ANALYSIS OF LINEAR SYSTEMS

by

DAVID K. CHENG

Department of Electrical Engineering
Syracuse University

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PREFACE

Among the myriad and ever-increasing subjects which an engineering student is required to master, few are more important than the techniques for analyzing linear systems. The study of linear systems is important for several reasons. First, a great majority of engineering situations are linear, at least within specified ranges. Second, exact solutions of the behavior of linear systems can usually be found by standard techniques. Third, the techniques remain the same irrespective of whether the problem at hand is one on electrical circuits, mechanical vibration, heat conduction, motion of elastic beams, or diffusion of liquids. Except for a very few special cases, there are no exact methods for analyzing nonlinear systems. Practical ways of solving nonlinear problems involve graphical or experimental approaches. Approximations are often necessary, and each situation usually requires special handling. Two essential steps are involved in the analysis of a physical system, namely, the formulation of mathematical equations that describe the system in accordance with physical laws, and the solution of these equations subject to appropriate initial or boundary conditions. This book attempts to furnish a thorough exposition of the techniques that are important in executing both of these steps for linear systems.

In the formulation of the equations that describe a physical system, emphasis is oriented toward electrical circuits. To deal with systems other than electrical, a chapter (Chapter 4) on analogous systems is included which treats in detail methods for drawing electrical circuits analogous to linear mechanical and electromechanical systems. This approach is advantageous because electrical engineers have developed a set of convenient symbols for circuit elements, so that a complex system can be set down with conventional symbols in the form of a circuit diagram from which the behavior of the system can be analyzed. Once the circuit diagram of the analogous electrical system is determined, it is possible to visualize and often predict important system behaviors by inspection. Moreover, electrical circuit-theory techniques, such as the use of the impedance concept and the various network theorems, can be applied in the actual analysis of the system.

One of the primary purposes of this book is to introduce the Laplace transform method of solving linear differential and integrodifferential equations. Fourier series and Fourier integral are first reviewed, and a discussion of Fourier transforms leads logically and directly to Laplace transforms. This method of introducing Laplace transformation is preferable to the unsatisfying approach of pulling the defining formulas out of thin air and applying them in a mechanical manner.

Although the Laplace transform method of solving linear differential and integrodifferential equations is, in many circumstances, simpler and more convenient to use than classical methods, I do not wish to minimize the importance of understanding the classical methods. I do not feel that the student should be led to believe that the Laplace transform method is superior to all other methods under all circumstances. First of all, there are definite limitations to the applicability of transform methods. For example, the Laplace transform method cannot conveniently be used to solve linear differential equations with variable coefficients even of the first order, while the classical approach can yield solutions to many such equations of practical importance. Second, when the known conditions of a problem are specified at values of the independent variable other than zero, the Laplace transform method becomes cumbersome to use even when the physical situation can be described by differential equations with constant coefficients. On the other hand, the application of classical methods is not modified by the way in which the known conditions are specified. Third, the separation of the general solution to a differential equation into a complementary function and a particular integral in the classical approach helps the understanding of the general nature of system response. It is not difficult to cite situations for which the complementary-function and particular-integral parts of the solutions can be written from the equations by inspection, while all steps in the formal procedure will have to be carried out in the transform method. Systems with constant or sinusoidal excitations are typical examples of such situations. A separate chapter (Chapter 2) is devoted to the discussion of classical methods for solving linear differential equations.

In applying the Laplace transform method, over-reliance on tables of transforms is discouraged. I strongly feel that a few fundamental transform pairs together with several important theorems should be remembered. It is realized that one cannot remember everything, but a good engineer or scientist should not be hopelessly ineffective without his tables or handbooks. The memory work involved is really very little. The tables of transforms in Appendix B are for reference purposes only, and students should not have to refer to them when they are learning the subject.

The complex Laplace inversion integral is derived in Chapter 6 from the Fourier integral, but evaluation of inverse Laplace transforms by contour integration in the complex plane is not attempted. This book does not include a chapter on the theory of functions of a complex variable. I am convinced that a superficial knowledge of the theory of functions of a complex variable serves no useful purpose in a book like this one. The inclusion of some introductory material on function theory may make the level of the book appear more advanced, but it would be a rather unnecessary and unrewarding digression. Experience has indicated that in order for a student to be able to evaluate inverse transforms of irrational functions (functions with branch points) a much better background on the theory of functions of a complex variable than could be offered in one or two short chapters is necessary. The use of function theory and Cauchy's residue theorem in connection with functions having pole singularities results in little advantage since Heaviside's expansion theorem can be applied with ease in these cases. I have chosen to discuss the inverse Laplace transforms of irrational functions in a direct manner (Sections 8-6 and 11-4) and carry the development far enough so that typical important systems with distributed parameters can be analyzed completely. A rigorous discussion of the intricacies of the inversion integral from the point of view of function theory is left to more advanced treatises.

The book begins with a chapter which explains in detail the characteristics of linear systems from both a physical and a mathematical viewpoint. General properties of linear differential equations are discussed. Chapter 2 presents the essentials of classical methods for solving linear differential equations.

Electrical circuit theory and methods of analyzing lumped-element electrical systems are carefully presented in Chapter 3, which should be well within the grasp of students in all branches of engineering, physics, and applied mathematics. Chapter 4 deals with analogous systems and discusses in detail methods for drawing electrical circuits analogous to linear mechanical and electromechanical systems.

Chapter 5 reviews Fourier series and Fourier integral. A discussion of Fourier transforms leads to Laplace transforms, which are introduced in Chapter 6.

Chapter 7 illustrates the applications of Laplace transformation. Impulse response, step response, convolution and superposition integrals, and other system concepts are discussed in Chapter 8. Inverse Laplace transforms of some irrational functions are also derived there.

Chapter 9 treats systems with feedback where both block-diagram and signal flow graph representations are used. System stability requirements are developed in detail.

Chapter 10 deals with sampled-data systems. Z transformation is introduced and stability requirements for sampled-data systems are examined. Also included are the method of solving difference equations by Z transformation and a modified Z transformation for determining the response between sampling instants.

Chapter 11 discusses systems with distributed parameters. Two appendixes, one on numerical solution of algebraic equations and the other containing transform tables, complete the book.

This book may be used by either advanced undergraduates or beginning graduate students. A few of the possible combinations of chapters that may serve various groups of students are suggested below:

1. Electrical engineering students with no prior knowledge of differential equations: Chapters 1, 2, 3, 5, 6, 7, 8.
2. Electrical engineering students who have had a course in ordinary differential equations: Chapters 1, 3, 4, 5, 6, 7, 8.
3. Electrical engineering graduate students: Chapters 1, 4, 6, 7, 8, 9, 10, 11.
4. Graduate students in mechanical engineering, physics, or applied mathematics: Chapters 1, 3, 4, 6, 7, 8, 9, 11.

Various other selections of material are of course possible, depending upon the nature of the course in the curriculum. The book provides enough material to prepare a student to go on to more advanced work in network theory, control systems, and vibrations.

This book was originally to be a joint project with Professor Norman Balabanian. Due to other commitments Professor Balabanian had to withdraw from this venture. I wish to thank him for a number of ideas which he contributed in the planning stage. To Professors William H. Huggins and William A. Lynch I wish to express my sincere appreciation for their many constructive suggestions. Thanks are also due Professor Richard A. Johnson, who reviewed parts of the manuscript and suggested improvements. The assistance of Mr. Mark Ma, who carefully read the galley proofs, is much appreciated. My special thanks go to my wife Enid, whose patience and understanding made the tedious book-writing task much easier to endure.

March, 1959 D. K. C.

CONTENTS

CHAPTER 1

CHARACTERISTICS OF A LINEAR SYSTEM

1-1 Introduction. The study of linear systems is important for two reasons: (1) a great majority of engineering situations are linear, at least within specified ranges; and (2) exact solutions of the behavior of linear systems can usually be found by standard techniques. Except for a very few special types, there are no standard methods for analyzing nonlinear systems. The practical ways of solving nonlinear problems involve graphical or experimental approaches. Approximations are often necessary, and each situation usually requires special handling. The present state of the art is such that there is neither a standard technique which can be used to solve nonlinear problems exactly, nor is there any assurance that a good solution can be obtained at all for a given nonlinear system. Hence, we are indeed fortunate that a great majority of engineering problems are linear and can be solved. However, we must realize that not all physical systems are linear without restrictions.

We are all familiar with the Ohm's law that governs the relation between the voltage across and the current through a resistor. It is a *linear* relationship because the voltage across a resistor is (linearly) proportional to the current through it. But even for this simple situation, the linear relationship does not apply under all conditions. For instance, as the current in a resistor is greatly increased, the value of its resistance will increase due to heat developed in the resistor, the amount of increase being dependent upon the magnitude of the current; and it is no longer correct to say that the voltage across the resistor bears a linear relationship to the current through it. The same can be said about Hooke's law, which states that the stress is (linearly) proportional to the strain of a spring. But this linear relationship breaks down when the stress on the spring is too great. When the stress exceeds the elastic limit of the material of which the spring is made, stress and strain are no longer linearly related. The actual relationship is much more complicated than the Hooke's law situation. We are therefore forewarned that restrictions always exist for linear physical situations; saturation, breakdown, or material changes will ultimately set in and destroy linearity. Under ordinary circumstances, however, physical conditions in many engineering problems stay well within the restrictions, and the linear relationship holds.

Ohm's law and Hooke's law describe only special linear systems. There exist systems that are much more complicated and so cannot be conveniently described by simple voltage-current or stress-strain relationships.

Other more universal criteria are necessary to establish that a system is linear. Linear systems are characterized by certain definite properties which make them simpler to describe physically and easier to solve mathematically. In the following sections, we shall examine the characteristics of a linear system from both a physical and a mathematical viewpoint.

1–2 Linear system from a physical viewpoint. An engineer's interest in a physical situation is very frequently the determination of the response of a system to a given excitation. Both the excitation and the response may be any physically measurable quantity, depending upon the particular problem. Figure 1–1 depicts such a situation. Suppose that an excitation function $e_1(t)$, which varies with time in a specified manner, produces a response function $w_1(t)$, and that a second excitation function $e_2(t)$ produces a second response function $w_2(t)$.

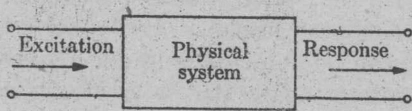

FIG. 1–1. A physical situation.

Symbolically, we may write

$$e_1(t) \rightarrow w_1(t), \tag{1-1}$$

$$e_2(t) \rightarrow w_2(t). \tag{1-2}$$

Then, for a linear system.

$$e_1(t) + e_2(t) \rightarrow w_1(t) + w_2(t). \tag{1-3}$$

Relation (1–3), in conjunction with (1–1) and (1–2), states that a superposition of excitation functions results in a response which is the superposition of the individual response functions. Hence, from a physical point of view, we may say that *a necessary condition for a system to be linear is that the principle of superposition applies.* We note in passing that the different excitations do not have to be applied on the same part of the system.

The validity of the principle of superposition means that the presence of one excitation does not affect the responses due to other excitations; there are no interactions among responses of different excitations within a linear system. To analyze the combined effect of a number of excitations on a linear system, we can start with the analysis of the effect of each individual excitation as if the other excitations were not present, and then combine (add, or superpose) the results.

If there are n identical excitations applied to the same part of the system, that is, if

$$e_1(t) = e_2(t) = \cdots = e_n(t), \tag{1-4}$$

then, for a linear system,

$$\sum_{k=1}^{n} e_k(t) = ne_1(t) \to \sum_{k=1}^{n} w_k(t) = nw_1(t). \tag{1-5}$$

Comparing relation (1–5) with (1–1), we see that n appears as a scale factor (a magnitude change). Hence, *a characteristic of linear systems is that the magnitude scale factor is preserved.* This characteristic is sometimes referred to as the property of *homogeneity.*

At this point the reader must be warned that although the "derivation" of (1–5) from (1–3) seemed flawless, there are situations in which we cannot automatically assume the property of homogeneity (1–5) when the principle of superposition (1–3) holds. This may be illustrated by the following example. Let Fig. 1–2 represent a *nonlinear* system in which the filters 1 and 2 separate the input signal or excitation into two nonoverlapping spectral bands. Then if the spectrum of $e_1(t)$ falls entirely inside the passband of filter 1 and that of $e_2(t)$ falls entirely inside the passband of filter 2, relation (1–3) would be satisfied and yet the system remains nonlinear. Here, then, we have a situation where relation (1–3) does not imply relation (1–5). It is for this reason that the properties of superposition and homogeneity should be regarded as two *separate* requirements for a linear system. *A system is linear if and only if both* (1–3) *and* (1–5) *are satisfied.*

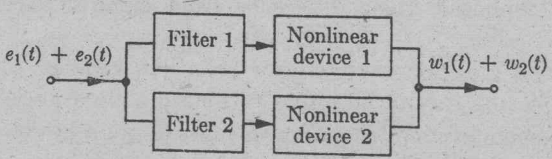

FIG. 1–2. A nonlinear system.

There is another physical aspect that characterizes a linear system with constant parameters. If the excitation function $e(t)$ applied to such a system is an alternating function of time with frequency f, then the steady-state response $w(t)$, after the initial transient has died out, appearing in any part of the system will also be alternating with the frequency f. We were aware of this fact when we solved a-c circuit problems. When a 60-cycle source is applied to a network of *fixed, linear* elements R, L, and C, the voltages and currents in all parts of the network will also be of

60-cycle frequency; no frequencies other than that of the source can exist in the network after transients have died out. In other words, *stationary (non-time-varying) linear systems create no new frequencies.* The qualification of *stationarity* implies that if

$$e(t) \to w(t) \tag{1-6}$$

then

$$e(t-\tau) \to w(t-\tau), \tag{1-7}$$

where τ is an arbitrary time delay. This qualification is to exclude situations with variable system parameters. A familiar example for such a situation is the carbon microphone circuit, in which a sinusoidal variation of the resistance in an R-L circuit will produce currents of harmonic frequencies. Another example is a linear radar system in which a moving target will cause a so-called Doppler frequency shift.

We occasionally hear the use of the terms "linear oscillators," "linear modulators," and "linear detectors." These are unfortunate choices of words. Oscillators are generators of definite frequencies, in which the only sources are d-c (zero frequency). Linear systems with constant parameters cannot do this. It is also evident that some sort of nonlinear process is there to limit the oscillation amplitude. Modulators inherently involve multiplication of frequencies and are not linear devices. The term "linear detectors" is rather commonly used for large-signal detectors where the detected output follows the envelope of the modulated carrier at the input. But large-signal detectors operate under class C conditions and are basically nonlinear. They are sometimes called "linear detectors" perhaps to emphasize their difference from small-signal or square-law detectors.

The physical viewpoints that have been discussed in this section will become clearer and can all be proved after we have examined the characteristics of a linear system from a mathematical point of view. This will be done in the next section.

1–3 Linear system from a mathematical viewpoint. In mathematical language we can define linear systems as systems whose behavior is governed by linear equations, whether linear algebraic equations, linear difference equations, or linear differential equations. Let us be more specific with a typical linear differential equation, since we shall be dealing with differential equations throughout this book:

$$\frac{d^2w}{dt^2} + a_1\frac{dw}{dt} + a_0w = e(t). \tag{1-8}$$

In Eq. (1-8), t is used as the independent variable,* e is the excitation function, and w is the response function. Coefficients a_1 and a_0 are system parameters determined entirely by the number, type, and arrangement of the elements in the system; they may or may not be functions of the independent variable t. Since there are no partial derivatives (there is only one independent variable) in Eq. (1-8), and the highest order of the derivative is 2, Eq. (1-8) is an ordinary differential equation of the second order.† Equation (1-8) is a *linear* ordinary differential equation of the second order because neither the dependent variable w nor any of its derivatives is raised to a power greater than one and because none of its terms contains a product of two or more derivatives of the dependent variable or a product of the dependent variable and one of its derivatives.

The validity of the principle of superposition here can be verified as follows. We assume that the excitations $e_1(t)$ and $e_2(t)$ give rise to responses $w_1(t)$ and $w_2(t)$ respectively, as before. Hence

$$\frac{d^2w_1}{dt^2} + a_1 \frac{dw_1}{dt} + a_0 w_1 = e_1, \tag{1-9}$$

$$\frac{d^2w_2}{dt^2} + a_1 \frac{dw_2}{dt} + a_0 w_2 = e_2. \tag{1-10}$$

Adding Eqs. (1-9) and (1-10), we have directly

$$\frac{d^2}{dt^2}(w_1 + w_2) + a_1 \frac{d}{dt}(w_1 + w_2) + a_0(w_1 + w_2) = (e_1 + e_2). \tag{1-11}$$

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* Although the symbol t is used here, the independent variable does *not* have to be time. It is just a mathematical symbol. What it is in a physical system depends upon the problem; it may be time, distance, angle, or some other physical quantity.

† The *degree* of a differential equation is the same as the power of the highest derivative in the equation. Hence Eq. (1-8) is of the first degree; an equation like

$$\left(\frac{d^2w}{dt^2}\right)^3 - 3\frac{d^2w}{dt^2}\frac{dw}{dt} + \left(\frac{dw}{dt}\right)^4 = 0$$

is of the third degree; and an equation like

$$w + t\frac{dw}{dt} = \sqrt{\frac{dw}{dt}},$$

which can be reduced to

$$\left(w + t\frac{dw}{dt}\right)^2 = \frac{dw}{dt}, \qquad \text{or} \qquad t^2\left(\frac{dw}{dt}\right)^2 + (2wt - 1)\frac{dw}{dt} + w^2 = 0,$$

is of the second degree.

Equation (1–11) states that the response of the system to an excitation $e_1(t) + e_2(t)$ is equal to the sum of the responses to the individual excitations, $w_1(t) + w_2(t)$. Note that *the principle of superposition applies and the system is linear even when the coefficients a_1 and a_0 are functions of the independent variable t.* The property of homogeneity (preservation of the magnitude scale factor) can also be easily verified.

The reader can satisfy himself in proving that the principle of superposition applies to *none* of the following equations:

$$3\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 2y = 5x^2, \tag{1–12}$$

$$\frac{du}{d\theta} + u + u^2 = \sin^3\theta, \tag{1–13}$$

$$t\left(\frac{d^2v}{dt^2}\right)^2 + 5\frac{dv}{dt} + t^2v = \epsilon^{-t}. \tag{1–14}$$

Equation (1–12) is nonlinear because the second term, $y(dy/dx)$ is a product of the dependent variable and its derivative; Eq. (1–13) is nonlinear because the third term, u^2, is a second power of the dependent variable; and Eq. (1–14) is nonlinear because the first term $t(d^2v/dt^2)^2$ contains a second power of a derivative of the dependent variable. *The existence of powers or other nonlinear functions of the independent variable does not make an equation nonlinear.*

1–4 General properties of linear differential equations. There exist certain properties which are characteristic of all linear differential equations. We shall discuss these general properties here without referring to any particular physical situation but with a view toward understanding the nature of linear systems better. We shall make no attempt to solve the equations in this section.

An ordinary linear differential equation of an arbitrary order n may be written as

$$a_n(t)\frac{d^nw}{dt^n} + a_{n-1}(t)\frac{d^{n-1}w}{dt^{n-1}} + \cdots + a_1(t)\frac{dw}{dt} + a_0(t)w = e(t), \tag{1–15}$$

where the coefficients $a_n(t), a_{n-1}(t), \ldots, a_1(t), a_0(t)$ and the right member of the equation, $e(t)$, are given functions of the independent variable t, determined by the system and the excitation function respectively. The equation is said to be *homogeneous* if $e(t) = 0$, and *nonhomogeneous* if $e(t) \neq 0$.

It is convenient to employ an abbreviated symbol for the long left side of the equation. Thus, if $e(t) = 0$, we represent the homogeneous linear differential equation as follows:

$$\left[a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t)\right]w(t) = 0.$$

Using the abbreviation

$$L = a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t), \quad (1\text{–}16)$$

we write

$$L[w] = 0, \quad (1\text{–}17)$$

where L can be regarded as an operator, operating on the dependent variable w.

(A) Since multiplying the dependent variable w by a constant multiplies each term in the equation by the same constant, we have

$$L[cw] = cL[w] \quad (1\text{–}18)$$

and

$$L[cw] = 0, \quad \text{if} \quad L[w] = 0. \quad (1\text{–}19)$$

Relations (1–18) and (1–19) state that if $w(t)$ is a solution of the homogeneous equation $L[w] = 0$, then so also is $cw(t)$.

(B) Since replacing w by $w_1 + w_2$ replaces each term by the sum of two similar terms, one in w_1 and one in w_2, we have

$$L[w_1 + w_2] = L[w_1] + L[w_2] \quad (1\text{–}20)$$

and

$$L[w_1 + w_2] = 0, \quad \text{if} \quad L[w_1] = 0 \quad \text{and} \quad L[w_2] = 0. \quad (1\text{–}21)$$

Relations (1–20) and (1–21) state that if $w_1(t)$ and $w_2(t)$ are solutions of the homogeneous equation $L[w] = 0$, then so also is $w_1(t) + w_2(t)$.

By combining the results in (A) and (B), we see that *if $w_1(t), w_2(t), \ldots, w_n(t)$ are solutions of the homogeneous linear differential equation $L[w] = 0$, then so also is a linear combination of them: $c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t)$, where the c's are arbitrary constants.*

(C) The solution $w_c(t) = c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t)$ with n (the order of the original differential equation) arbitrary constants is a general solution of the homogeneous equation (1–17) provided the n individual solutions $w_1(t), w_2(t), \ldots, w_n(t)$ are linearly independent. The solutions are linearly independent if none of them can be expressed as a linear combination of the others.* This general solution of the homogeneous equation is called the *complementary function* of the nonhomogeneous equation (1–15).

(D) If $w_p(t)$ is any particular solution of the nonhomogeneous equation such that

$$L[w_p] = e(t), \tag{1-23}$$

then the sum of this particular solution (called a *particular integral*) and the complementary function

$$\begin{aligned} w(t) &= w_c(t) + w_p(t) \\ &= c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t) + w_p(t) \end{aligned} \tag{1-24}$$

is the *general* or *complete* solution of the nonhomogeneous equation (1–15). In other words, any solution whatsoever of Eq. (1–15) can be written as a combination of the complementary function and a particular integral as

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* The n solutions $w_1, w_2, \ldots, w_n$ are *linearly dependent* if constants $b_1, b_2, \ldots, b_n$ (which are not all zero) can be found such that

$$b_1w_1 + b_2w_2 + \cdots + b_nw_n = 0. \tag{1-22}$$

Hence $w_1 = \epsilon^{-(1+j2)t}$, $w_2 = \epsilon^{-(1-j2)t}$, and $w_3 = \epsilon^{-t}\sin(2t - \pi/4)$ are linearly dependent because

$$\begin{aligned} w_3 &= \epsilon^{-t}(\cos\pi/4\sin 2t - \sin\pi/4\cos 2t) = \frac{\epsilon^{-t}}{\sqrt{2}}(\sin 2t - \cos 2t) \\ &= \frac{\epsilon^{-t}}{\sqrt{2}}\left[\frac{1}{2j}(\epsilon^{j2t} - \epsilon^{-j2t}) - \frac{1}{2}(\epsilon^{j2t} + \epsilon^{-j2t})\right] \\ &= -\frac{\epsilon^{-t}}{2\sqrt{2}}[(1+j)\epsilon^{j2t} + (1-j)\epsilon^{-j2t}] \\ &= -\frac{1}{2\sqrt{2}}[(1+j)w_2 + (1-j)w_1] \end{aligned}$$

or

$$(1-j)w_1 + (1+j)w_2 + 2\sqrt{2}\,w_3 = 0.$$

Compared with Eq. (1–22), we have $b_1 = (1-j)$, $b_2 = (1+j)$, and $b_3 = 2\sqrt{2}$.

There is an elegant method for testing whether a set of n solutions are linearly independent. They are linearly independent if their *Wronskian* does not vanish. The Wronskian of n solutions $w_1(t), w_2(t), \ldots, w_n(t)$ is the determinant formed by these functions and their first $n-1$ derivatives. A detailed discussion of this method is beyond the scope of this book. Interested readers are referred to E. L. Ince, *Ordinary Differential Equations*, Sec. 5.2, Dover Publications, 1944.

Eq. (1-24). For, if w is any solution of Eq. (1-15) and w_p is a particular integral, then, from Eq. (1-20), we can write

$$\begin{aligned} L[w - w_p] &= L[w] - L[w_p] \\ &= e(t) - e(t) = 0. \end{aligned}$$

Hence $w - w_p = w_c$ is a solution of the homogeneous equation (1-17), which, by property (C), must be expressible as

$$w - w_p = c_1 w_1(t) + c_2 w_2(t) + \cdots + c_n w_n(t).$$

Transferring w_p to the right side, we obtain the solution in the form of (1-24).

(E) The n arbitrary constants $c_1, c_2, \ldots, c_n$ in the complete solution (1-24) are determined by n known values* of the response function or its derivatives for specific values of the independent variable.

Remarks (A) through (E) above apply to general linear differential equations of an arbitrary order. If all the coefficients $a_n, a_{n-1}, \ldots, a_1$, and a_0 are constants, we have a linear differential equation with constant coefficients. Linear differential equations with constant coefficients are of extreme importance because they characterize a large number of physical and engineering situations. They are of such a nature that transformation methods can be applied with advantage. They will receive our prime attention throughout this book.

1-5 Illustrative examples. A few examples are given below to illustrate the properties of linear differential equations and their solutions.

EXAMPLE 1-1. Verify that the function

$$y = c_1 \sin x + c_2 \cos x - \tfrac{1}{2} x \cos x$$

is a general solution of the linear differential equation

$$\frac{d^2y}{dx^2} + y = \sin x. \qquad (1\text{-}25)$$

Solution. Let us examine the complementary function and the particular integral separately: $y = y_c + y_p$.

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* These are commonly referred to as *initial conditions*, but this term is inappropriate when the independent variable is not time. Even when the independent variable is time, this term does not always apply because final conditions or conditions given at any t are just as useful as initial conditions in determining the arbitrary constants.