CARIS

ELEMENTS OF ALGEBRA

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FOR COLLEGES

By

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PREFACE

Elements of Algebra has been written for college students who have taken only one year of high-school algebra or who, for any reason, find their knowledge of high-school algebra insufficient. It provides an adequate foundation for college courses in mathematics, general science, business administration, and other technical and scientific subjects in which algebra is needed. Also, for students who elect nontechnical and nonscientific courses where algebra is not required it presents those mathematical essentials which are generally considered part of a well-rounded education. Thus this textbook meets the needs of two groups of students,—those who must bridge the gap between high-school algebra and college courses requiring algebra, and those who desire a well-balanced grounding in useful mathematics as an end course.

The text is simple in style and natural in sequence, so that it may be read with understanding and profit by those for whom it is written. Among the considerations underlying the organization, development, and content of the book, the following seem important.

1. Since the students are far away from their first course in algebra, the fundamental principles are reviewed.
2. Emphasis has been placed upon skill in computation and on methods of reasoning. The ability to translate problems stated in words into the symbols of algebra is developed by careful training in the language peculiar to mathematics.
3. There are an unusual number of illustrative examples to aid the student. Other help is provided by notes which caution against common mistakes or give hints about methods.
4. In order that the ordinary procedures of algebra shall not be handled in purely mechanical fashion, rules have been given only after explanations and illustrations have shown a reason for their use.
5. Because the students are adults, with broader experiences and greater reasoning ability than high-school students, the explanations are brief and direct, and the problems have a practical background suited to the students' fields of experience.

6. A number of problems taken from geometry and mensuration give not only practice in algebraic methods but also a valuable review. The same is true of problems in percentage, interest, square root, motion, mixtures, and other topics.

7. The book has enough problem and exercise material to provide ample choice for both home work and class work. There are also groups of review exercises at the ends of chapters, which may be used as the instructor finds advisable and as the time allotted to the course may dictate.

8. A discussion of related variables and functions lays the foundation for the functional point of view which is fundamental in advanced mathematics and in life. Graphs also are used to provide visual assistance in understanding abstract number relations.

9. Several topics usually found in more advanced college textbooks are discussed, primarily for the benefit of those who are taking the subject as an end course.

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ELEMENTS OF ALGEBRA

i FUNDAMENTAL IDEAS

1. Specific and General Numbers

Besides the specific number symbols 1, 2, 3, 4, etc., of arithmetic, it is characteristic of algebra that a letter of the alphabet or some other symbol is used to represent an unknown number or one to which various values may be assigned. Such letters or symbols are called *general numbers*, or *literal numbers*.

Thus, if we represent the side of a square by s and the perimeter of the square by P, we may state a general rule for finding the perimeter of any square as

$$P = 4 \times s.$$

Such a statement in symbols is called a *formula*.

By means of formulas we can give rules very briefly. Also, if the value of either P or s in the formula above is known for some definite square, the value of the other letter, representing the unknown, can be easily found, because the formula states the relation between the known number and the unknown.

2. Signs of Operation

Plus and minus signs, together with the symbols $\times$ and $\div$, are used as symbols of the same fundamental operations for which they are used in arithmetic.

The symbol $\times$ for multiplication is, however, not ordinarily used between two numbers in algebra except when neither of the numbers is general. When one or both numbers are general, multiplication is indicated either by the absence of any sign or by the use of a dot between the numbers.

Illustration. $a \cdot b$, $3 \cdot x$, $4\,a$, and mn are indicated products.

As in arithmetic, a fraction, such as $\frac{3}{4}$ or $\frac{a}{b}$, is a number in its own right and, at the same time, an indicated division.

3. Algebraic Expressions

Any single general number or any set of numbers, specific or general, between which processes of operation are indicated, is called an *algebraic expression.*

Illustration 1. In the formula for finding the perimeter of a triangle (see Fig. 1),

$$P = a + b + c,$$

both the single number, P, on the left side of the equality and the indicated sum, $a + b + c$, on the right are algebraic expressions.

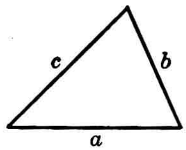

Fig. 1

Illustration 2. Suppose we have the three equal-sided figures of Fig. 2 and are to find the sum, S, of their perimeters. Evidently,

$$S = 3\,a + 4\,b + 6\,c,$$

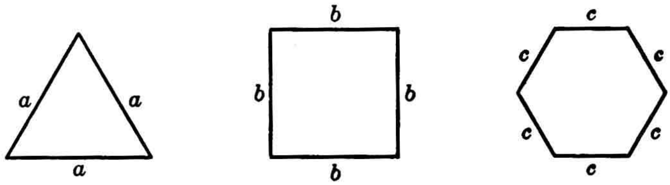

Fig. 2

and here we have an algebraic expression in which both multiplication and addition are indicated but may not be actually performed until we know the values of a, b, and c in definite-sized figures.

Illustration 3. By the rule for finding the volume of a rectangular solid ($V = lwh$), and by addition, we may find the sum, S, of the volumes of the two solids of Fig. 3.

Thus, $$S = a \cdot a \cdot a + b \cdot c \cdot d.$$

What are the two algebraic expressions in this illustration?

When an algebraic expression is made up of parts joined together by plus or minus signs, as are some of the expressions in the illustrations above, or when the expression consists of a single part, each such part, with the plus or minus sign preceding it, is called a *term*.

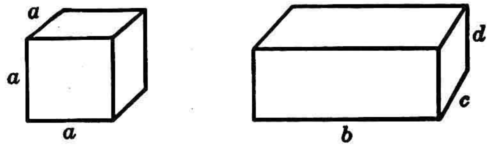

Fig. 3

The complete expressions are classified according to the number of terms as *monomials*, *binomials*, and *trinomials* for one-term, two-term, and three-term expressions, respectively.

Illustration. P and S, in the illustrations on page 4, are monomials, $a \cdot a \cdot a + b \cdot c \cdot d$ is a binomial, and $3a + 4b + 6c$ is a trinomial.

An expression which has more than one term is called a *multinomial*.

EXERCISE 1

1. How do specific numbers differ from general numbers?
2. Give two uses of general numbers.
3. If a, b, and c represent three numbers, what represents their sum?
4. If d represents the number of dollars that one book costs, what represents the cost of 10 of these books?
5. If we let y represent a man's age and x his son's age, how can we say in symbols that the man is twice as old as his son?
6. If x represents the side of an equilateral triangle, what represents the perimeter?
7. If x represents the number of bushels of grain raised on 1 A. (acre), what represents the number raised on 5 A.?
8. If one book costs x dollars and another costs y dollars, what is the cost of both?
9. If an automobile goes x miles per hour, how far will it travel in 5 hr.?

4. The Parts of a Term: Factors, Coefficients, Exponents

1. Factors. When two or more numbers are multiplied together to give a product we say that the numbers are *factors* of the product.

Illustration. 2, a, and b are factors of their product $2ab$.

In arithmetic such numbers as 3, 7, 11, etc., are known as *prime numbers*, since they cannot be broken up into smaller integral factors.

We likewise consider a single letter, representing an unknown number, as a prime number even though it may be discovered to represent a factorable number.

2. Coefficients. Whenever a term may be broken up into two factors, in any manner, each factor is called the *coefficient* of the other.

Illustration. In the term $-3\,abc$, $-3\,a$ is the coefficient of bc and bc is the coefficient of $-3\,a$; or, with different grouping, abc is the coefficient of -3, and -3 is the coefficient of abc.

In $-3\,abc$, the -3, being a specific number, is called the *numerical coefficient* of abc; however, in common usage, the single word *coefficient* means the numerical coefficient unless otherwise indicated.

3. Exponents. Referring to the equilateral triangle of Fig. 2, we see that its perimeter is $3\,a$. Here 3, the coefficient of a, also tells the *number of a's which are added* in giving the perimeter.

In Fig. 3, the volume of the cube is $a \cdot a \cdot a$, which we write in the shortened form, a^3. In this term, the 3, written at the upper right of a, tells the *number of a's which are multiplied together*, and is called the *exponent of a*.

When a number is used as a factor, in this manner, any number of times, the resulting expression is called a *power* of that number. Thus when x is multiplied by itself $(x \cdot x)$ we say that it is raised to the second power. In like manner, $x \cdot x \cdot x$ is the third power of x, $x \cdot x \cdot x \cdot x$ is the fourth power of x, etc. These powers are written x^2, x^3, x^4, etc., respectively, and may be read in several ways. x^2 may be read: "x to the second power" or "x with the exponent 2," but is usually read "x square." x^3 may be read: "x to the third power" or "x with the exponent 3," but is usually read "x cube." In like manner the higher powers of x may be read in the first two ways indicated above but are usually read "x fourth," "x fifth," etc.

Any number, either specific or general, may be used either as a coefficient or as an exponent; however, the number 1, used in either of these ways, is understood without being written. That is, $1 \cdot x$ and x^1 are both simply written as x.

4. Degree of a term. Exponents are also used to determine the *degree of a term*, which is defined as the sum of the exponents of the letters of the term.

Illustration. The term $4\,x^2y^3$ is of fifth degree.

We sometimes say, however, that a term like $4\,x^2y^3$ is of second degree with respect to x and of third degree with respect to y.

The *degree of a multinomial* is considered as the same as that of the term of highest degree in the multinomial.

Illustration. The multinomial $x^3y + x^2y^2 - 3\,xy^4$ is of fifth degree, since the exponents of the letters in the last term add up to 5, which is more than the sum of the exponents of the letters in either of the other terms.

Each term in the following illustrations is written in two ways; both forms have identical value and almost the same meaning. The right-hand form is known as the *factored form*, since all the prime factors are shown separately. Except for the specific numbers, which are actually factored, the illustrations show merely two ways of writing the same thing.

Illustrations.

	Term	*Factored form*
1.	30	$= 2 \cdot 5 \cdot 3$
2.	$6\,xy$	$= 2 \cdot 3 \cdot x \cdot y$
3.	$9\,x^2y$	$= 3 \cdot 3 \cdot x \cdot x \cdot y$
4.	$15\,a^3$	$= 3 \cdot 5 \cdot a \cdot a \cdot a$
5.	$3^2\,xy^2$	$= 3 \cdot 3 \cdot x \cdot y \cdot y$
6.	$8\,x^4y^3$	$= 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y$

In each of Illustrations 1 to 6, the degree of the term may be found from either form of the term,—in the form on the left by adding the exponents of all the general factors, and in the form on the right by counting the separate general factors. Thus, in Illustration 1, the term is of zero degree, since there are no general factors; and the term in Illustration 6 is of seventh degree, as shown by either method.

EXERCISE 2

Write each of the following in factored form:

1. 24. 2. $12\,xy^2$. 3. $7\,m^2n$. 4. $26\,m^3n^2$.

Express each of the following in simpler form, using exponents:

5. $2\,aabb$. 6. $xxxyyyy$. 7. $4 \cdot a \cdot a \cdot a \cdot a$. 8. $2 \cdot 5 \cdot x \cdot x \cdot x \cdot y \cdot y$.

9. $xyyz$. 10. $cccddxx$. 11. $2 \cdot 5 \cdot x \cdot x \cdot y$. 12. $2\,abcabcab$.

Read each of the following:

13. $6x^3$. 14. $3x^2y^4$. 15. $\frac{1}{2}mv^2$. 16. $\frac{1}{2}gt^2$.

17. $5mn$. 18. $4mn^2x^2y^2$. 19. $3abc$. 20. $\frac{bd^3}{12}$.

Give the degree of each expression, and the number of terms in each:

21. $xy - 4x$. 22. $x^2 - y^2 + x - y$. 23. $7x^2y$.

24. $4x + 3y$. 25. $4x^3 + 3x + 2$. 26. $2xy$.

27. In the expression $7x^2y$, what is the coefficient of x^2?

5. Fundamental Laws of Operation

Our experience in dealing with specific numbers will be a sufficient basis for accepting the truth of these laws.

I. *The Commutative Law of Addition.* The order in which several numbers are added does not affect the result.

Illustrations.

1. $2 + 3 + 4 = 4 + 2 + 3.$

2. $a + b + c = b + a + c.$

II. *The Associative Law of Addition.* Several numbers to be added may be grouped in any way and the groups then added without changing the sum.

Illustration.

$$5 + 3 + 2 + 4 + 6 = (5 + 3 + 2) + (4 + 6) = 10 + 10 = 20.$$

This example illustrates a convenient way of grouping numbers in tens for easy addition; however the grouping might be entirely different, as, for instance,

$$5 + 3 + 2 + 4 + 6 = (5 + 3) + (2 + 4) + 6 = 8 + 6 + 6 = 20.$$

NOTE ▶ The parenthesis marks, ( ), indicating the groups above, show that each such group is to be used as a single number.

III. *The Commutative Law of Multiplication.* The order in which a set of numbers is multiplied does not change the product.

Illustrations.

1. $3 \cdot 4 \cdot 5 = 4 \cdot 5 \cdot 3.$

2. $x \cdot y = y \cdot x.$

3. $abc = bca.$

IV. *The Associative Law of Multiplication.* In whatever way a set of factors may be arranged in groups, if the factors of each group are first multiplied and these results are then multiplied, the resulting product will be the same.

Illustration. $3 \cdot 4 \cdot 5 \cdot 2 = (3 \cdot 4) \times (5 \cdot 2) = 12 \cdot 10 = 120;$
or $3 \cdot 4 \cdot 5 \cdot 2 = (3 \cdot 2) \times (4 \cdot 5) = 6 \cdot 20 = 120.$

V. *The Distributive Law of Multiplication with Respect to Addition.* If a multinomial is to be multiplied by a number, each term of the multinomial may be multiplied by the number and the results then added.

EXAMPLE 1. Multiply: $3(2 + 3 + 4).$

SOLUTION. *Method 1.* By the usual arithmetic method,

$$3(2 + 3 + 4) = 3 \cdot (9) = 27.$$

Method 2. By the Distributive Law,

$$3(2 + 3 + 4) = 6 + 9 + 12 = 27.$$

EXAMPLE 2. $K(a + b + c) = Ka + Kb + Kc.$

NOTE ▶ The Distributive Law need not be applied in multiplying specific numbers (Example 1, Method 1); the ordinary arithmetic method is impossible with general numbers (Example 2).

EXERCISE 3

In each case, what is the law which justifies the statement?

1. $2 \cdot 3 \cdot 5 = 3 \cdot 2 \cdot 5.$
2. $3(a + b + c) = 3a + 3b + 3c.$
3. $a + (b + c + d) = (a + b) + (c + d).$
4. $(a + b + c) + d = (c + d) + (b + a).$

In each problem arrange the numbers in convenient groups and add:

5. $2 + 3 + 6 + 4 + 5 + 8 + 9 + 2 + 1.$
6. $7 + 9 + 8 + 3 + 1 + 2.$
7. $4 + 2 + 3 + 1 + 7 + 8 + 5.$
8. What is the law used in exercises 5, 6, and 7?

Perform each of the indicated multiplications below in two ways:

9. $2(4 + 8 + 10).$
10. $(3 + 4 + 2) \cdot x.$
11. Perform the indicated multiplication: $3(x + y + z)$. How does this exercise differ from exercises 9 and 10?

6. Fundamental Operations with General Numbers

1. Addition and subtraction. Numerically it is possible to add or subtract only those things to which the same name may be given. In considering general numbers we think of the general part of a term as being the name of the term. For example, since $5x$ and $3x$ have the same name, x, they may be added or subtracted, as shown in the following illustrations.

Illustrations.

1. $5x + 3x = 8x.$
2. $5x - 3x = 2x.$
3. $7xy + 6xy = 13xy.$
4. $12ab - 5ab = 7ab.$

In the case of $4x + 3y$, however, there is no common name and this sum must be left in the form of an indicated addition.

If the general parts of two or more terms are exactly alike, the terms are called *similar*, or *like*, terms. From the illustrations above, we have the following

RULE FOR ADDITION OR SUBTRACTION OF SIMILAR TERMS

Add (or subtract) the numerical coefficients, and after the result write the general part, or name.

2. Multiplication. In arithmetic, multiplication is described as a short method of addition. With this reasoning,

$$3 \times 4 \text{ means three 4's, or } 4 + 4 + 4, \text{ or } 12.$$

Similarly, $$3 \times 5ab = 5ab + 5ab + 5ab = 15ab.$$

In the multiplication of a term of several factors by another such term the Commutative Law permits the factors to be multiplied in any order.

EXAMPLE. $$3ab \cdot 4xy = 3 \cdot a \cdot b \cdot 4 \cdot x \cdot y$$
$$= 3 \cdot 4 \cdot a \cdot b \cdot x \cdot y = 12abxy.$$

This illustrates the

RULE FOR MULTIPLYING TERMS

Multiply the numerical coefficients, and indicate the product of the remaining factors.

NOTE ▶ It is usually found convenient to arrange letters of a product in the order of the alphabet.