Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

790

Warren Dicks

Groups, Trees and Projective Modules

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

790

Warren Dicks

Groups, Trees and Projective Modules

Springer-Verlag
Berlin Heidelberg New York 1980

Author
Warren Dicks
Department of Mathematics
Bedford College
London NW1 4NS
England

AMS Subject Classifications (1980): 16A50, 16A72, 20E06, 20J05

ISBN 3-540-09974-3 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09974-3 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Dicks, Warren, 1947- Groups, trees, and projective modules. (Lecture notes in mathematics; 790) Bibliography: p. Includes indexes. 1. Associative rings. 2. Groups, Theory of. 3. Trees (Graph theory) 4. Projective modules (Algebra) I. Title. II. Series: Lectures notes in mathematics (Berlin); 790. QA3.L28 no. 790. [QA251.5]. 510s. [512'.4]. 80-13138

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

To the memory
of my mother

PREFACE

For 1978/9 the Ring Theory Study Group at Bedford College rather naively set out to learn what had been done in the preceding decade on groups of cohomological dimension one. This is a particularly attractive subject, that has witnessed substantial success, essentially beginning in 1968 with results of Serre, Stallings and Swan, later receiving impetus from the introduction of the concept of the fundamental group of a connected graph of groups by Bass and Serre, and recently culminating in Dunwoody's contribution which completed the characterization. Without going into definitions, one can state the result simply enough: For any nonzero ring R (associative, with 1) and group G, the augmentation ideal of the group ring R[G] is right R[G]-projective if and only if G is the fundamental group of a graph of finite groups having order invertible in R.

These notes, a (completely) revised version of those prepared for the Study Group, collect together material from several sources to present a self-contained proof of this fact, assuming at the outset only the most elementary knowledge - free groups, projective modules, etc. By making the rôle of derivations even more central to the subject than ever before, we were able to simplify some of the existing proofs, and in the process obtain a more general "relativized" version of Dunwoody's result, cf IV.2.10. An amusing outcome of this approach is that we here have a proof of one of the major results in the theory of cohomology of groups that nowhere mentions cohomology - which should make this account palatable to hard-line ring theorists. (Group theorists

PREFACE

will notice we have not touched upon the fascinating subject of ends of groups, usually one of the cornerstones of this topic, cf Cohen [72]; happily, an up-to-date outline of the subject of ends is available in the recently published lecture notes of Scott-Wall [79].)

There are four chapters. Chapter I covers, in the first six sections, the basics of the Bass-Serre theory of groups acting on trees (using derivations to prove the key theorem, I.5.3), and then in I.§8, I.§9 gives an abstract treatment of Dunwoody's results on groups acting on partially ordered sets with involution. Chapter II gives the standard classical applications of the Bass-Serre theory, including a proof of Higgins' generalization of the Grushko-Neumann theorem (based on a proof by I.M.Chiswell). Chapter III presents the Dunwoody-Stallings decomposition of a group arising from a derivation to a projective module, and gives Dunwoody's accessibility criteria. Finally, in Chapter IV, the groups of cohomological dimension one are introduced and characterized; the final section describes the basic consequences for finite extensions of free groups.

A reader interested mainly in the projectivity results of IV.§2 can pursue the following course: Chapter I:§§1-6,§8,§9; Chapter II:3.1,3.3,3.5; Chapter III:1.1,1.2,§2,§3,4.1-4.8,4.11, Chapter IV:§1,§2.

Since the subject is quite young, and the notation to some extent still tentative, we have felt at liberty to introduce new terminology and notation wherever it suited our needs, or satisfied our category-theoretic prejudices. At these points, we have made an effort to indicate the notations used by other authors.

PREFACE

Through ignorance, we have been unable to give much in the way of historical remarks, and those we have given may be inaccurate, since, as both Cohen and Scott have remarked, it is difficult to attribute, with any precision, results which existed implicitly in the literature before being made explicit.

The computer microfilm drawings, pp 13, 25, were produced by the CDC 7600 at the University of London Computer Centre, using their copyrighted software package DIMFILM. I thank Chris Cookson and Phil Taylor for their helpful technical advice in using this package.

I thank all the participants of the Study Group for their kind indulgence in this project, and especially Yuri Bahturin and Bill Stephenson for relieving me (and the audience) by giving many of the seminars.

I gratefully acknowledge much useful background material (and encouragement) from the experts at Queen Mary College, I.M. Chiswell and D.E. Cohen for Chapters I-II and III-IV respectively.

Bedford College
London
January 1980

Warren Dicks

NOTATION AND CONVENTIONS

The following notation will be used:

ϕ	for the empty set;
$\mathbb{Z}$	for the ring of integers;
$\mathbb{Q}$	for the field of rational numbers;
$\mathbb{C}$	for the field of complex numbers;
$A - B$	for the set of elements in A not in B;
$\|A\|$	for the cardinal of A;
B^A	for the set of all functions from A to B, the elements thought of as A-tuples with entries chosen from B;
$A \times B$, $\prod_{\alpha \in A} B_\alpha$	for the Cartesian product;
$A \vee B$, $\bigvee_{\alpha \in A} B_\alpha$	for the disjoint union of sets;
$A \oplus B$, $\bigoplus_{\alpha \in A} B_\alpha$	for the direct sum of modules.

Functions are usually, but not always, written on the right of their arguments.

All theorems, propositions, lemmas, corollaries, remarks and conventions are numbered consecutively in each section, thus 4.3 CONVENTION follows 4.2 DEFINITION in section I.4 (and outside Chapter I they are referred to as I.4.3 and I.4.2). The end of each subsection is indicated by □.

References to the bibliography are by author's name and the last two digits of the year of publication, thus Serre [77], with primes to distinguish publications by the same author in the same year.

CONTENTS

CHAPTER I: GROUPS ACTING ON GRAPHS 1

I.1 Graphs 1
I.2 Graph morphisms and coverings 4
I.3 Group actions 7
I.4 Graphs of groups 10
I.5 A tree 15
I.6 The structure theorem 21
I.7 An example: $SL_2(\mathbb{Z})$ 24
I.8 Fixed points 27
I.9 Trees and partial orders 31

CHAPTER II: FUNDAMENTAL GROUPS 35

II.1 The trivial case - free groups 35
II.2 Basic results 38
II.3 The faithful case 43
II.4 Coproducts 49

CHAPTER III: DECOMPOSITIONS 55

III.1 Decomposing a group 55
III.2 Cuts 62
III.3 Decomposition theorems 68
III.4 The relationship with derivations 74
III.5 Accessibility 82

CHAPTER IV: COHOMOLOGICAL DIMENSION ONE 101

IV.1 Projective augmentation modules 102
IV.2 Pairs of groups 107
IV.3 Finite extensions of free groups 118

BIBLIOGRAPHY AND AUTHOR INDEX 121

SUBJECT INDEX 125

SYMBOL INDEX 127

CHAPTER I

GROUPS ACTING ON GRAPHS

1. GRAPHS

By a <u>graph</u> X we mean a set X that is given as the disjoint union $V \vee E$ of two sets $V = V(X) \neq \phi$ and $E = E(X)$, given with two maps $\iota, \tau : E \to V$. The elements of V are called the <u>vertices</u> of X, and the elements of E the <u>edges</u> of X. For $e \in E$, the vertices ιe, τe are called the <u>initial</u> and <u>terminal</u> vertices of e, respectively. An edge will usually be depicted

$$\iota e \xrightarrow{\quad e \quad} \tau e$$

although we also allow the possibility that $\iota e = \tau e$, in which case e is called a <u>loop</u>.

Let us fix a graph, X.

For any subset S of X we write $V(S) = S \cap V(X)$, and $E(S) = S \cap E(X)$. If for each $e \in E(S)$ we have $\iota e, \tau e \in V(S)$, then we say S is a <u>subgraph</u> of X.

For each edge e of X we define formal symbols e^1, e^{-1}, to be thought of as travelling along e the right way and the wrong way, respectively. We set $\iota e^1 = \tau e^{-1} = \iota e$, $\tau e^1 = \iota e^{-1} = \tau e$.

By a <u>path</u> P in X is meant a finite sequence,

$$(1) \qquad P = v_0, e_1^{\varepsilon_1}, v_1, \ldots, e_n^{\varepsilon_n}, v_n$$

usually abbreviated $e_1^{\varepsilon_1}, e_2^{\varepsilon_2}, \ldots, e_n^{\varepsilon_n}$, where $n \geq 0$, $\varepsilon_i = \pm 1$, and $\iota e_i^{\varepsilon_i} = v_{i-1}$, $\tau e_i^{\varepsilon_i} = v_i$ for $i = 1, \ldots, n$. We shall call v_0 the initial vertex of P, and v_n the terminal vertex of P, and say P is a path from v_0 to v_n of length n.

Two elements of X are said to be connected if there is a path in X containing both of them. This defines an equivalence relation on X. An equivalence class of this relation is called a connected component of X (or simply, a component), and it is easily seen to be a subgraph of X. We say that X is connected if it has only one component.

Let P be a path in X as in (1). We say that P is reduced if for each $i = 1,\ldots,n-1$, if $e_{i+1} = e_i$ then $\varepsilon_{i+1} \neq -\varepsilon_i$, that is, $\varepsilon_{i+1} = \varepsilon_i$. If P is not reduced then for some $i = 1,\ldots,n-1$, we have $e_{i+1} = e_i$ and $\varepsilon_{i+1} = -\varepsilon_i$; in this case we say that a simple reduction of P gives the path

$$e_1^{\varepsilon_1},\ldots,e_{i-1}^{\varepsilon_{i-1}},e_{i+2}^{\varepsilon_{i+2}},\ldots,e_n^{\varepsilon_n}.$$

By successive simple reductions we can transform P to a reduced path, called the reduced form of P. It is in fact unique, as can be shown by induction on the length of P, noting that any two simple reductions of P either give equal paths, or each can be followed by a suitable simple reduction to give equal paths.

A circuit at a vertex v of X is a reduced path from v to itself of length at least 1. A graph with no circuits is called a forest, and a connected forest is called a tree. In a tree there is, by the above, a unique reduced path between any pair of vertices; this path will be called a geodesic between the vertices.

By Zorn's Lemma there is a subgraph X' of X having $V(X') = V(X)$ and maximal with the property that X' is a forest.

By maximality, no two connected components of X' can be joined by an edge of X, so two vertices connected in X must already be connected in X'. In particular, if X is connected then so is X', in which case X' is a tree, called a spanning tree or a maximal subtree of the connected graph X.

Having assembled all these definitions, let us conclude this section by giving an algebraic characterization of trees.

For any ring R and set S we write $R[S]$ for the R-bi-module freely generated by the R-centralizing set S. Thus $R[S] = \bigoplus_{s \in S} Rs$, with $r.s = s.r$ for all $r \in R$, $s \in S$. The elements of $R[S]$ will be expressed $\sum_{s \in S} r_s . s = \sum_{s \in S} s . r_s$, where $r_s \in R$, almost all zero.

1.1 PROPOSITION. Let R be a nonzero ring, and X a graph. Write $E = E(X)$, $V = V(X)$. There is a sequence of R-bimodules

$$(2) \qquad 0 \to R[E] \xrightarrow{\partial} R[V] \xrightarrow{\varepsilon} R \to 0$$

determined by $(e)\partial = \iota e - \tau e$, $(v)\varepsilon = 1$ $(e \in E,\ v \in V)$.

(i) The sequence is exact at $R[V]$ if and only if X is connected.

(ii) The sequence is exact at $R[E]$ if and only if X is a forest.

(iii) The sequence is exact if and only if X is a tree. In this event, for any vertex v_0 of X, the map ∂ has an R-bimodule right inverse $X(-,v_0): R[V] \to R[E]$ determined by, for $v \in V$, $X(v,v_0) = \varepsilon_1 e_1 + \ldots + \varepsilon_n e_n$ where $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ is the geodesic from v to v_0 in the tree X.

Proof. (i) The cokernel of ∂ is the R-bimodule presented on

generators v that centralize R, $v \in V$;

relations saying $\iota e = \tau e$ for all $e \in E$.

Thus Coker ∂ is $R[C]$, where C is the set of components of X. Since R is nonzero, (i) follows.

(ii), (iii) If X is not a forest then X has some circuit $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ with no repeated edges, and then $\varepsilon_1 e_1 + \ldots + \varepsilon_n e_n$ is a nonzero element of Ker ∂ so (2) is not exact at $R[E]$. Conversely, if X is a forest then each component of X is a tree, and it suffices to consider the case where X itself is a tree. Here, for any edge e of X, $X(-,v_0)$ sends $e\partial$ $(= \iota e - \tau e)$ to $X(\iota e, v_0) - X(\tau e, v_0) = X(\iota e, \tau e) = e$. That is, $X(-,v_0)$ is a right inverse of ∂ as desired, and this verifies all the claims. □

2. GRAPH MORPHISMS AND COVERINGS

Let Γ, X be graphs.

A <u>morphism of graphs</u> $\alpha : \Gamma \to X$ is the disjoint union of two maps $V(\alpha) : V(\Gamma) \to V(X)$, $E(\alpha) : E(\Gamma) \to E(X)$ which have the property that for each edge e of Γ, $\alpha(\iota e) = \iota(\alpha e)$, $\alpha(\tau e) = \tau(\alpha e)$. Thus

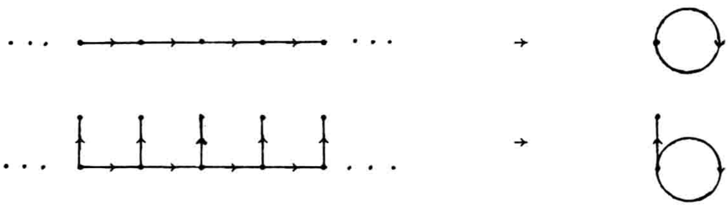
$$\alpha\left(\overset{v}{\bullet} \xrightarrow{\;e\;} \overset{w}{\bullet} \right) \quad = \quad \overset{\alpha v}{\bullet} \xrightarrow{\;\alpha e\;} \overset{\alpha w}{\bullet} \quad .$$

We use the terms <u>isomorphism</u> and <u>automorphism</u> of graphs in the natural way.

For any vertex v of Γ, we define

$$(3) \qquad \text{star}(v) = \{e \in E(\Gamma) \mid \iota e = v\} \vee \{e \in E(\Gamma) \mid \tau e = v\}.$$

We say that α is <u>locally surjective</u> if for each vertex v of Γ, the induced map $\text{star}(v) \to \text{star}(\alpha v)$ is surjective; and we define <u>locally injective</u> analogously. If α is both locally injective and locally surjective then it is said to be a <u>local isomorphism</u>. For example, the morphisms



are both local isomorphisms.

2.1 PROPOSITION. <u>Let</u> $\alpha:\Gamma \to X$ <u>be a locally surjective graph morphism, and</u> v <u>be a vertex of</u> Γ. <u>Any subtree</u> X' <u>of</u> X <u>containing</u> αv <u>lifts back to a subtree</u> Γ' <u>of</u> Γ <u>containing</u> v, <u>that is</u>, $\alpha:\Gamma' \to X'$ <u>is an isomorphism</u>.

Proof. By Zorn's Lemma there exists a maximal connected subgraph Γ' of Γ containing v such that $\alpha:\Gamma' \to X'$ is injective. Notice that Γ' is then a tree. If α is not an isomorphism then there is a path P in X' starting at αv such that P does not lie entirely in $\alpha(\Gamma')$. Since X' is a tree, P traverses some edge e that does not lie in $\alpha(\Gamma')$ and has exactly one vertex in $\alpha(\Gamma')$. But as α is locally surjective, we can find a preimage of e in Γ connected to Γ', and this contradicts the maximality of Γ'. Thus α is an isomorphism. □

2.2 COROLLARY. <u>If</u> $\alpha:\Gamma \to X$ <u>is a locally surjective graph morphism, and</u> X <u>is connected then</u> α <u>is surjective</u>. □

Notice that in the situation of 2.2 we can choose a maximal subtree T of the connected graph X and lift it back to a subtree Γ' of Γ by 2.1, and for each edge e of X not in T we can, by considering star(ιe), choose an edge f in Γ such that $\alpha f = e$ and $\iota f \in \Gamma'$. This will give us a subset S of Γ such that S is a _transversal_ for α (that is, $\alpha : S \to X$ is bijective). Further, S has the property that any two vertices of S are joined by a path all of whose terms are in S, and for any edge e of S we have $\iota e \in S$; such a subset will be said to be _connected_. It is clear what we mean by a _maximal subtree_ of S, so we can state the foregoing as follows.

2.3 PROPOSITION. _Let_ $\alpha : \Gamma \to X$ _be a locally surjective graph morphism and_ X _be connected_. _Any maximal subtree_ T _of_ X _lifts back to a subtree_ Γ' _of_ Γ, _and there exists a connected transversal_ S _of_ α _which has_ Γ' _as maximal subtree_. □

A local isomorphism $\alpha : \Gamma \to X$ between connected graphs is called a _covering_, or a _covering of_ X. The covering is said to be _universal_ if Γ is a tree. By 2.2 any covering is surjective; if X is a tree we can say more.

2.4 PROPOSITION. _Any covering of a tree is an isomorphism_.

Proof. Let X be a tree and $\alpha : \Gamma \to X$ be a covering. We have seen that α is surjective so it remains to show that α is injective. Suppose that two vertices of Γ are mapped by α to a single vertex v of X. Then the reduced path between them in

Γ is mapped to a reduced path in X from v to v, since α is locally injective. But X is a tree so the path has length zero. This shows that α is injective on vertices. Since α is locally injective, it is therefore injective on edges also. Hence α is an isomorphism. □

3. GROUP ACTIONS

Let G be a group.

We call a set X a G-set, or say that G acts on X, if there is given a group homomorphism from G to Sym_X, the group of all permutations of X. (As mappings the permutations are viewed as being written on the left of their arguments.) The image of an element g of G will usually be thought of as left multiplication by g, and denoted $x \mapsto gx$ $(x \in X)$. For any $x \in X$, the stabilizer of x is defined to be the subgroup $G_x = \{g \in G \mid gx = x\}$, and the orbit of x is defined to be the G-subset $Gx = \{gx \mid g \in G\}$ of X. Notice that for $g \in G$, $x \in X$, we have $G_{gx} = gG_xg^{-1}$, so the stabilizers of two points in the same orbit are conjugate. We write h^g for $g^{-1}hg$, so $G_{gx} = G_x^{g^{-1}}$. For $x \in X$, the set of left cosets of G_x in G, $G/G_x = \{gG_x \mid g \in G\}$, is in bijective correspondence with Gx, under $gG_x \leftrightarrow gx$. This is actually an isomorphism of G-sets, if G/G_x is given its natural G-action by left multiplication. The set of orbits is denoted $G\backslash X$. There is a natural surjection $X \to G\backslash X$, $x \mapsto \bar{x} = Gx$. Choose a transversal S in X for the G-action (that is, S is a transversal for $X \to G\backslash X$). Attach to each element of $G\backslash X$ a subgroup of G by writing, for each $x \in S$, $G(\bar{x}) = G_x$. Then as G-set, $X \simeq \bigvee_{\bar{x} \in G\backslash X} G/G(\bar{x})$.

In some situations we shall want G to act on some set X on the *right*, in which case X will be called a *right G-set*. The set of orbits is then denoted X/G, a notation which generalizes the use of G/G_x above.

Let X be a graph, and write $V = V(X)$, $E = E(X)$.

We say that G *acts on* X, or that X is a G-*graph*, if there is given a group homomorphism from G to the group of all graph automorphisms of X. Then V, E are G-sets in such a way that $g\iota e = \iota ge$, $g\tau e = \tau ge$ for all $e \in E$, $g \in G$. Pictorially,

$$g\left(\underset{\bullet}{v}\xrightarrow{e}\underset{\bullet}{w}\right) = \underset{\bullet}{gv}\xrightarrow{ge}\underset{\bullet}{gw}.$$

3.1 EXAMPLE. Let A be a subset of G. The *Cayley graph* $\Gamma = \Gamma(G,A)$ is defined as follows: $V(\Gamma) = G$, $E(\Gamma) = G\times A$, and the incidence maps are given by $\iota(g,a) = g$, $\tau(g,a) = ga$ ($(g,a) \in E(\Gamma)$). Here G acts on Γ in a natural way by left multiplication on the vertices and on the first components of the edges. Some examples are illustrated on p. 13. We remark that $\Gamma(G,A)$ is connected if and only if A generates G. □

Let G act on X.

We write $G\backslash X$ for the graph with $V(G\backslash X) = G\backslash V$, $E(G\backslash X) = G\backslash E$, where the incidence maps are given by $\iota(Ge) = G\iota e$, $\tau(Ge) = G\tau e$, clearly well-defined.

There is then a natural surjective morphism of graphs, $X \to G\backslash X$, $x \mapsto \bar{x} = Gx$. For any vertex v of X, the induced map