4 Distributions Derived from the Normal Distribution                                    | 487        |
| B.4.1 Chi-Square Distributions  | 487        |
| B.4.2 Quadratic Forms of Gaussian Vectors   | 490        |
| B.4.3 Student Distributions   | 492        |
| B.4.4 Fisher Distributions  | 495        |
| B.5 Stochastic Processes  | 496        |
| B.5.1 Definition  | 496        |
| B.5.2 Types of Convergence of Random Vectors  | 497        |
| B.5.3 An Example of Stationary Processes: The Autoregressive Process of Order 1           | 499        |
| B.5.4 Convergence in Distribution   | 501        |
| B.5.5 Poisson Processes   | 502        |
| B.6 Asymptotic Theorems   | 504        |

|       |  |     |
|-------|--|-----|
| B.6.1 | Laws of Large Numbers                                  | 504 |
| B.6.2 | Central Limit Theorems                                 | 506 |
| B.6.3 | Applications to Continuous Differentiable<br>Functions | 509 |
| B.6.4 | Other Asymptotic Theorems                              | 510 |

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# *Introduction to Tests of Hypotheses*

## 14.1 Testing Theory and Modelling

The purpose of the methods described in the first volume is to specify and estimate, on the basis of the data, a model of which the validity is not questioned. On the contrary, in the theory of hypothesis testing, which is the general topic of the second volume the validity of the model is now challenged.

For instance, one may wonder whether the specified model is not too “large,” i.e., whether a submodel defined by a subset of the family  $\mathcal{P}$  of possible probability distributions is not preferable. This is the basis of *significance tests*. Conversely, one may wonder whether the specified model is not too restrictive, i.e., whether the true distribution that has generated the observations actually belongs to  $\mathcal{P}$ . In the latter case, one frequently talks about *specification tests*. As a matter of fact, we shall not make a distinction between these two kinds of testing situations, for the approach that is generally considered in specification testing is to nest  $\mathcal{P}$  in a larger family and to examine whether  $\mathcal{P}$  is an acceptable restriction of this larger family. Hence the second problem reduces to the first problem. There exists, however, another approach to the problem of specification testing. This will be discussed in Chapter 22 when studying *nonnested hypotheses tests*.

Over the last fifty years, the statistical methods of the theory of hypothesis testing have considerably developed under the impulse of statisticians such as J. Neyman, E. Lehman and A. Wald. As for the theory of statistics in general, this development has its source in the increasing role of probabilistic modelling as a scientific tool. Another reason, however, which may be more fundamental and specific to the

theory of hypothesis testing, is the progressive disappearance of the idea that a model can be validated with certainty on the basis of the data. Such an idea, which was frequently held during the nineteenth century, was gradually forsaken especially at the beginning of the twentieth century when physicists started to question the theory of classical mechanics considered up to then as the definitive theory. The relative value of a model in a collection of competing models then proved to be a valuable concept. As a natural consequence, the theory of hypothesis testing, whose main purpose is to arbitrate among models, received an increasing interest.

## 14.2 Hypotheses

A testing problem is defined by a statistical model  $(\mathcal{Y}, \mathcal{P})$  and by a partitioning of the family  $\mathcal{P}$  into two subfamilies  $\mathcal{P}_0$  and  $\mathcal{P}_1 = {}^c\mathcal{P}_0$ . These two subfamilies define respectively two *hypotheses* about the true distribution  $P_0$  generating the observations, namely

$$H_0 : P_0 \in \mathcal{P}_0$$

and

$$H_1 : P_0 \in \mathcal{P}_1.$$

It is frequently convenient to identify  $H_0$  with  $\mathcal{P}_0$  and  $H_1$  with  $\mathcal{P}_1$ . Although the two hypotheses  $H_0$  and  $H_1$  play a symmetric role in this section, they are given hereafter two different names:  $H_0$  is called the *null hypothesis* while  $H_1$  is called the *alternative hypothesis*. The union of  $H_0$  and  $H_1$  defines the hypothesis  $H : P_0 \in \mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ , which is called the *general* or *maintained hypothesis*.

**Definition 14.1:** *A hypothesis is called simple if it contains a unique probability distribution. It is called composite otherwise.*

In a parametric model  $(\mathcal{Y}, \{P_\theta, \theta \in \Theta\})$  the hypotheses  $H_0$  and  $H_1$  are defined, in general, by two subsets  $\Theta_0$  and  $\Theta_1 = {}^c\Theta_0$  of  $\Theta$ . When the model is identified, such a definition of the null and alternative hypotheses is identical to that based on a partition of  $\mathcal{P}$  into  $\mathcal{P}_0$  and  $\mathcal{P}_1$ . This is because the mapping that associates  $P_\theta$  to  $\theta$  is a one-to-one and onto mapping, i.e., a bijective mapping from  $\Theta$  to  $\mathcal{P}$ . When the model is not identified, however, a difficulty arises. Specifically, there may exist values for the parameter  $\theta_0 \in \Theta_0$  and  $\theta_1 \in \Theta_1$  leading to the same probability distribution, i.e., such that  $P_{\theta_0} = P_{\theta_1}$ . In this case the subsets  $\mathcal{P}_0 = \{P_\theta, \theta \in \Theta_0\}$  and  $\mathcal{P}_1 = \{P_\theta, \theta \in \Theta_1\}$  are no longer disjoint.

**Definition 14.2:** A testing problem defined by  $\Theta_0$  and  $\Theta_1 = {}^c\Theta_0$  is identified if  $P_{\theta_0}$  is different from  $P_{\theta_1}$  for every  $\theta_0 \in \Theta_0$  and  $\theta_1 \in \Theta_1$ .

It is obvious that a model is identified if and only if every testing problem is identified. As Example 14.4 illustrates, however, *some* testing problems can be identified even though the model is not identified.

## 14.3 Examples

**Example 14.1:** A machine produces steel balls whose diameters are independently and identically distributed as  $N(\theta, \sigma_0^2)$ . It is assumed that the accuracy  $\sigma_0^2$  of the machine is a characteristic known to the investigator and that the mean diameter  $\theta$  of the produced steel balls is a parameter that can be chosen. One observes  $n$  diameters  $Y_1, \dots, Y_n$  and one wishes to test whether the tuning of the machine corresponds to the posted value  $\theta_0$ .

In this example the statistical model is

$$(\mathbb{R}^n, \{(N(\theta, \sigma_0^2))^{\otimes n}, \theta \in \mathbb{R}^+\}).$$

The null hypothesis of good tuning is  $H_0: \theta = \theta_0$  and the alternative hypothesis is  $H_1: \theta \neq \theta_0$ . The null hypothesis is simple while the alternative hypothesis is composite.

**Example 14.2:** It is assumed that the production  $Q_t$  of a given commodity at time  $t$ ,  $t = 1, \dots, T$ , can be modelled by the Cobb–Douglas production function

$$\log Q_t = a + b \log N_t + c \log K_t + u_t,$$

where  $N_t$  denotes the quantity of labor input and  $K_t$  denotes the quantity of capital input. It is assumed that the random disturbances  $u_t$ ,  $t = 1, \dots, T$ , are independently and identically distributed as  $N(0, \sigma^2)$ .

The model is parametric. If  $\log Q_t$ ,  $t = 1, \dots, T$ , are viewed as the observations, the model is

$$\left( \mathbb{R}^T, \left\{ \bigotimes_{t=1}^T N(a + b \log N_t + c \log K_t, \sigma^2), (a, b, c, \sigma^2) \in \mathbb{R}^3 \times \mathbb{R}^+ \right\} \right).$$

One may want to test the hypothesis  $H_0$  of constant returns, i.e., the property that multiplying labor input and capital input by a same factor leads to multiplying production by this factor. Such a hypothesis

is identical to the condition that the Cobb–Douglas production function is homogenous of degree one. In terms of the parameters, this translates into the condition  $b + c = 1$ . It is clear that the two hypotheses  $H_0$  and  $H_1$  are composite. For instance,  $H_0$  is given by

$$\Theta_0 = \{(a, b, c, \sigma^2) \in \mathbb{R}^3 \times \mathbb{R}^+, b + c = 1\}.$$

One may also want to question the hypothesis of independence among the  $u_t$ 's. A method for dealing with such a “specification” testing situation is to nest the preceding model into a larger model where the disturbances  $u_t$ 's satisfy the first-order autoregressive process

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad |\rho| < 1$$

and the  $\varepsilon_t$ 's are independently and identically distributed as  $N(0, \sigma_\varepsilon^2)$ . The null hypothesis of independence of the  $u_t$ 's is then characterized by the condition  $\rho = 0$ .

**Example 14.3:** A consumption survey provides observations on health expenditures and incomes of  $n$  households,  $(C_i, R_i)$ ,  $i = 1, \dots, n$ . It is assumed that the pairs  $(C_i, R_i)$  are independently and identically distributed with unknown density  $f(c_i, r_i)$  with respect to the Lebesgue measure  $\lambda_2^+$  on  $\mathbb{R}^{+2}$ . The statistical model is defined by the family

$$\mathcal{P} = \left\{ \prod_{i=1}^n f(c_i, r_i) \cdot \lambda_{2n}^+, f \text{ arbitrary on } \mathbb{R}^{+2} \right\}.$$

Suppose that one wishes to test whether health expenditures are independent of incomes. The null hypothesis corresponds to the subfamily

$$\mathcal{P}_0 = \left\{ \prod_{i=1}^n g(c_i)h(r_i) \cdot \lambda_{2n}^+, g \text{ and } h \text{ arbitrary on } \mathbb{R}^+ \right\}.$$

The model is nonparametric. The null and alternative hypotheses are composite.

**Example 14.4:** At time  $t = 1, \dots, T$ , the quantity exchanged  $Q_t$  and the price  $p_t$  of an agricultural product are determined by the demand equation

$$Q_t = \alpha p_t + \beta x_{t-1} + \gamma z_{t-1} + \delta + u_t,$$

and the supply equation

$$Q_t = \alpha p_t^* + b + v_t,$$



where  $x_{t-1}$  and  $z_{t-1}$  are some variables treated as nonstochastic,  $u_t$  and  $v_t$  are zero-mean random errors uncorrelated contemporaneously and over time. The variable  $p_t^*$  denotes the producers' expectation at time  $t-1$  of price at time  $t$ . In addition, it is assumed that this expectation is a function of past exogenous variables and is given by

$$p_t^* = \phi_1 x_{t-1} + \phi_2 z_{t-1} + \phi_3.$$

Using this expression in the supply equation, one obtains

$$\begin{cases} Q_t = \alpha p_t + \beta x_{t-1} + \gamma z_{t-1} + \delta + u_t, \\ Q_t = \beta_1 x_{t-1} + \beta_2 z_{t-1} + \beta_3 + v_t, \end{cases}$$

where

$$\begin{aligned} \beta_1 &= a\phi_1, \\ \beta_2 &= a\phi_2, \\ \beta_3 &= b + a\phi_3. \end{aligned}$$

The parameters  $a, b, \phi_1, \phi_2$ , and  $\phi_3$  are not first-order identified although the parameters  $\beta_1, \beta_2$ , and  $\beta_3$  are.

One wishes to test whether price expectations are “rational,” i.e., whether they coincide with the optimal predictions  $p_t^* = E_{t-1}p_t$ , where  $E_{t-1}p_t$  is the conditional expectation of  $p_t$  given the variables known at time  $t-1$ . Taking first the conditional expectation of the demand and supply equations and then, the difference between the resulting equations, one obtains

$$p_t^* = E_{t-1}p_t = \frac{\beta}{a-\alpha}x_{t-1} + \frac{\gamma}{a-\alpha}z_{t-1} + \frac{\delta-b}{a-\alpha}.$$

Thus the null hypothesis of “rational” expectations can be written as

$$H_0 : \phi_1 = \frac{\beta}{a-\alpha}, \quad \phi_2 = \frac{\gamma}{a-\alpha}, \quad \phi_3 = \frac{\delta-b}{a-\alpha}.$$

The hypothesis  $H_0$  is not identified since  $\phi_1, \phi_2$ , and  $\phi_3$  are not identified.

The null hypothesis  $H_0$ , however, implies

$$H_0^* : \frac{\phi_1}{\phi_2} = \frac{\beta}{\gamma},$$

i.e.

$$H_0^* : \frac{\beta_1}{\beta_2} = \frac{\beta}{\gamma}.$$