BARRY M. McCOY AND TAI TSUN WU

THE TWO-DIMENSIONAL ISING MODEL

Harvard University Press   Cambridge, Massachusetts   1973

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Distributed in Great Britain by Oxford University Press, London
Library of Congress Catalog Card Number 77–188972
SBN 674–91440–6
Printed in the United States of America

PREFACE

Of all the systems in statistical mechanics on which exact calculations have been performed, the two-dimensional Ising model is not only the most thoroughly investigated; it is also the richest and most profound. In 1925, Ising introduced the statistical system which now bears his name and studied some of its properties in one dimension. Although the generalization of Ising's system to higher dimensions was immediately obvious, it was not until 1941 that a quantitative statement about the phase transition in the two-dimensional case was made when Kramers and Wannier and also Montroll computed the Curie (or critical) temperature. However, the most remarkable development was made in 1944 when Onsager was able to compute the thermodynamic properties of the two-dimensional lattice in the absence of a magnetic field. Onsager's approach was greatly simplified by Kaufman in 1949, and in a companion paper Kaufman and Onsager studied spin correlation functions. The spontaneous magnetization was first published, without derivation, by Onsager in 1949, and the first derivation was given by Yang in 1952. For the next decade no new result of fundamental significance was derived, but a great deal was accomplished in simplifying the mathematics of these pioneering papers. The work of Kac, Kasteleyn, Montroll, Potts, Szegö, and Ward, among others, has been especially significant.

The methods of Onsager, Kaufman, and Yang, although very beautiful and powerful, are also extremely complicated. Thus, the two-dimensional Ising model has acquired a notorious reputation for difficulty whereas, in fact, the simplified methods developed by 1963 have reduced the analysis to the point where it may be readily understood. Since then we have actively used these methods as the basis for computing many

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more quantities of physical interest. Our original concern was with the spin correlation functions of the two-dimensional Ising model. However, it soon became apparent that much more could be studied. In particular, we found that the Ising model has properties which exhibit a hysteresis behavior. Moreover, we discovered that exact results can be obtained even in a much more complicated situation where the interaction between the spins is allowed to be a random variable. On the basis of these results, a quantitative study of the influence of impurities on phase transitions has been carried out. This influence is large and has experimental consequences that have yet to be fully explored.

Since the two-dimensional Ising model forms the basis of much of our theoretical understanding of phase transitions, it is unfortunate that these recent developments have not been easily accessible to the general community of physicists. Perhaps as a result of its notoriety, most physicists tend to think of the two-dimensional Ising model as a closed problem that was completely solved by Onsager, Kaufman, and Yang. Moreover, once a physicist does become aware of the wide variety of open questions there is no convenient place where he can find the known facts collected together and explained in an organized fashion. Furthermore, even if one has the patience to trace the references back to Kasteleyn's paper of 1961, the usual result is a feeling of confusion. This confusion arises not out of any errors in the published work, but out of the fact that in journal articles many things must be omitted owing to lack of space. Therefore, points that can be straightened out and rigorously shown to cause no problems are often treated very briefly. The careful reader therefore has questions that he must resolve for himself and the resolutions are frequently quite time consuming.

The study of the two-dimensional Ising model requires the use of mathematics from such apparently widely separated areas as the theory of determinants and integral equations. Few physicists are knowledgeable in all these branches of mathematics. Therefore, a formula that may have been well known to a mathematician of 100 years ago may be totally unknown to a physicist of today. It is quite impossible to discuss such a formula in a journal article. One must call it "well known," give a reference, and go on. But the reference is often useless because, while correct, it usually is so arranged that the reader must spend an inordinate amount of time in mastering a lot of notation which is mostly superfluous if he wants to derive only one particular formula. For example, in our study of Ising-model spin correlation functions we make extensive use of the theory of Wiener-Hopf sum equations. Except as an afterthought to the theory of Wiener-Hopf integral equations, these sum equations are rarely discussed in the literature. This circumstance often leads one to believe that the sum equations are harder than the integral equations. In fact they are simpler.

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For these reasons we feel that it is at this time most desirable to write a book on the two-dimensional Ising model that has the following three goals:

(1) It should be completely up to date and be crystal clear in its statement of what is known and, more important, what is as yet unknown.

(2) It should discuss all topics in complete detail. No significant point should be dismissed with the casual remark "it can be shown."

(3) It should strive to be self-contained. All mathematical statements that are not known by the average graduate student in physics should be proved.

The present book, which sets forth the theory of the two-dimensional Ising model with nearest-neighbor interactions as it has developed through the end of 1969, is our attempt at meeting these three goals. In particular, we have tried to make this book complete in such a manner that the physicist can read it without consulting any additional source. To this end we have assumed that the reader knows no statistical mechanics at all and have included at the beginning a chapter that develops all the statistical mechanics needed for the entire book. Furthermore, though we do not feel justified in including a chapter on special functions, we have not assumed a familiarity with mathematics beyond a basic understanding of complex-variable theory.

For the sake of readability we abandon altogether any attempt at preserving the historical development of the Ising model and, in fact, give no exposition whatsoever of the original work of Onsager, Kaufman, and Yang. The development we follow instead should be obvious from the chapter headings in the table of contents. Only three points deserve special attention. First, the interrelations between the chapters are shown in Fig. 0.1. Secondly, we have chosen to give a thorough treatment of

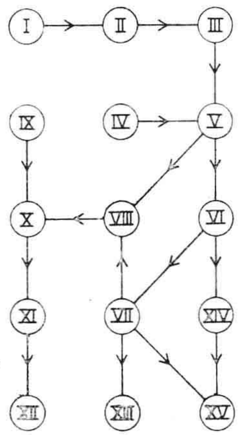

Fig. 0.1. The interrelations of the chapters.

boundary effects before any study is made of bulk spin correlation functions. We have made this choice because the calculations for the boundary are much easier to understand and are more complete than the corresponding calculations for the bulk. Lastly, we wish to call the reader's attention to Chapter IV. This is the most crucial chapter in the book because everything that is done later depends on it. For this reason, in Chapter IV we attempt to give a complete and detailed discussion of all the fine points. However, several of these details have to do only with straightening out certain + and − signs associated with the various boundary conditions that may be imposed on the lattice. Accordingly, if the reader is willing to accept the conclusions of Sections 4 and 5 of Chapter IV, he may omit the derivations without impairing his ability to read the rest of the book. In fact, we will suggest that the book may be profitably read with the omission of Chapter IV altogether, even though this is the most crucial chapter, because the results of this chapter are much more easily stated and used than they are proved. Moreover, the several open questions we will arrive at already incorporate the combinatorics of Chapter IV in their formulation. Therefore, it is perfectly possible to appreciate the current status of the physics of the Ising model without a full understanding of the combinatorial problem involved. Indeed, this was precisely the route that we took ourselves when we first entered the field.

If the authors of any scientific book are to be fair to the reader, it is as important for them to indicate what is omitted as to explain what is covered. Not a book, but an encyclopedia, results if an effort is made to include all related topics, related related topics, and so on. For this book we mention the conspicuous omission of the following five related topics: (1) high- and low-temperature expansions, (2) Padé approximants, (3) the circle theorem of Lee and Yang, (4) decorated lattices, and (5) theorems that prove the existence of limits or of analyticity without actually computing the quantity involved.

One of us (TTW) would like to express special gratitude to Professor Ronold W. P. King and Professor Elliott W. Montroll. Because of the insistence of Professor King that each student must be allowed to decide his own interest and pick his own topic, it has been a most rewarding experience to write a doctorial dissertation under his guidance. Furthermore, without his continual encouragement and help in every respect of this present book, its completion would be impossible. Professor Montroll taught us the Pfaffian approach to the two-dimensional Ising model. The influence of his two papers, "Lattice Statistics" (Chapter IV, *Applied Combinatorial Mathematics*, ed. E. F. Beckenbach, Wiley, New York, 1964) and "Correlations and Spontaneous Magnetization of the Two-Dimensional Ising Model" (with R. B. Potts and J. C. Ward, J. Math. Phys. 4, 308, 1963), is particularly evident in Chapter IV and Chapter VIII of this book.

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In the course of our study of the Ising model, we have had the benefit of the advices of many friends. In particular, we wish to thank Professor H. Cheng, Professor F. Dyson, Professor K. Huang, Professor T. D. Lee, Professor H. Levine, Professor H. McKean, Professor A. Pais, Professor G. C. Rota, and Professor C. P. Yang for many helpful discussions.

Much valuable assistance in the preparation of this book was given by Miss M. K. Ahern and Mr. J. D. Elder, science editor of the Harvard University Press. We thank them especially for their patience and understanding.

We are particularly grateful to Miss Margaret Owens who expertly read the proofs of the entire volume.

In Chapters VI, VII, XI, XII, XIII, XIV, and XV, we draw heavily on material we have published separately, jointly, and with Professor H. Cheng in *The Physical Review* and *The Physical Review Letters*. We wish to thank the National Science Foundation and the Alfred P. Sloan Foundation for supporting the research reported in these papers, and the editors of *The Physical Review* and *The Physical Review Letters* for allowing us to use this copyrighted material. We also wish to thank the Pergamon Press Ltd. (Oxford, England) for permitting us to employ the proof of Privalov's theorem as it appears in the book *Trigonometric Series* by N. K. Bary.

Finally, special thanks are due Professor Chen Ning Yang, who has been a constant source of inspiration and who tries to make theoretical physicists out of us.

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State University of New York
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November 1972

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CHAPTER I
Introduction

1. INTRODUCTION

Statistical mechanics is an old and venerable branch of physics which has received the attention of many physicists from the time of Gibbs. Since then it has developed in several directions so that at present it is possible to distinguish at least three different theoretical approaches to the subject: (1) the foundational approach, which is concerned with establishing the general properties of, and proving existence theorems for, statistical mechanical systems by rigorous mathematical means, (2) the phenomenological approach, which is concerned with correlating and quantitatively explaining the results of experiments by any available method, and (3) the model-building approach, which attempts to gain insight into practical situations by studying simple models in which at least some physically interesting quantities may be exactly computed. Each of these approaches has made such valuable contributions to our understanding of statistical mechanics that it is neither feasible nor desirable to separate them completely. However, because each approach has developed such a large body of literature, it is likewise not possible to give an adequate treatment of all of them in a single book. Therefore, while we will attempt to make this book self-contained by giving a brief discussion of the foundation of statistical mechanics, and while we will attempt to place the book in a somewhat broader context by making contact with the existing experimental situation, we will concentrate our efforts on the study of certain solvable models.

The number of exactly solvable problems in a field depends on the complexity of the subject. For example, there are innumerable solvable problems in classical mechanics, whereas, at the other extreme, very few problems in relativistic quantum field theory have ever been exactly solved. The scarcity of solvable models in both statistical mechanics and relativistic quantum field theory is due basically to the fact that, in a system with a very large number of particles, each particle may indirectly interact with an enormous number of others even if the fundamental interaction is two-body and of short range. However, it is the purpose of statistical mechanics to study systems with a large number of particles, and the phenomena of greatest interest are precisely those which are not present in the classical or quantum mechanics of a small number of particles. Therefore, a criterion for the usefulness of any model in statistical mechanics is its capability of giving us insight into the new phenomena characteristic of a large number of particles.

The most characteristic feature of statistical mechanical systems is the existence of phase transitions. Surely the most familiar phase transition is either the condensation of steam into water or the freezing of water into ice. Only slightly less familiar is the ferromagnetic phase transition that takes place at the Curie temperature, which, as an example, is roughly 1043°K for iron. Of the several existing models which exhibit a phase transition, the most famous is the Ising model. In three dimensions the model is so complicated that no exact computation has ever been made, while in one dimension the Ising model does not undergo a phase transition. However, it is one of the most beautiful discoveries of twentieth-century physics that in two dimensions the Ising model not only has a ferromagnetic phase transition but also has very many physical properties which may be exactly computed. Indeed, despite the restriction on dimensionality, the two-dimensional Ising model exhibits all of the phenomena peculiar to magnetic systems near the Curie temperature. For that reason, the two-dimensional Ising model forms the basis of almost all our theoretical understanding of the phase transition to the ferromagnetic state.

2. THE ISING MODEL

The model introduced by Ising[1] consists of a lattice of "spin" variables σ_α, which may take on only the values $+1$ and -1. Any two of these "spins" have a mutual interaction energy

$$-E(\alpha, \alpha')\sigma_\alpha\sigma_{\alpha'}. \tag{2.1}$$

The meaning of (2.1) is as follows: the mutual interaction energy is $-E(\alpha, \alpha')$ when σ_α and $\sigma_{\alpha'}$ are both $+1$ or both -1, but is $+E(\alpha, \alpha')$ in the two cases where $\sigma_\alpha = +1$, $\sigma_{\alpha'} = -1$ and $\sigma_\alpha = -1$, $\sigma_{\alpha'} = +1$. In

1. E. Ising, *Z. Physik* **31**, 253 (1925). Some people prefer to refer to this as the Lenz-Ising model because Lenz introduced the model in *Physik. Z.* **21**, 613 (1920). However, Lenz never computed any of the model's properties. Therefore, we will follow the practice of Onsager, Kaufman, and Yang and refer to the model by Ising's name alone.

addition, a spin may interact with an external magnetic field H with an energy

$$-H\sigma_\alpha. \tag{2.2}$$

Like (2.1), (2.2) means that the interaction energy is $-H$ or $+H$ according as the spin σ_α is $+1$ or -1. Throughout this book we will consider only the case where $E(\alpha, \alpha')$ vanishes unless the locations α and α' are nearest neighbors on the lattice. Furthermore, we will restrict ourselves to the square lattice, where all the spins σ_α are situated at the intersections of a square grid. These two restrictions are of quite different natures and deserve comment.

In two dimensions, the restriction to nearest-neighbor interactions has proved essential if we wish to perform exact calculations valid for all temperatures. In one dimension, however, the same remark does not apply, and, in fact, explicit computations on the linear Ising chain have been carried out with interactions which include several neighbors. More important, it has been shown by Dyson[2] that, whereas no phase transition can exist if all interactions are finite and of finite range, a phase transition does exist if the interactions are of infinite range and decrease sufficiently slowly as the separation between the spins becomes large. At present only the existence of this phase transition has been established, but none of its properties have been computed. This one-dimensional work is extremely interesting but does not fall within the scope of this book. It does, however, lead us to believe that in two dimensions a generalization of the interaction to include more than nearest neighbors will change the nature of the phase transition qualitatively only if the range of interaction is infinite. Because dipole-dipole forces are of long range, this is physically an interesting topic, but unfortunately nothing is exactly known at present.

There are numerous two-dimensional lattices other than the square lattice. For example, the triangular, hexagonal, and decorated lattices have all been considered. However, the square lattice is the one which has been most thoroughly studied. Furthermore, the existing work on all of these lattices has been performed by methods quite closely related to those we will develop for the square lattice. Also, with the exception of decorated lattices, most of the physical properties of these lattices reveal no new phenomena not already exhibited by the square lattice. Consequently, for reasons of concreteness and convenience, we will, with one exception in Chapter VIII, not consider these lattices in this book.

With these restrictions, we may now write the total energy of the two-dimensional Ising model as

$$\mathscr{E} = -\sum_j \sum_k \{E_1(j, k)\sigma_{j,k}\sigma_{j,k+1} + E_2(j, k)\sigma_{j,k}\sigma_{j+1,k} + H\sigma_{j,k}\}, \tag{2.3}$$

2. F. Dyson, *Communications in Math. Physics* **12**, 91, 212 (1969).

where the first subscript of σ labels the rows and the second subscript labels the columns of the square lattice. However, further simplifying assumptions on $E_1(j, k)$ and $E_2(j, k)$ are still needed if we wish to obtain explicit results. Throughout the first thirteen chapters of this book we will consider the case first studied by Onsager,[3] namely, the case where $E_1(j, k)$ does not depend on j and k and where $E_2(j, k)$ also does not depend on j and k. We will refer to the square lattice with these conditions on E_1 and E_2 as *Onsager's lattice* and write its total energy explicitly as

$$\mathscr{E} = -E_1 \sum_j \sum_k \sigma_{j,k}\sigma_{j,k+1} - E_2 \sum_j \sum_k \sigma_{j,k}\sigma_{j+1,k} - H \sum_j \sum_k \sigma_{j,k}. \tag{2.4}$$

Onsager's lattice has the property that (ignoring for the moment possible complications at the boundary) each site is equivalent to every other site. In the last two chapters of this book, the restriction to Onsager's lattice is relaxed. Instead, we study lattices where $E_2(j, k)$ is allowed to depend on j but not k although $E_1(j, k)$ is still independent of both j and k.

Our definition of the Ising model is still incomplete, because we have not yet specified the situation at the boundary. In this book, several different choices appear, depending on the physical quantity of interest.

It must be pointed out that the Ising model is a useful model for several physical phenomena other than ferromagnetism. For example, Lee and Yang[4] have used it to study the liquid-gas transition, and it has also proved to be of great value in understanding the order-disorder transition of alloys such as β-brass. However, for the sake of concreteness of interpretation and because the considerations of the last three chapters do not make physical sense with any other interpretation, we will unabashedly think of the Ising model as a ferromagnet.

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We conclude this chapter with a chronological list of the literature on the two-dimensional Ising model that is referred to in the course of this book.

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